# Analytic ideals and rigidity of their quotients

This is a preliminary version. All remarks are welcome.

Ilijas Farah

December 7, 2025

Dedicated to my wonderful daughter, Gala Vera.

Once again we are happy to be here in Sankei Hall playing for you, because we have a lot of interesting things to present. Like this tune we are about to play now is a new one, written by our pianist, Joe Zawinul. It's a kind of thing based on. . . things, in general.

There are a lot things happening in our country, most of them bad right now, but everything is gonna be alright. The name of this tune is *Mercy, mercy, mercy*.

Julian 'Cannonball' Adderley in Tokyo, August 1966

# Contents

# Introduction

This text is based on [40], which is in turn a revised version of [36]. While [40] was obtained from [36] by removing some material (Chapters I, IV, and V) and then indiscriminately adding a fair amount of new material over the period of a few months, the present revision took almost three decades (ok, I was doing other things), involved removing Chapters 3 and 5 (more on this below), rearranging the remaining Chapters, incorporating results obtained in the meantime ([51], [50], [102], [101], [21], [57]) and improving the main results. The short Chapter 3 of [40] on $\mathrm{OCA_T}$-reflection became obsolete and Chapter 5 was removed because it generated virtually no interest or further work, apart from the elegant [172].

The main objective of [40] was to develop structure theory for quotient Boolean algebras over analytic ideals on $\mathbb{N}$, as follows. We first equip the power set $\mathcal{P}(\mathbb{N})$ of $\mathbb{N}$ with the Cantor set topology, and an ideal on $\mathbb{N}$ is called *analytic* if it is an analytic subset of this compact metric space. The 'structure theory' hinged primarily on proving two strong lifting theorems that reduce questions about embeddability and isomorphisms between such quotients to questions about Rudin–Keisler reductions between the underlying ideals. Under suitable assumptions, an isomorphism $\Phi$ between quotients has a *completely additive lifting*, i.e., a map determined by a function $h\colon \mathbb{N} \to \mathbb{N}$ as in Figure 1. A quick review of history, well-known to many

$$\mathbb{N} \xleftarrow{\quad h \quad} \mathbb{N}$$

$$\begin{array}{ccc} \mathcal{P}(\mathbb{N}) & \xrightarrow{A \mapsto h^{-1}(A)} & \mathcal{P}(\mathbb{N}) \\ \downarrow{\scriptstyle \pi_{\mathcal{I}'}} & & \downarrow{\scriptstyle \pi_{\mathcal{I}}} \\ \mathcal{P}(\mathbb{N})\mathcal{I}' & \xrightarrow{\quad \Phi \quad} & \mathcal{P}(\mathbb{N})/\mathcal{I} \end{array}$$

FIGURE 1. A completely additive lifting of $\Phi$.

readers but so essential that it cannot be omitted, is in order. An isomorphism between $\mathcal{P}(\mathbb{N})/\mathcal{I}$ and $\mathcal{P}(\mathbb{N})/\mathcal{J}$ is *trivial* if it has a completely additive lifting. In [136], Walter Rudin proved that the Continuum Hypothesis implies that $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ has $2^{\mathfrak{c}}$ nontrivial automorphisms. This result belongs to the dark side and will be discussed further in §11. A considerably deeper result, that in some forcing extension all automorphisms of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ are trivial, was proved by Shelah in [139]. This resulted the entire subject and our main results (lifting theorems of Chapters 4, 6, and 9) are its direct descendants. Shelah's model is an oracle-cc forcing extension devised specifically for this purpose (and for the purpose of proving that the measure algebra does not have a Borel lifting, also see [139]). An outline of this remarkable proof can be found in [59, §7.1]. The next breakthrough came with [141], where it was realised that because gaps in $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ can be 'frozen' the delicate oracle-cc approach can be replaced by robust forcing axioms, specifically PFA. In [164] it was shown that Shelah's conclusion follows from $\mathrm{OCA_T}$ and MA, removing the need for explicit use of forcing and making the proof accessible to anyone willing to accept these axioms. Finally, in [21] it was shown that $\mathrm{OCA_T}$ alone implies all automorphisms of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ are trivial, but I am getting ahead of myself. Back in the 90's, motivated by a problem of Erdös and Ulam, Just produced an oracle-cc extension in which there is no isomorphism between quotients over $\mathcal{Z}_0$ and $\mathcal{Z}_{\log}$ ([93], [92]). Soon after the $\mathrm{OCA_T}$-breakthrough of [162], Just derived these results from $\mathrm{OCA_T}$ ([94]). In the $\mathrm{OCA_T}$ lifting theorem of [40] it was shown that $\mathrm{OCA_T}$ and MA imply that every isomorphism between quotients over analytic P-ideals have continuous lifting, and even that every homomorphism from $\mathcal{P}(\mathbb{N})$ into such quotient is decomposable ([40, Theorems 3.3.5 and 3.3.6]). In [50, Theorem 4] it was proved that PFA implies every isomorphism over quotients over countably 3204-determined ideals ([50, Theorem 4]) has a continuous lifting. [57, Theorem 1] it was shown that $\mathrm{OCA_T}$ and MA($\sigma$-linked) imply the same conclusion for 80-determined ideals (see Chapter 6).

The lesser of two main results of [40], Theorem 1.9.1, was about homomorphisms with topologically simple liftings. A map from $\mathcal{P}(\mathbb{N})$ to $\mathcal{P}(\mathbb{N})$ is *topologically simple* if it is Baire measurable (i.e., measurable with respect to the $\sigma$-algebra of sets that have the Property of Baire), Borel-measurable, or continuous. It is well-known that for a homomorphism $\Phi$ the existence of a Baire measurable lifting is equivalent to the existence of a continuous lifting.[1] For concreteness, we adopt[2] the terminology from [40] and talk about Baire measurable liftings and Baire-embeddability of quotients, but we also adopt the terminology from [51] (also [59]) and talk about topologically simple homomorphisms. A completely additive lifting is continuous, and in [40] it was proved that if $\mathcal{I}$ is countably generated or a nonpathological analytic P-ideal then every homomorphism from $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ into $\mathcal{P}(\mathbb{N})/\mathcal{I}$ that has a Baire measurable lifting has a completely additive lifting ([40, Theorem 1.9.1]). The special case when $\mathcal{I} = \operatorname{Fin}$ was extracted from [140] in [161], and Just proved important partial results in [92] and [94]. In §4 we present an extension of this result to a larger class of Fubini ideals, following the work of Kanovei and Reeken ([102] and [101]). Just's method of stabilisers that permeated much of [40] is no longer used.

The main result of [40], OCA lifting theorem ([40, Theorems 3.3.5 and 3.3.6]), asserts that under forcing axioms every homomorphism from $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ into $\mathcal{P}(\mathbb{N})/\mathcal{I}$ has a topologically simple lifting for every countably generated ideal and every analytic P-ideals. By combining this result with the analysis of topologically simple homomorphisms, in case when ideals are nonpathological or countably generated, the lifting can be chosen so that it is completely additive. In [50] and [57] this result was improved, first by extending the class of ideals to which it applies to countably 3024-determined ideals (Definition 1.10.1) and then by weakening the

[1] In [69] and [101] it was shown that these are equivalent to the existence of a Haar-measurable lifting.

[2] Or rather, we don't renounce.

forcing axioms used and extending the class of ideals to those that are countably 80-determined. All of these ideals are $F_{\sigma\delta}$ and to the best of my knowledge all $F_{\sigma\delta}$ ideals may turn out to be countably $\infty$-determined but this is rather embarrassing and I'll move on. In the case of [57] the result was proved for homomorphisms from $\mathcal{P}(\mathbb{N})$ into $\mathcal{P}(\mathbb{N})/\mathcal{I}$; there is a good reason for doing this, more the merits of this technically minor strengthening below. We present these results in §6, largely following [57]. Thanks to a simple trick, the original agonising proof of [40, Theorem 3.3.5] that occupied most of Chapter 3 is now almost, but not quite, entirely obsolete. A revised version of the part of this proof where MA is used to 'multiply' a bad homogeneous set is included in §A.7, for reasons that are not entirely sentimental.

We prove another set of lifting results that extends the results of [40, Chapter 4], and have strong implications to the structure of Čech–Stone remainders of locally compact, non-compact, Polish spaces. Only the zero-dimensional case is proved here, but the general case is a theorem of [167], proved by using lifting theory for *-homomorphisms between coronas of separable C*-algebras ([165], [125]). Regrettably, this text is too short to get started on C*-algebras (but see [54]).

**Diagrams.** Some of our results are presented in figures 3–6, see also the legend in Fig. 2. In each of the diagrams, two 'blobs' are connected if ideals in the lower one

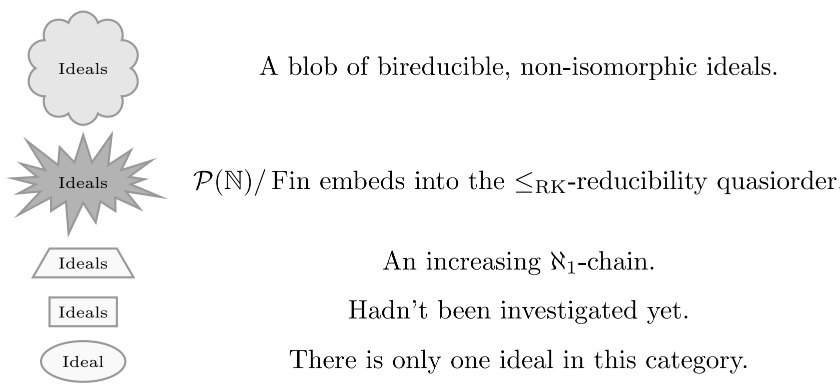


Figure 2. Legend for Fig. 3–6.

are RK-below those in the upper one, and two 'blobs' are not connected if and only if there is no RK-reduction between their elements. For example, by Corollary 2.8.5, Fin is the only ideal RK-reducible to a density ideal and a summable ideal.

Figure 3 depicts the structure of the Rudin–Blass order on some analytic ideals. By Lemma 2.6.1, every ideal RK-reducible to a summable ideal is summable. By Proposition 2.6.5, every non-dense summable ideal is reducible to every other summable ideal (other than Fin), and a dense ideal is not reducible to an ideal which is not dense. Reductions between summable ideals are studied further in §2.6.2. By Theorem 2.6.10, there is an embedding of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ into the poset of dense summable ideals with respect to $\leq_{\mathrm{RB}}$. Proposition 2.7.18 implies that the classes of generalised density ideals and LV-ideals are hereditary with respect to $\leq_{\mathrm{RB}}$. Theorem 2.7.12 implies that the asymptotic density zero ideal $\mathcal{Z}_0$ is the terminal

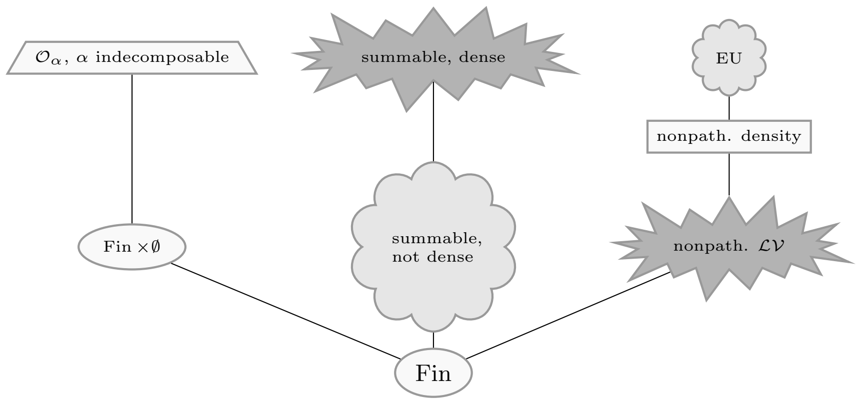


FIGURE 3. RK-reductions between ideals.

object in the category of nonpathological generalised density ideals, and by Proposition 2.7.10 all EU-ideals are $\leq_{\mathrm{RK}}$-bireducible, but Theorem 2.7.16 implies that there are many non-isomorphic EU-ideals. By [146], Fin $\times \emptyset$ is RB-reducible to every analytic ideal that is not a P-ideal. For ordinal ideals $\mathcal{O}_\alpha$ (Definition 1.9.1), where $\alpha$ is an indecomposable countable ordinal, it is straightforward to prove that $\alpha \leq \beta$ implies $\mathcal{O}_\alpha \leq_{\mathrm{RB}} \mathcal{O}_\beta$. On the other hand, [171, Proposition 2.1 and Corollary 2.3] asserts that $\mathcal{O}_\alpha$ is a $\Sigma^0_\alpha$ ideal and that if $\alpha$ is an odd ordinal then it is $\Pi^0_\alpha$ complete. This implies that the ideals $\mathcal{O}_\alpha$, for countable indecomposable ordinal $\alpha$, form an increasing $\aleph_1$-chain with respect to $\leq_{\mathrm{RK}}$. Since the orbit equivalence relation associated to a dense analytic P-ideal is turbulent ([81]), [104] implies that none of the ideals on the right-hand side of Fig. 3 is RK-reducible to any $\mathcal{O}_\alpha$. The picture

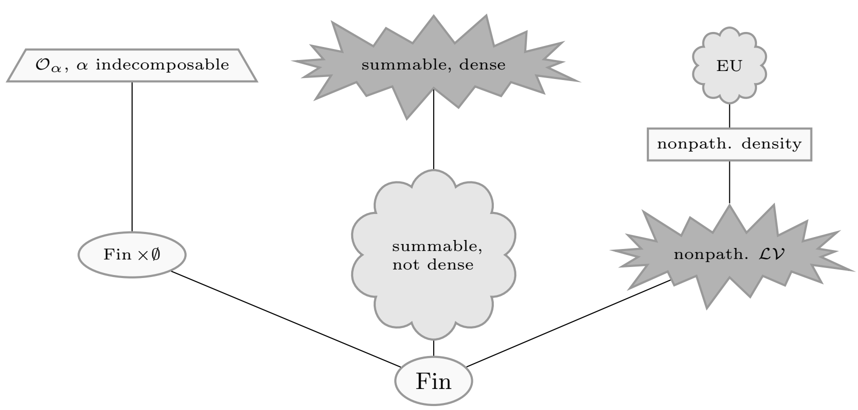


FIGURE 4. Baire embeddability between quotients. The diagram is identical to the one depicting RK-reductions (Fig. 3), but that is exactly the point. It is, perhaps, not entirely impossible that the orders (and diagrams) disagree on pathological ideals.

is incomplete, for example it does not include Mazur's complicated family of $F_\sigma$ ideals ([123]) or any pathological ideals. Since all ideals appearing in Fig. 3 have the Fubini property, and therefore the Radon–Nikodym property (Theorem 4.1.2), Baire-embeddability of their quotients is equivalent to Rudin–Keisler reducibility of the underlying ideals; this is Fig. 4.

Under the Continuum Hypothesis, $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ is universal for quotients over ideals that have the property of Baire and include Fin (Proposition 11.1.9). Thus CH implies that $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ is both initial and final object of the category of quotients over analytic ideals that include Fin and the diagram for embeddability of quotients collapses (Fig. 5). There are however many analytic quotients that are, provably in ZFC, non-isomorphic (see Proposition 5.1.7, and, for continuum many nonisomorphic quotients, [131]).

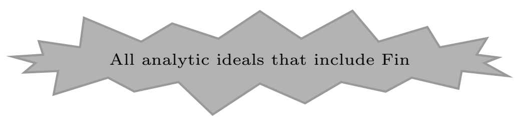


FIGURE 5. Embeddability between quotients over analytic ideals that include Fin under the Continuum Hypothesis.

Finally, if one assumes forcing axioms (such as $\mathrm{OCA_T}$ and MA($\sigma$-linked)), then the embeddability between quotients over ideals $\mathcal{I}$ that are 80-countably determined (this includes all known $F_{\sigma\delta}$ ideals, and in particular all ideals that appear in our figures) almost, but not quite, coincides with the Baire-embeddability (Fig. 6). To be precise, in this situation for an analytic ideal $\mathcal{J}$, there is an embedding from $\mathcal{P}(\mathbb{N})/\mathcal{J}$ into $\mathcal{P}(\mathbb{N})/\mathcal{I}$ if and only if there is a Baire-embedding of $\mathcal{P}(\mathbb{N})/\mathcal{I}$ into $\mathcal{P}(A)/(\mathcal{I} \restriction A)$ for some positive set $A$. By results of [60] and [73], in some forcing extensions this applies to all analytic ideals $\mathcal{J}$. The question whether forcing axioms—any forcing axioms—imply that every homomorphism of $\mathcal{P}(\mathbb{N})$ into $\mathcal{P}(\omega^2)/\mathcal{O}_{\omega^2}$[3] has a continuous almost lifting is wide open.

**Chapter by chapter.** Various ideals to which our lifting results apply are introduced in §1. Rudin–Keisler and Rudin–Blass orders are studied in §2.

In §3 we study largeness properties of hereditary subsets of $\mathcal{P}(\mathbb{N})$, in particular nonmeagerness and being ccc over Fin. These results are used in the proof of the OCA lifting theorem.

§8 is devoted to one of the few properties of $\beta\mathbb{N}$ and other '$\beta\mathbb{N}$-spaces' that one can prove in ZFC. (Following van Douwen, we say that a space is a $\beta\mathbb{N}$ space if the closure of every countable discrete subspace is homeomorphic to $\beta\mathbb{N}$.) If $f$ is a continuous function from any product of compact Hausdorff spaces into a $\beta\mathbb{N}$-space, then the domain can be partitioned into finitely many clopen sets such that the restriction of $f$ to each one of them depends on at most one coordinate (Theorem 8.1.3). This phenomenon, in a weaker form, was first observed by van Douwen ([25]), and this section is a revision of [47], [48], and [45] (the latter one especially benefited from being revised).

§9 is a revision of Chapter 4 of [40], with addition of results of Dow and Hart ([30] and [31]). It extends lifting results of §6 to quotients of 'Polish Boolean algebras' over countable ideals. By Stone duality, this gives an extension principle

[3]The ideal $\mathcal{O}_{\omega^2}$ is isomorphic to $\operatorname{Fin} \times \operatorname{Fin}$ and is arguably the simplest analytic ideal that is not $F_{\sigma\delta}$.

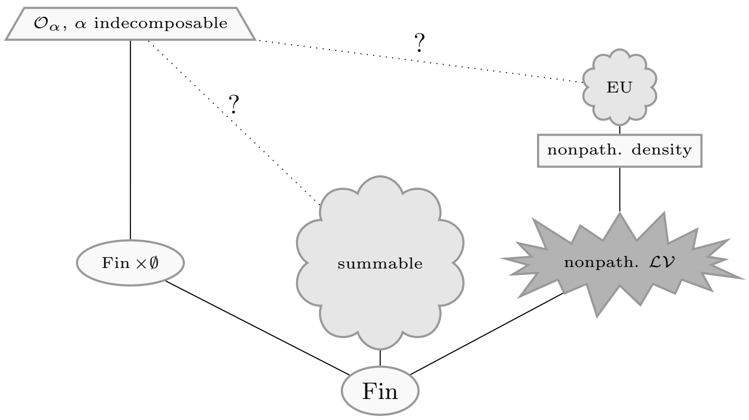


FIGURE 6. Embeddability of quotients assuming $\mathrm{OCA_T}$ and MA($\sigma$-linked). Relations along the dotted lines are not known to be refuted by any forcing axioms, but they do not exist in models constructed in [60] and [73].

for maps between Čech–Stone remainders of locally compact, Polish spaces (Theorem 9.1.4). New methods have lead to a major simplification of the proofs (the proof no longer follows the diagram on p. 136 of [40]). We finally include a proof of [40, Proposition 4.10.1] (in the meantime it appeared in [167] but more about this below).

§10 is devoted to applications of §9 to Čech–Stone remainders and their products. Since the new lifting result no longer requires the kernel of the homomorphism to include Fin, we also prove (and slightly improve) the main result of [31], that under $\mathrm{OCA_T}$ the measure algebra does not embed into $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$.

On the dark side (not to be confused with the dark side of non-Polishable Borel equivalence relations, see [81]), in §11 we (finally!) provide a transparent proof of a 1984 theorem of Just and Krawczyk ([95]), that the Continuum Hypothesis (CH) implies that all quotients over EU-ideals are isomorphic, as well as (equally transparent) proofs of [49, Corollary 5.4], that under CH there are only two isomorphism classes of quotients over dense density ideals (Theorem 11.2.3) and [49, Theorem 5.5], that under CH all quotients over dense LV-ideals are isomorphic (Theorem 11.2.6). We also include a discussion of layered ideals and saturation of the associated quotients.

**Corrections to [40].** During the admittedly hasty and careless preparation of [40], proofreading has been largely waived for the sake of adding more theorems. This peculiar approach resulted in a haphazardly (dis)organized text. It also resulted in a number of typos, obscurities, and outright mistakes. I take this opportunity to correct the ones that I am aware of.

Density ideals and generalised density ideals had been defined in [40], [49], and [51], but the definitions did not quite agree. I decided to stick with the original definition from [40] (Definition 1.7.1; see also the discussion preceding it). In [49], in the definition of LV-ideals it was not explicitly required that they are dense, but it is clear from [49, Theorem 5.5] that it was implicitly assumed that all LV-ideals are dense.

In [50] I claimed that the quotients over the ideals $\mathsf{null}$ and $\mathcal{Z}_s$ (Definition 1.8.7 and Definition 1.8.6) are not isomorphic and that this follows from [32], but I hadn't been able to reconstruct the proof.

[40, Theorem 1.13.3] is wrong, only because the author of [40] confused 'equal' and 'isomorphic'; the paper [157] addresses this issue. Note that Lemma 2.7.4, implicit in the proof of [40, Theorem 1.13.3], improves the results of See [157, Theorem 3.7] that addresses this issue. The corrected version of [40, Theorem 1.13.3] is Theorem 2.7.8. Although the old proof works, it has been rewritten and made more transparent.

The last part of Proposition 3.5.4 has been taken for granted in [40]; it is actually not obvious that a homomorphism that has a continuous lifting on a nonmeager (or even ccc over Fin) ideal decomposes into a homomorphism with a continuous lifting and one with a nonmeager kernel.

After adopting the Kanovei–Reeken Fubini property in §4, the discussion of Ulam stability, [40, §1.8 and §1.9], was removed as it became obsolete.

Finally, the readers of [40] may appreciate some information on how the notation has changed. Instead of

$$\|A\|_\varphi = \limsup_n \varphi(A \setminus n) = \lim_n \varphi(A \setminus n),$$

I will use $\varphi_\infty(A)$.

In [40] virtually ideals were denoted $\mathcal{I}_{☺}$, for some choice of ☺ or another, as I was assuming that the information provided by this symbol will be sufficiently clear to help avoid confusion in every context. Over the intervening years I have learned that nothing is sufficiently clear and that confusion lurks around every corner, and decided to denote the ordinal ideals $\mathcal{O}_\alpha$ instead of $\mathcal{I}_\alpha$ (Definition 1.9.1).

An abbreviation for '$A \subseteq B$ and $A$ is finite' is

$$A \Subset B.$$

Given how many bad things are happening in the world right now, it only made sense to commence this text with the strangely uplifting 1966 Cannonball Adderley quote.

**Acknowledgements.** I am indebted to Alan Dow and K.P. Hart for noticing an issue with the bibliography and several suggestions for how to fix it.

CHAPTER 1

# Ideals

## 1.1. Preliminaries

Absolutely nothing happens in this section, to be read only when necessary. By $|X|$ we denote the cardinality of a set $X$. We accept the convention that $0 \in \mathbb{N}$.[1] We follow von Neumann's convention and define natural number $n$ to be the set of all smaller natural numbers (thus $0 = \emptyset$, $1 = \{\emptyset\}$, and so on, yielding a not completely trivial reading to the formula $|n| = n$ for every natural number $n$). Therefore, $n$ is both an element and a subset of $\mathbb{N}$. To avoid ambiguity, we will write $f[X]$ to denote the pointwise image of a set $X$ under $f$.

By identifying sets of integers with their characteristic functions, we equip $\mathcal{P}(\mathbb{N})$ (the power set of $\mathbb{N}$) with the Cantor space topology and therefore we can assign the topological complexity to the ideals of sets of integers.

**Definition 1.1.1.** An *ideal* on $\mathbb{N}$ is a subset $\mathcal{I}$ of $\mathcal{P}(\mathbb{N})$ that is closed under taking finite unions and subsets of its elements. In other words, $\mathcal{I}$ is ideal in the Boolean ring $(\mathcal{P}(\mathbb{N}), \cup, \cap, \emptyset, \mathbb{N})$. An ideal is *proper* if $\mathbb{N} \notin \mathcal{I}$. An ideal is *Borel* if it is Borel as a subset of $\mathcal{P}(\mathbb{N})$. Ideals which are $F_\sigma$, $F_{\sigma\delta}$, analytic,... are defined analogously. By Fin ideal of all finite subsets of $\mathbb{N}$, the so-called *Fréchet ideal*.

Although we will normally consider only the ideals which include Fin, it will be useful to consider the empty set, $\emptyset$, as an ideal, since it will serve as a building block for some important ideals. If $X$ is an infinite set, we will write $\mathrm{Fin}(X)$ for the ideal of finite subset of $X$. When there is no danger of confusion, we will write Fin for $\mathrm{Fin}(X)$.

**Definition 1.1.2.** if $\mathcal{I}$ is an ideal on $\mathbb{N}$ then on $\mathcal{P}(\mathbb{N})$ we define the following relations on $\mathcal{P}(\mathbb{N})$

1. $A$ and $B$ are *disjoint modulo* $\mathcal{I}$ (in symbols, $A \perp^{\mathcal{I}} B$) if $A \cap B \in \mathcal{I}$.
2. $A$ and $B$ are *equal modulo* $\mathcal{I}$ (in symbols, $A =^{\mathcal{I}} B$) if their symmetric difference $A \Delta B$ belongs to $\mathcal{I}$.
3. $A$ *includes* $B$ *modulo* $\mathcal{I}$ (in symbols, $A \supseteq^{\mathcal{I}} B$) if $B \setminus A \in \mathcal{I}$.
4. $A$ is *included in* $B$ *modulo* $\mathcal{I}$ (in symbols $A \subseteq^{\mathcal{I}} B$,) if $A \setminus B \in \mathcal{I}$.
5. For $m \in \mathbb{N}$ write $A =^m B$ if $A \setminus m = B \setminus m$, $A \subseteq^m B$ if $A \setminus m \subseteq B$ and $A \supseteq^m B$ if $A \cup m \supseteq B$.

If $\mathcal{I} = \mathrm{Fin}$ then we write $\perp, =^*, \subseteq^*, \supseteq^*$ for $\perp^{\mathcal{I}}, =^{\mathcal{I}}, \subseteq^{\mathcal{I}}, \supseteq^{\mathcal{I}}$.

Clearly $=^m$, $=^*$, and $=^{\mathcal{I}}$ are equivalence relations and $A =^* B$ if and only if $A =^m B$ for a large enough $m$. Analogous remarks apply to $\subseteq^m$ and $\supseteq^m$, and clearly $A \subseteq^\dagger B$ if and only if $B \subseteq^\dagger A$, where $\dagger$ is any of $*, \mathcal{I}$, or $m$.

[1] This is a departure from the setup used in [40], justified by simplified notation.

For $A \subseteq \mathbb{N}^2$ and $m, n \in \mathbb{N}$ the corresponding vertical and horizontal sections of $A$ are

$$A_m = \{n : \langle m, n\rangle \in A\}.$$
$$A^n = \{m : \langle m, n\rangle \in A\}.$$

Although the ideals that we are interested live on $\mathbb{N}$, it will be convenient to allow ideals on an arbitrary countable set. The following few definitions provide some justification for this convention, with more to come.

**1.1.1. Direct sum, $\mathcal{I} \oplus \mathcal{J}$.** For two ideals $\mathcal{I}$, $\mathcal{J}$ define their *sum*, $\mathcal{I} \oplus \mathcal{J}$, to be the ideal on $\mathbb{N} \times \{0, 1\}$ given by the following.

$$A \in \mathcal{I} \oplus \mathcal{J} \qquad \Leftrightarrow \qquad \{n : \langle n, 0\rangle \in A\} \in \mathcal{I} \ \ \& \ \ \{n : \langle n, 1\rangle \in A\} \in \mathcal{J}.$$

**1.1.2. The Fubini product, $\mathcal{I} \times \mathcal{J}$.** For $A \subseteq X \times Y$, $x \in X$ and $y \in Y$ it will be convenient to write

$$A_x = \{y \in Y : (x, y) \in A\},$$
$$A^y = \{x \in X : (x, y) \in A\}.$$

The *Fubini product*, $\mathcal{I} \times \mathcal{J}$,, of ideals $\mathcal{I}$ and $\mathcal{J}$ on sets $X$ and $Y$, respectively, is the ideal on $X \times Y$ given by:

$$A \in \mathcal{I} \times \mathcal{J} \qquad \Leftrightarrow \qquad \{x : A_x \notin \mathcal{J}\} \in \mathcal{I}.$$

**1.1.3. Restriction.** If $\mathcal{I}$ is a subset of $\mathcal{P}(X)$, with $\mathcal{I}$ not necessarily (but typically) an ideal and $X$ not necessarily (but typically) $\mathbb{N}$, and $A \subseteq X$, let

$$\mathcal{I} \restriction A = \{B \cap A : B \in \mathcal{I}\}.$$

This notation applies whenever $\mathcal{I}$ is a subset of $\mathcal{P}(\mathbb{N})$, not necessarily an ideal. If $\mathcal{I}$ is an ideal on a set $X$, then we consider the coideal $\mathcal{I}_+$ of $\mathcal{I}$-positive subsets of $X$ and the dual filter $\mathcal{I}_*$, defined as follows.

$$\mathcal{I}_+ = \{A \subseteq X : A \notin \mathcal{I}\}$$
$$\mathcal{I}_* = \{A \subseteq X : X \setminus A \in \mathcal{I}\}.$$

This also serves as a definition of a coideal on $X$ and of a filter on $X$ (any set of the form $\mathcal{I}_+$ or $\mathcal{I}_*$, respectively, for an ideal $\mathcal{I}$). Axiomatisation of these notions is left to the reader.

**1.1.4. Quantifiers $\forall^{\mathcal{I}}$ and $\exists^{\mathcal{I}}$.** If $\mathcal{I}$ is an ideal on a set $X$, then $\forall^{\mathcal{I}}$ and $\exists^{\mathcal{I}}$ denote the quantifiers 'for all but $\mathcal{I}$ many $x \in X$' and 'there exists $\mathcal{I}_+$ many $x \in X$'. To be precise,

$$(\forall^{\mathcal{I}} x)\varphi(x) \qquad \Leftrightarrow \qquad \{x \in X : \neg\varphi(x)\} \in \mathcal{I}$$
$$(\exists^{\mathcal{I}} x)\varphi(x) \qquad \Leftrightarrow \qquad \{x \in X : \varphi(x)\} \in \mathcal{I}_+$$

If $\mathcal{I}$ is the Fréchet ideal, we then write $\forall^\infty$ and $\exists^\infty$ for $\forall^{\mathrm{Fin}}$ and $\exists^{\mathrm{Fin}}$. These are the familiar quantifiers "for all but finitely many" and "there exist infinitely many".

**1.1.5. Quotients.** Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. The elements of the quotient $\mathcal{P}(\mathbb{N})/\mathcal{I}$ are the equivalence classes

$$[A]_{\mathcal{I}} = \{C \subseteq \mathbb{N} : A \Delta C \in \mathcal{I}\}$$

for $A \in \mathcal{P}(\mathbb{N})$. Since this equivalence relation is a congruence, the algebraic operations on the quotient are defined in the natural way.

## 1.2. Closed approximations and $F_\sigma$ ideals

The following notion first appeared in [94].

**Definition 1.2.1.** A set $\mathcal{K} \subseteq \mathcal{P}(\mathbb{N})$ is *hereditary* if $A \subseteq B \in \mathcal{K}$ implies $A \in \mathcal{K}$. If $\mathcal{K}$ and $\mathcal{L}$ are families of subsets of $\mathbb{N}$, then

$$\mathcal{K} \underline{\cup} \mathcal{L} = \{K \cup L : K \in \mathcal{K} \text{ and } L \in \mathcal{L}\}$$

and

$$\mathcal{K}^k = \mathcal{K} \underline{\cup} \dots \underline{\cup} \mathcal{K} = \{A_1 \cup A_2 \cup \dots \cup A_k : A_i \in \mathcal{K} \text{ for } i \leq k\}.$$

Note that $\mathcal{K} \underline{\cup} \operatorname{Fin} = \bigcup_n \{A : A \setminus n \in \mathcal{K}\}$. We say that $\mathcal{K} \subseteq \mathcal{P}(\mathbb{N})$ is an *approximation to* $\mathcal{I}$ if it is hereditary and $\mathcal{I} \subseteq \mathcal{K} \underline{\cup} \operatorname{Fin}$. An approximation is closed if it is closed (as a subset of the Cantor space).

If $\mathcal{K}$ is a hereditary subset of $\mathcal{P}(\mathbb{N})$, then for $A$ and $B$ in $\mathcal{P}(\mathbb{N})$ we write

$$A =^{\mathcal{K}} B$$

if $A \Delta B \in \mathcal{K}$. If $\mathcal{K}$ is not an ideal then this is not an equivalence relation.

We will be primarily interested in closed approximations. Here is an easy fact to help the definitions settle in.

**Lemma 1.2.2.** *If $\mathcal{K}$ is a closed hereditary set, then $a \notin \mathcal{K} \underline{\cup} \operatorname{Fin}$ if and only if there are disjoint intervals $J(n)$, for $n \in \mathbb{N}$, such that $a \cap J(n) \notin \mathcal{K}$ for all $n$.*

Proof. This proof involves one of the key trivialities that will be reoccurring often in what is to come. Sets of the form

$$[J, s] = \{A : A \cap J = s\}$$

for $J \Subset \mathbb{N}$ and $s \subset J$ comprise a basis for the topology on $\mathcal{P}(\mathbb{N})$. A moment of though shows that the sets of the form $[J, s]$ for $J$ a finite interval and $s \subseteq s$ also comprise a basis. Finally, if $\mathcal{K}$ is hereditary and $[J, s] \cap \mathcal{K} = \emptyset$ then $[J, \emptyset] \cap \mathcal{K} = \emptyset$.

Now assume $a \notin \mathcal{K} \underline{\cup} \operatorname{Fin}$. By using the observations from the previous paragraph, recursively find disjoint intervals $J(n)$, for $n \in \mathbb{N}$, such that we have $a \cap J(0) \notin \mathcal{K}$ and $(a \setminus \max(J(n))) \cap J(n+1) \notin \mathcal{K}$. These intervals are as required. □

**Lemma 1.2.3.** *An ideal $\mathcal{I}$ on $\mathbb{N}$ is an $F_\sigma$-ideal if and only if it has a closed approximation $\mathcal{K}$ such that $\mathcal{I} = \mathcal{K} \underline{\cup} \operatorname{Fin}$.*

Proof. Only the direct implication requires a proof. If $\mathcal{K} \subseteq \mathcal{P}(\mathbb{N})$ is closed, then its hereditary closure $\hat{\mathcal{K}} = \bigcup_{a \in \mathcal{K}} \mathcal{P}(a)$ is also closed. Therefore an $F_\sigma$-ideal $\mathcal{I}$ can be written as a union of closed hereditary sets, $\mathcal{I} = \bigcup_n \mathcal{K}_n$.

Let $\mathcal{K} = \{a \setminus n : a \in \mathcal{K}_n\}$. This is a closed hereditary set included in $\mathcal{I}$, and clearly $\mathcal{I} = \mathcal{K} \underline{\cup} \operatorname{Fin}$. □

Lemma 1.2.3 appears in [114, Lemma 6.3] and it also follows from [123, Lemma 1.2(c)]; see Theorem 1.4.6 below.

We end with two properties of closed hereditary sets needed in §1.10.

**Lemma 1.2.4.** *Suppose that $\mathcal{K} \subseteq \mathcal{P}(\mathbb{N})$ is hereditary. Then the following are equivalent.*

*(1) $\mathcal{K}$ is closed*

*(2) For every sequence $A_n \in \mathcal{K}$ such that $A_n \subseteq A_{n+1}$ for all $n$ we have $\bigcup_n A_n \in \mathcal{K}$.*

*(3) For every sequence $A_n \in \mathcal{K} \cap \mathrm{Fin}$ such that $A_n \subseteq A_{n+1}$ for all $n$ we have $\bigcup_n A_n \in \mathcal{K}$.*

Proof. It suffices to prove that the third condition implies $\mathcal{K}$ is closed. Assume that $B_n \in \mathcal{K}$ converge pointwise to $A \subseteq \mathbb{N}$. For every $n$ there is $A_n \subseteq n$ such that $A_n = B_j \cap n = A \cap n$ for all but finitely many $j$. Since $\mathcal{K}$ is hereditary, $A_n \in \mathcal{K}$ for all $n$. Clearly $A_n \subseteq A_{n+1}$ and $A = \bigcup_n A_n$. We therefore have $A \in \mathcal{K}$ as required, □

**Lemma 1.2.5.** *If $\mathcal{K}_n$ are closed and hereditary for all $n$ and $\mathcal{K}_n \cap \mathcal{P}(n) = \{\emptyset\}$ for all $n$, then $\mathcal{K} = \bigcup_n \mathcal{K}_n$ is closed and hereditary.*

Proof. The union of a family of hereditary sets is hereditary. Any pointwise convergent sequence in $\mathcal{K}$ that is not included in a finite union of $\mathcal{K}_n$'s converges pointwise to $\emptyset$. □

## 1.3. P-ideals and dense (tall) ideals

An ideal $\mathcal{I}$ on a set $X$ is called *nontrivial* if $\mathcal{I} \neq \{\emptyset\}$ and $\mathcal{Y} \neq \mathcal{P}(X)$. An ideal on $X$ is *proper* if $\mathcal{I} \neq \mathcal{P}(X)$.

**Definition 1.3.1.** An ideal $\mathcal{I}$ on a set $X$ is *dense* (or *tall*) if every infinite subset of $X$ has an infinite subset in $\mathcal{I}$.

**Definition 1.3.2.** An ideal $\mathcal{I}$ is a *P-ideal* if for every sequence $A_n$, for $n \in \mathbb{N}$ of sets in $\mathcal{I}$ there is a single set $A_\infty$ in $\mathcal{I}$ such that $A_n \subseteq^* A_\infty$ for all $n$.

An ideal $\mathcal{I}$ is a *$P^+$-ideal* if for every $\subseteq$-decreasing sequence $A_n$, for $n \in \mathbb{N}$ of $\mathcal{I}$-positive sets there is a $\mathcal{I}$-positive set $A_\infty$ in $\mathcal{I}$ such that $A_n \supseteq^* A_\infty$ for all $n$.

In other words, $\mathcal{I}$ is a P-ideal if the partial order $\mathcal{I}, \subseteq^*$ is $\sigma$-directed Note that trivial ideals are (trivially) P-ideals. The following is well-known, but we include its proof as it anticipates some of the ideas used in proofs that quotients over layered ideals are $\aleph_1$-saturated given in §11.1.

**Lemma 1.3.3.** *(1) The intersection of a countable family of dense ideals is a dense ideal.*

*(2) The intersection of a countable family of P-ideals is a P-ideal.*

Proof. (1) Suppose that $\mathcal{I}_n$, for $n \in \mathbb{N}$, are dense ideals. Let $\mathcal{I} = \bigcap_n \mathcal{I}_n$ and fix an infinite subset $A$ of $\mathbb{N}$. Since each $\mathcal{I}_n$ is dense, we can find a $\subseteq$-decreasing sequence $A_n$, for $n \in \mathbb{N}$, of subsets of $\mathbb{N}$ such that $A_n \in \mathcal{I}_n$ for all $n$. Let $m(n) \in A_n$, for $n \in \mathbb{N}$, be such that $m(n) < m(n+1)$ for all $n$. Then the set $B = \{m(n) : n \in \mathbb{N}\}$ is infinite, $B \setminus A_n$ is finite for all $n$, and $B|in \bigcap_n \mathcal{I}_n$. Since $A$ was arbitrary, this shows that $\bigcap_n \mathcal{I}_n$ is dense.

(2) Suppose that $\mathcal{I}_n$, for $n \in \mathbb{N}$, are P-ideals. Let $\mathcal{I} = \bigcap_n \mathcal{I}_n$ and choose $A_n \in \mathcal{I}$, for $n \in \mathbb{N}$. Since $\mathcal{I}_n$ is a P-ideal, we can choose $B_n \in \mathcal{I}_n$ such that $A_m \setminus B_n$ is finite

for all $n$. By replacing $A_n$ with $\bigcup_{j\leq n} A_j$ and $B_n$ with $\bigcap_{j\leq n} B_j$ we may assume that $A_n \subseteq A_{n+1}$ and $B_{n+1} \subseteq B_n$ for all $n$. Let

$$C = \bigcup_n (A_n \cap \bigcap_{j\leq n} B_j).$$

Then $A_n \subseteq^* C$, $C \subseteq^* B_n$ for all $n$, and $C \in \mathcal{I}$. Since the sequence $(A_n)$ was arbitrary, this proves that $\mathcal{I}$ is a P-ideal. □

An $\aleph_0$-*limit* in a Boolean algebra is a strictly decreasing sequence that has the greatest lower bound or a strictly increasing sequence that has the lowest upper bound.

**Lemma 1.3.4.** *An ideal $\mathcal{I}$ is a $P^+$-ideal if the quotient $\mathcal{P}(\mathbb{N})/\mathcal{I}$ has no $\aleph_0$-limits.*

Proof. A counterexample to being a $\mathrm{P}^+$-ideal is clearly an $\aleph_0$-limit. On the other hand, if there is an $\aleph_0$-limit in $\mathcal{P}(\mathbb{N})/\mathcal{I}$ then by taking intersections (and complements if the sequence is increasing) one obtains a violation of being a $\mathrm{P}^+$-point. □

**Example 1.3.5** (Ideals $\mathrm{Fin}\times\emptyset$ and $\emptyset\times\mathrm{Fin}$)**.** The ideal $\mathrm{Fin}\times\emptyset$ is the Fubini product of the ideals Fin and $\emptyset$, therefore some $A \subseteq \mathbb{N}^2$ belongs to $\mathrm{Fin}\times\emptyset$ if $\forall^\infty n(A_n = \emptyset)$. This ideal is $F_\sigma$, being equal to the union of the closed sets $\mathcal{P}(n \times \mathbb{N})$.

The ideal $\emptyset \times \mathrm{Fin}$ is the Fubini product of $\emptyset$ and Fin, and therefore some $A \subseteq \mathbb{N}^2$ is in $\emptyset \times \mathrm{Fin}$ if $\forall n(A_n \in \mathrm{Fin})$. This ideal is $F_{\sigma\delta}$, because $A \in \emptyset \times \mathrm{Fin}$ if and only if

$$(\forall m)(\exists n)(\forall k > n)(m, k) \notin A.$$

Taking $n = f(m)$ in the previous line, we obtain an alternative equivalent definition of $\emptyset \times \mathrm{Fin}$. It is the ideal of all subsets or $\mathbb{N}^2$ generated by the sets below the graph of a function. To wit, if $f\colon \mathbb{N} \to \mathbb{N}$, the set of all pairs below the graph of $f$ is

$$\Gamma_f = \{(m, k) : k \leq f(m)\}.$$

Then $A \in \emptyset \times \mathrm{Fin}$ if and only if $A \subseteq \Gamma_f$ for some $f$.

**Lemma 1.3.6.** *Suppose that $\mathcal{I}$ is an ideal on a set $X$.*

*(1) If $\mathcal{I}$ does not include a cofinite subset of $\mathbb{N}$ then $\mathrm{Fin}\times\mathcal{I}$ is not a P-ideal.*
*(2) If $\mathcal{I}$ is a P-ideal then $\emptyset \times \mathcal{I}$ is a P-ideal.*
*(3) If a proper ideal includes $\mathrm{Fin} \times \mathrm{Fin}$ then it is not a P-ideal.*

*In particular, $\mathrm{Fin}\times\emptyset$ is not a P-ideal and $\emptyset \times \mathrm{Fin}$ is a P-ideal.*[2]

Proof. (1) Suppose $\mathcal{I}$ does not include a cofinite subset of $\mathbb{N}$. Then $m \times \mathbb{N}$ is in $\mathrm{Fin}\times\mathcal{I}$ for all $m$. If $A \subseteq \mathbb{N}^2$ is such that $m \times \mathbb{N} \subseteq^* A$ for all $m$, then $A_n$ is a cofinite subset of $\mathbb{N}$ for all $n$, hence $A \notin Fin \times \mathcal{I}$.

(2) Fix $A(m) \in \emptyset \times \mathcal{I}$ for $m \in \mathbb{N}$. Since $\mathcal{I}$ is a P-ideal, for every $n$ there is $B(n) \in \mathcal{I}$ such that $\{n\} \times B(n) \supseteq^* A(m)_n$ for all $n$. We may also assure that $\{n\} \times B(n) \supseteq A(j)$ for all $j \leq n$. Let $B = \bigcup_n \{n\} \times B(n)$. Then $B \in \emptyset \times \mathcal{I}$. For every $m$ we have that $A(m)_n \setminus B$ is finite for all $n$ and $A(m)_n \setminus B = \emptyset$ for all $n > m$. Therefore $A(m) \setminus B$ is finite for all $m$, as required.

(3) This is because if a set that includes all $m \times \mathbb{N}$ modulo finite, then its complement belongs to $\emptyset \times \mathrm{Fin}$ and hence to $\mathrm{Fin} \times \mathrm{Fin}$. □

[2]In set-theorese this translates as $\mathfrak{b} \geq \aleph_1$.

Note that if $\mathcal{I}$ is a proper ideal that includes a cofinite subset of $\mathbb{N}$, $B$, (for example if $\mathcal{I} = \mathcal{P}(\mathbb{N} \setminus \{0\})$) then $\mathrm{Fin} \times \mathcal{I}$ is isomorphic to Fin.

Here is a generalisation of the second part of Lemma 1.3.6 whose proof is, being a minor modification of the given proof, omitted.

**Lemma 1.3.7.** *Suppose $\mathcal{I}(n)$, for $n \in \mathbb{N}$, is a P-ideal on set $X(n)$. Then*

$$\{A \subseteq \bigcup_n \{n\} \times X(n) : (\forall n) A_n \in \mathcal{I}(n)\}$$

*is a P-ideal on $\bigcup_n \{n\} \times X(n)$.* □

### 1.4. Lower semicontinuous submeasures and analytic P-ideals

Apart from the definitions, this section contains the statement (but not the proof) of Solecki's and Mazur's characterisation of analytic P-ideals (Theorem 1.4.6) and its consequence, that an analytic P-ideal $\mathcal{I}$ is either $F_\sigma$ or satisfies $\emptyset \times \mathrm{Fin} \leq_{\mathrm{RB}} \mathcal{I}$ (Corollary 1.4.8).

#### 1.4.1. Submeasures.

**Definition 1.4.1.** If $\mathbb{B}$ is a Boolean algebra, then $\varphi\colon \mathbb{B} \to [0, \infty]$ is is a *submeasure* on $\mathbb{B}$ if

$$\varphi(\emptyset) = 0,$$
$$\varphi(A) \leq \varphi(A \cup B) \leq \varphi(A) + \varphi(B),$$

for all $A, B$ in $\mathbb{B}$. If $\mathbb{B} = \mathcal{P}(X)$ for some set $X$ then $\varphi$ is called a *submeasure on* $X$. Since we will not need submeasures on $\mathcal{P}(\mathbb{B})$ when $\mathbb{B}$ is a Boolean algebra, this terminology is unambiguous. [3]

**Definition 1.4.2.** If $\varphi$ is a submeasure on a set $X$, then its *support* is

$$\mathrm{supp}(\varphi) = \{n \in X : \varphi(\{n\}) \neq 0\}.$$

Submeasures $\varphi$ and $\psi$ on $X$ are *orthogonal* if they have disjoint supports. Let

$$\mathrm{supp}(\varphi) = \{x : \varphi(\{x\}) > 0\}$$
$$\|\varphi\| = \varphi(\mathrm{supp}(\varphi)),$$
$$\mathrm{at}^+(\varphi) = \sup_{k \in \mathrm{supp}(\varphi)} \varphi(\{k\}),$$
$$\mathrm{at}^-(\varphi) = \inf_{k \in \mathrm{supp}(\varphi)} \varphi(\{k\}).$$

**Definition 1.4.3.** If $\varphi$ is a submeasure on $\mathbb{N}$, let

$$\mathrm{Fin}(\varphi) = \{A : \varphi(A) < \infty\},$$
$$\mathrm{Exh}(\varphi) = \{A : \lim_n \varphi(A \setminus n) = 0\}.$$

Since the function $n \mapsto \varphi(A \setminus n)$ is nonincreasing, $\mathrm{Exh}(\varphi) = \{A : \inf_n \varphi(A \setminus n) = 0\}$.

If $\varphi$ is a submeasure which is Borel-measurable as a function on $\mathcal{P}(\mathbb{N})$ then $\mathrm{Fin}(\varphi)$ and $\mathrm{Exh}(\varphi)$ are Borel ideals on $\mathbb{N}$. Conversely, every Borel ideal $\mathcal{I}$ on $\mathbb{N}$ is equal to $\mathrm{Fin}(\varphi)$ for the Borel submeasure $\varphi$ defined by $\varphi(A) = \infty$ if $A \notin \mathcal{I}$ and $\varphi(A) = 0$ if $A \in \mathcal{I}$.[4]

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[3] One could talk about submeasures on the Stone spaces of $\mathbb{B}$, but the route taken here appears to be more reasonable.

[4] This apparently silly fact will be of use in §4.2.

It will be convenient to consider ideals and submeasures on countable sets other than $\mathbb{N}$. This change does not affect the definition (or properties) of $\mathrm{Fin}(\varphi)$, and $\mathrm{Exh}(\varphi)$ is defined as follows[5]):

$$\mathrm{Exh}(\varphi) = \{A : \inf_{F \Subset X} \varphi(A \setminus F) = 0\}.$$

A submeasure $\varphi$ on $\mathbb{N}$ is called *lower semicontinuous* if every $A \subseteq \mathbb{N}$ satisfies

$$\varphi(A) = \lim_{n\to\infty} \varphi(A \cap (n+1)).$$

This is equivalent to $\varphi$ being lower semicontinuous in the Cantor set topology on $\mathcal{P}(\mathbb{N})$.

**Lemma 1.4.4.** *If $\varphi$ is a lower semicontinuous submeasure supported by $\mathbb{N}$, then $\mathrm{Exh}(\varphi)$ is an $F_{\sigma\delta}$ P-ideal, and $\mathrm{Fin}(\varphi)$ is an $F_\sigma$ ideal which includes $\mathrm{Exh}(\varphi)$.*

*Also, $\mathrm{Exh}(\varphi)$ is dense if and only if $\limsup_n \varphi(\{n\}) = 0$.*

Proof. If $\varphi$ is lower semicontinuous then $\mathrm{Fin}(\varphi) = \bigcup_n \{A : \varphi(A) \leq n\}$ is $F_\sigma$ and $\mathrm{Exh}(\varphi) = \bigcap_m \bigcup_n \{A : \varphi(A \setminus n) \leq 1/(m+1)\}$ is $F_{\sigma\delta}$. The fact that $\mathrm{Fin}(\varphi) \subseteq \mathrm{Exh}(\varphi)$ follows by the Cauchy criterion for convergence of a series.

It only remains to see that $\mathrm{Exh}(\varphi)$ is a P-ideal. Assume $A_i$ ($i \in \mathbb{N}$) is a sequence of sets in $\mathrm{Exh}(\varphi)$. Fix $n_i$, for $i \in \mathbb{N}$, such that $\varphi(A_i \setminus n_i) \leq 2^{-i}$ for every $i$, and let $A_\infty = \bigcup_i (A_i \setminus n_i)$. Fix $n \in \mathbb{N}$ and let $m$ be such that $\varphi\big(\bigcup_{i\leq n} A_i \setminus m\big) \leq 2^{-n-1}$. Then $\varphi(A_\infty \setminus m) \leq 2^{-n}$. Since $n$ was arbitrary, $A_\infty$ is in $\mathrm{Exh}(\varphi)$.

To prove the second statement, note that $\limsup_n \varphi(\{n\}) > 0$ if and only if the set $A_\varepsilon = \{n : \varphi(\{n\}) \geq \varepsilon\}$ is infinite for some $\varepsilon > 0$, and that a set has no infinite subset in $\mathcal{I}$ if and only it is equal modulo finite to $A_\varepsilon$ for some $\varepsilon > 0$. $\square$

Every analytic P-ideal is of the form $\mathrm{Exh}(\varphi)$ for a lower semicontinuous submeasure $\varphi$ (Theorem 1.4.6). Here are some explicit examples. They will not be used, and the proofs are left to the reader.

**Example 1.4.5** (Explicit submeasures)**.**

(1) The ideal $\emptyset \times \mathrm{Fin}$ is equal to $\mathrm{Exh}(\varphi)$ for the lower semicontinuous submeasure $\varphi$ defined on $\mathbb{N}^2$ by

$$\varphi(A) = \frac{1}{\min\{n : A_n \neq \emptyset\} + 1}.$$

(2) For summable ideal $\mathcal{I}_f$ associated with $f\colon \mathbb{N} \to \mathbb{R}_+$ (§1.5.1) we have $\mathcal{I}_f = \mathrm{Fin}(\mu_f)$, where

$$\mu_f(A) = \sum_{i\in A} f(i)$$

is a lower semicontinuous measure (every lower semicontinuous measure on $\mathbb{N}$ is of this form).

(3) EU-functions and EU-ideals had been defined in §1.7.1. If $f\colon \mathbb{N} \to \mathbb{R}_+$, we write $\mu_f(A) = \sum_{n\in A} f(n)$. If $f$ is called an EU-function if $\mu_f(\mathbb{N}) = \infty$ and $\lim_n f(n)/\mu_f(n+1) = 0$. Define[6]

$$\varphi_f(A) = \sup_n \frac{\mu_f(A \cap n)}{\mu_f(n)}.$$

[5] We write $F \Subset X$ for '$F \subseteq X$ and $F$ is finite'.

[6] The displayed formula presents a rare occasion when identifying $n$ with $\{0, \dots, n-1\}$ may cause some confusion.

This is a lower semicontunuous submeasure. Recall that the $f$-density was defined as $d_f(A) = \lim_n \frac{\mu_f(A\cap n)}{\mu_f(n)}$ and that $\mathcal{EU}_f = \{A : d_f(A) = 0\}$. It is not difficult to check that $\mathcal{EU}_f = \mathrm{Exh}(\varphi_f)$.

(4) Suppose that $\mathcal{I}(n)$, for $n \in \mathbb{N}$, is a P-ideal on set $X(n)$. By Lemma 1.3.7,

$$\mathcal{J} = \{A \subseteq \bigcup_n \{n\} \times X(n) : (\forall n) A_n \in \mathcal{I}(n)\}$$

is a P-ideal on $Y = \bigcup_n \{n\} \times X(n)$. If $\mathcal{I}(n) = \mathrm{Exh}(\varphi_n)$ then define a submeasure $\psi$ on $Y$ as follows. First let $\tilde{\varphi}_n = \min(\varphi_n, 1/n)$; this operation does not change the Exh of the submeasure. Then let

$$\psi(A) = \sup_n \tilde{\varphi}(A_n).$$

It is an exercise to verify that $\mathcal{J} = \mathrm{Exh}(\psi)$.

Not every ideal of the form $\mathrm{Fin}(\varphi)$ for a lower semicontinuous submeasure is a P-ideal. For example, define $\varphi$ on $\mathbb{N}^2$ by

$$\varphi(A) = \sup\{n : A \setminus (n \times \mathbb{N}) \neq \emptyset\}.$$

Then $\mathrm{Exh}(\varphi) = \mathrm{Fin}(\mathbb{N}^2)$ and $\mathrm{Fin}(\varphi) = \mathrm{Fin} \times \emptyset$ (see Example 1.3.5).

### 1.4.2. Characterisation of analytic P-ideals and some consequences.

**Theorem 1.4.6** (Mazur, Solecki)**.** *Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. Then*

*(1) $\mathcal{I}$ is an $F_\sigma$ ideal if and only if $\mathcal{I} = \mathrm{Fin}(\varphi)$ for some lower semicontinuous submeasure $\varphi$.*

*(2) $\mathcal{I}$ is an analytic P-ideal if and only if $\mathcal{I} = \mathrm{Exh}(\varphi)$ for some lower semicontinuous submeasure $\varphi$.*

*(3) $\mathcal{I}$ is an $F_\sigma$ P-ideal if and only if $\mathcal{I} = \mathrm{Fin}(\varphi) = \mathrm{Exh}(\varphi)$ for some lower semicontinuous submeasure $\varphi$.* □

Proof of the first part is in [123, Lemma 1.2] (it can be deduced from Lemma 1.2.3). Proofs of the other two parts are in [146, Theorem 2.1]. Solecki proved the following more precise statement; it follows by combining Theorem 1.4.6 with Lemmas 2.2–2.5 in [146] (for closed approximations see §1.2).

**Proposition 1.4.7.** *If $\mathcal{I}$ is an analytic ideal on $\mathbb{N}$, then the following are equivalent.*

*(1) $\mathrm{Fin} \times \emptyset \not\leq_{\mathrm{RB}} \mathcal{I}$.*

*(2) If $\mathcal{K}_n$, for $n \in \mathbb{N}$, are closed hereditary subsets of $\mathcal{P}(\mathbb{N})$ none of which is an approximation to $\mathcal{I}$, then $\bigcup_n \mathcal{K}_n$ is not an approximation to $\mathcal{I}$.*

*(3) $\mathcal{I} = \mathrm{Exh}(\varphi)$ for a lower semicontinuous measure $\varphi$.* □

Part (2) implies that the family of closed hereditary subsets of $\mathcal{P}(\mathbb{N})$ that are not an approximation to $\mathcal{I}$ is, in the terminology of [107], a $\sigma$-ideal of compact sets. By [107], this set is $G_\delta$ in the Vietoris topology and an argument analogous to the Birkhoff–Kakutani metrisation theorem can be used to prove (3).

Corollary 1.4.8 below is also a consequence of [144]. It refers to the Rudin–Blass order defined in §2 that is surely familiar to many readers.

**Corollary 1.4.8.** *Suppose that $\mathcal{I}$ is an analytic P-ideal. Then the following are equivalent.*

*(1) $\mathcal{I}$ is not $F_\sigma$.*

*(2) There is a partition of $\mathbb{N}$ into $\mathcal{I}$-positive sets, $\mathbb{N} = \bigsqcup_n A_n$, such that every $B \subseteq \mathbb{N}$ satisfies $B \in \mathcal{I}$ if and only if $B \cap A_n \in \mathcal{I}$ for all $n$.*

*(3)* $\emptyset \times \mathrm{Fin} \leq_{\mathrm{RB}} \mathcal{I}$.

Proof. By Theorem 1.4.6, $\mathcal{I} = \mathrm{Exh}(\varphi)$ for some lower semicontinuous $\varphi$. If there is $\varepsilon > 0$ such that $\varphi(A) < \varepsilon$ implies $A \in \mathcal{I}$, then

$$\mathcal{K} = \{B \subseteq \mathbb{N} : \varphi(B) \leq \varepsilon\}$$

is a closed approximation to $\mathcal{I}$ such that $\mathcal{I} = \mathcal{K} \underline{\cup}\, \mathrm{Fin}$, hence $\mathcal{I}$ is $F_\sigma$.

It therefore suffices to prove that the following is equivalent to (2) and (3).

(4) For every $\varepsilon > 0$ some $\mathcal{I}$-positive set $X$ satisfies $\varphi(X) < \varepsilon$.

Assume (4). Since every $\mathcal{I}$-positive set can be partitioned into two $\mathcal{I}$-positive sets, we can find disjoint $\mathcal{I}$-positive sets $A_n$, for $n \in \mathbb{N}$, such that $\varphi(A_n) < 2^{-n}$ for all $n \geq 1$. We may assume $\bigsqcup_n A_n = \mathbb{N}$. For $B \subseteq \mathbb{N}$ such that $B \cap A_n \in \mathcal{I}$ for all $n$ we have that $A \in \mathrm{Exh}(\varphi)$, and therefore (2) holds.

If (2) holds, then since Corollary 3.2.3 implies $\mathrm{Fin} \leq_{\mathrm{RB}} \mathcal{I} \restriction A_n)$ for all $n$, (3) follows.

Finally, assume (3) holds. Since $\emptyset \times \mathrm{Fin}$ is not $F_\sigma$ and RB-reduction is continuous, $\mathcal{I}$ cannot be $F_\sigma$ and (4) follows. □

**1.4.3. Submeasures on Boolean algebras.** This should not lead to confusion. A submeasure on a Boolean algebra $\mathcal{B}$ is *strictly positive* if it assigns a strictly positive value to every nonzero element. A submeasure on a set $X$ is *strictly positive* if it is strictly positive as a submeasure on $\mathcal{P}(X)$. The following definition will be needed only sporadically.

**Definition 1.4.9.** A submeasure $\varphi$ on a Boolean algebra $\mathbb{B}$ is *continuous* if for every decreasing sequence $A_n$, for $n \in \mathbb{N}$, in $\mathbb{B}$ we have $\varphi(\bigcap_n A_n) = \inf_n \varphi(A_n)$.

A submeasure on an atomless Boolean algebra is called *pathological* if it does not majorise any nonzero finitely additive functional. The existence of a continuous pathological submeasure on the algebra of clopen subsets of $\mathcal{P}(\mathbb{N})$ is a deep theorem ([**151**]). With the above definition, no lower semicontinuous submeasure on a set $X$ could be pathological. The following (nonstandard) definition is taken from [**40**, p. 21].

**Definition 1.4.10.** A submeasure $\varphi$ on $\mathbb{N}$ is *pathological* if there is a subset $A$ of its support such that for every measure $\mu$ on $\mathbb{N}$, $\mu \leq \varphi$ implies $\mu(A) < \varphi(A)$.

An analytic P-ideal $\mathcal{I}$ is *pathological* if every submeasure $\varphi$ such that $\mathcal{I} = \mathrm{Exh}(\varphi)$ is pathological $P(\varphi) = \infty$ and *non-pathological* otherwise.

The importance of this notion largely stemmed from the approach to lifting theorems of [**40**] that relied on stabilisers and Ulam-stability of approximate homomorphisms . Some analytic P-ideals are pathological, as a consequence of Theorem 4.4.1 using [**40**, Theorem 1.9.2] and Theorem 4.4.2. We do not reproduce these theorems here because the old approach to lifting theorems is now replaced with the Fubini property.

## 1.5. Examples of $F_\sigma$ P-ideals

**1.5.1. Summable ideals.** These ideals were introduced by Mathias ([**120**]) and studied, among others, by Mazur ([**123**]). The summable ideal $\mathcal{I}_{1/n}$ was also considered by Erdös and Monk (see [**26**]).

Summable ideals are ideals of the form

$$\mathcal{I}_f = \{A : \sum_{n\in A} f(n) < \infty\}$$

for some positive function $f$ such that $\sum_n f(n) = \infty$. For a function $f\colon \mathbb{N} \to \mathbb{R}_+$ let $\mu_f$ denote the measure associated to $f$,

$$\mu_f(s) = \sum_{k\in s} f(k)$$

so that $\mathcal{I}_f = \{A : \mu_f(A) < \infty\} = \{A : \lim_n \mu_f(A \setminus n) = 0\}$. In other words, with $\varphi(A) = \sum_{n\in A} f(n)$, we have $\mathcal{I}_f = \mathrm{Exh}(\varphi) = \mathrm{Fin}(\varphi)$ (see Theorem 1.4.6).

Thus, every summable ideal is an $F_\sigma$ P-ideal. This can be seen explicitly by noting that it is equal to the union of closed sets $\{A \subseteq \mathbb{N} : \sum_{n\in A} f(n) \leq k\}$, for $k \in \mathbb{N}$. Also, Cauchy's criterion for convergence of a sequence easily implies that $\mathcal{I}_f$ is a P-ideal.

We will return to summable ideals in §2.6.

**1.5.2. Simple examples of non-summable $F_\sigma$ P-ideals.** If $\mathbb{N} = \bigsqcup_n J_n$, all sets $J_n$ are finite, and $\varphi_n$ is a submeasure on $J_n$, then $\sum_n \varphi_n$ is a lower semi-continuous submeasure and $\mathrm{Exh}(\varphi) = \mathrm{Fin}(\varphi)$ is an $F_\sigma$ P-ideal. At some point it was not clear whether there exists an $F_\sigma$ P-ideal which is not summable.

Pathological submeasures have already been used to construct $F_\sigma$ ideals with peculiar properties. Mazur ([**123**, Theorem 1.9]) constructs an $F_\sigma$ ideal which is not included in any summable ideal using one of the most frequently rediscovered examples of a pathological submeasure (see [**156**], [**150**], [**98**]; a much deeper example of a pathological submeasure on an atomic Boolean algebra was constructed in [**151**]). Mazur's ideal is not a P-ideal, but the following gives a non-pathological $F_\sigma$ P-ideal which is not summable.

**Example 1.5.1.** There is a non-pathological $F_\sigma$ P-ideal $\mathcal{I}$ on $\mathbb{N}$ which is not summable. For $k \in \mathbb{N}$ let $I_k = [2^k, 2^{k+1})$ and define $\psi_k\colon \mathcal{P}(I_k) \to \mathbb{R}_+$, $\psi\colon \mathcal{P}(\mathbb{N}) \to \mathbb{R}_+$ and an ideal $\mathcal{I}$ by

$$\psi_k(s) = \frac{\min(k, |s|)}{k^2},$$
$$\psi(A) = \sum_{k=0}^{\infty} \psi_k(A \cap I_k),$$
$$\mathcal{I} = \{A : \psi(A) < \infty\}.$$

We claim that $\mathcal{I}$ is not equal to a summable ideal $\mathcal{I}_f$ for any function $f$. Assume this is not true, and let $f\colon \mathbb{N} \to \mathbb{R}_+$ be such that $\mathcal{I}_f = \mathcal{I}$. We claim there is an integer $M$ such that

$$\bigcup\{I_k : \mu_f(I_k)/\psi(I_k) < M\} \in \mathcal{I}. \tag{1.1}$$

Assume this fails for all $M$. We will choose a sequence of finite disjoint sets $w_j$ such that for all $j$ (recall the notation $\mu_f(A) = \sum_{n\in A} f(n)$):

$$\frac{\mu_f(\bigcup_{i\in w_j} I_i)}{\psi(\bigcup_{i\in w_j} I_i)} \geq j,$$

$$\frac{1}{j^2} \leq \psi\left(\bigcup_{i\in w_j} I_i\right) < \frac{2}{j^2}.$$

This sequence is constructed recursively as follows: If $w_1, \dots, w_m$ had been chosen, let $l \geq m+1$ be such that $2^l \geq \max \bigcup_{j=1}^m w_m$, and consider the set $A_{m+1}$ of all

$k \geq l$ for which $\mu_f(I_k)/\psi(I_k) \geq m+1$. Then this set is not in the ideal, and, since $\psi(I_k) < 1/(m+1)^2$ for all $k \in A_{m+1}$, there is $w_{m+1} \Subset A_{m+1}$ such that $1/(m+1)^2 \leq \psi(\bigcup_{k\in w_{m+1}} I_k) \leq 2/(m+1)^2$. Let $W = \bigcup_j w_j$. Then the set

$$X = \bigcup_{i\in W} I_i$$

belongs to $\mathcal{I} \setminus \mathcal{I}_f$. This is because for all $j$ we have

$$\mu_f\left(\bigcup_{i\in w_j} I_j\right) \geq j \cdot \tfrac{1}{j^2} = \tfrac{1}{j}, \ \& \ \psi\left(\bigcup_{i\in w_j} I_j\right) \leq \tfrac{1}{j^2}.$$

This finishes the proof that there is $M$ satisfying (1.1). Fix such $M$ and let $A = \{k : \mu_f(I_k)/\psi(I_k) \leq M\}$. Then $\sum_{k\in A} 1/(k+1) = \infty$. For every $k \in A$ there is a $k$-element set $s_k \subseteq I_k$ such that

$$\mu_f(s_k) \leq \frac{M}{k} \cdot \frac{k}{2^k} = \frac{M}{2^k}$$

and therefore $Y = \bigcup_{k\in A} s_k \in \mathcal{I}_f \setminus \mathcal{I}$ which contradicts $\mathcal{I} = \mathcal{I}_f$.

This proof gives a more general fact (see Definition 1.4.3 for $\mathrm{Exh}(\varphi)$).

**Proposition 1.5.2.** *If $\{\varphi_n\}$ is a sequence of submeasures with pairwise disjoint finite supports, then the following are equivalent:*

*(1) The ideal $\mathrm{Exh}\big(\sum_n \varphi_n\big)$ is summable.*
*(2) There is a sequence of measures $\{\nu_n\}$ such that $\mathrm{supp}(\nu_n) = \mathrm{supp}(\varphi_n)$ and $\sum_n \sup_A |\nu_n(A) - \varphi_n(A)| < \infty$.* □

## 1.6. More examples of $F_\sigma$ ideals

*Solecki's ideal.* In [147], Solecki defined the following ideal on a countable set $\Omega$. By $\mathrm{Clop}(X)$ we denote the Boolean algebra of clopen subsets of a topological space $X$.

**Definition 1.6.1.** Let $\lambda$ denote the Haar measure on $\{0,1\}^{\mathbb{N}}$ and let $\mathcal{S}$ be the ideal on $\Omega = \{U \in \mathrm{Clop}(\{0,1\}^{\mathbb{N}}) : \lambda(U) = 1/2\}$ generated by sets whose intersection is nonempty.

The following is well-known but the literature appears to contain pointers to places where it was proved instead of a (simple) proof that we include for reader's convenience.

**Lemma 1.6.2.** *The ideal $\mathcal{S}$ is a proper, dense, $F_\sigma$ ideal.*

Proof. This is a proper ideal because for every $F \Subset \{0,1\}^{\mathbb{N}}$ there is $U \in \Omega$ disjoint from $F$. To prove that it is $F_\sigma$, for $n \in \mathbb{N}$ let

$$\mathcal{X}(n) = \{A \subseteq \Omega : \text{there is } f\colon A \to n, \bigcap f^{-1}(\{j\}) \neq \emptyset \text{ for all } j < n\}.$$

Clearly $\mathcal{S} = \bigcup_n \mathcal{X}(n)$ and it will suffice to prove that each $\mathcal{X}(n)$ is closed. For this compactness argument, fix $A \in \mathcal{X}(n)$ such that $A \cap k \in \mathcal{X}(n)$ for all $k$. Let

$$T = \{(s,k) : s\colon A\cap k \to n, \bigcap s^{-1}(\{j\}) \neq \emptyset \text{ for all } j < n\}$$

and order $T$ by $(s,k) \sqsubseteq (t,l)$ if $k \leq l$ and $t \restriction k = s$. Then $T$ is a finitely-branching (to be precise, each node has at most $n$ immediate successors) tree. For each $k \in \mathbb{N}$, the $k$-th level of $T$ is nonempty because $A \cap k \in \mathcal{X}(n)$. By König's lemma, $T$ has an infinite branch and this branch defines a partition of $A$ into $n$ sets each of which has the finite intersection property. Since all sets in $\Omega$ are compact, each of these pieces has nonempty intersection.

A different (only cosmetically but arguably, in light of the Mazur–Solecki characterisation of $F_\sigma$ ideals Theorem 1.4.6, more appropriate) proof that $\mathcal{S}$ is $F_\sigma$ proceeds by defining a submeasure $\varphi\colon \mathcal{P}(\mathbb{N}) \to \mathbb{N}$ by $\varphi(A) = \min\{n : A \in \mathcal{X}(n)\}$, with $\min \emptyset = \infty$. The above compactness argument shows that $\varphi$ is lower semicontinuous and therefore $\mathcal{S} = \mathrm{Fin}(\varphi)$ is $F_\sigma$.

To prove that $\mathcal{S}$ is dense, let $A \subseteq \Omega$ be infinite. For each $n$ the set

$$F(n) = \{x \in \{0,1\}^{\mathbb{N}} : |\{U \in A : x \in U\}| \leq n\}$$

is closed, and $F(n) \subseteq F(n+1)$ for all $n$. If $x \in \{0,1\}^{<\mathbb{N}} \setminus \bigcup_n F(n)$, then the set $\{U \in A : x \in U\}$ is infinite, included in $A$, and it belongs to $\mathcal{S}$. It therefore suffices to prove that $\bigcup_n F(n)$ does not cover $\{0,1\}^{\mathbb{N}}$. Assume otherwise. For every $n \geq 1$ we will find finite $G(n) \subseteq F(n)$ and for $x \in G(n)$ distinct clopen sets $U(x,j) \in A$, for $j < n$ in $A$ such that $x \in \bigcap_{j<n} U(x,j)$ and $\bigcup_{x\in G(n)} \bigcap_{j<n} U(x,j) \supseteq F(n)$. For $n = 1$ the existence of these objects is a consequence of compactness. Note that $\bigcup_{x\in G(1)} \bigcap_{j<1} U(x,j) = F(1)$, and in particular both $F(1)$ and $F(2) \setminus F(1)$ are clopen. We can therefore again use compactness to find $G(2)$ and sets $U(x,j) \in A$ for $x \in G(2)$ and $j < 2$ such that $\bigcup_{x\in G(2)} \bigcap_{j<2} U(x,j) = F(2) \setminus F(1)$. Then $F(3) \setminus (F(2) \cup F(1))$ is clopen, and we can proceed like this.

Since $\bigcup_n F(n) = \{0,1\}^{\mathbb{N}}$, a large enough $n$ satisfies $\lambda(F(n)) > 1/2$. Since $A$ is infinite, there is $V \in A \setminus \bigcup_{k\leq n}\{U(x,j) : x \in G(k), j \leq k\}$. As $\lambda(V) = 1/2$, we have $V \cap F(n) \neq \emptyset$. If $k \leq n$ and $x \in V \cap F(k)$ then $V \neq U(x,j)$ for $j < k$; contradiction. □

*Eventually different ideal(s).* The following ideal resembles Fin × Fin and plays an important role in study of the Katětov order.

**Lemma 1.6.3.** *The* eventually different *ideal is the ideal on* $\mathbb{N}^2$ *defined as*

$$\mathcal{ED} = \{A \subseteq \mathbb{N}^2 : \limsup_n |A_n| < \infty\}$$

*and* $\mathcal{ED}_{\mathrm{Fin}}$ *is the ideal on* $\prod_m m = \{(m,n) \in \mathbb{N}^n : n < m\}$ *defined by*

$$\mathcal{ED}_{\mathrm{Fin}} = \{A \subseteq \textstyle\prod_m m : \limsup_n |A_n| < \infty\}.$$

*Each of these ideals is a proper, dense,* $F_\sigma$ *ideal.*

Proof. It clearly suffices to prove these claims for $\mathcal{ED}$. It is clearly proper and dense. For $n \in \mathbb{N}$ let

$$\mathcal{X}(n) = \{A \subseteq \mathbb{N}^2 : (\forall k \geq n)|A_k| \leq n\}.$$

The complement of $\mathcal{X}(n)$ is open and $\mathcal{ED} = \bigcup_n \mathcal{X}(n)$. □

For more see [82, §3.2].

1.6.0.1. *Kanovei–Lyubetskiy ideals.* The following is based on a class of ideals introduced in [100]. Let $D$ be a countable elementary submodel of the Boolean algebra $\mathcal{P}(\mathbb{N})$ and let $\mathbb{I}$ be the set of all tuples of the form $(m, X, A_s : s \in X)$ where $m \in \mathbb{N}$, $X$ is a nonempty and proper subset of $\mathcal{P}(m)$, $A_s \in D$ is such that $A_s \cap m = s$ for $s \in X$, and moreover for all $s$ and $t$ in $\mathcal{P}(m)$ we have the following.

(KL1) If $s \cap t \in X$ then $A_s \cap A_t = A_{s\cap t}$.
(KL2) If $m \setminus s \in X$ then $\mathbb{N} \setminus A_s = A_{m\setminus s}$.

Clearly the set $\mathbb{I}$ is countable. For an element $p$ of $\mathbb{I}$ write $p = (m^p, X^p, A^p_s : s \in X)$ and $U^p = \{A \in \mathcal{P}(\mathbb{N}) : A \cap m^p \in X^p\}$. Let

$$\mathcal{KL}_{\mathcal{P}(\mathbb{N})} = \{Z \subseteq \mathbb{I} : \text{ there is } F \Subset \mathcal{P}(\mathbb{N}) \text{ such that } F \setminus U^p \neq \emptyset \text{ for all } p \in Z\}.$$

The proof of the following is similar to that for Lemma 1.6.2.

**Lemma 1.6.4.** *The ideal $\mathcal{KL}_{\mathcal{P}(\mathbb{N})}$ is a proper $F_\sigma$ ideal.* □

Kanovei and Lyubetskiy defined analogous ideal $\mathcal{KL}_G$ for an abelian Polish group $G$, but with conditions (KL1) and (KL2) replaced with the analogous condition involving the group operation. The ideal $\mathcal{KL}_G$ is analytic for every abelian Polish group $G$, but if $G$ is $\sigma$-compact then it $\mathcal{KL}_G$ is an $F_\sigma$ ideal. The motivation for this definition is that these ideals fail a variant of the Radon–Nikodym property (§4.1). According to [**100**], these ideals were inspired by Solecki's ideal (Definition 1.6.1), but the idea is also closely related to the examples in [**37**, Theorem 7] and [**41**, Theorem 3.3], also defined in order to provide counterexamples to the Radon–Nikodym property.

A large family of $F_\sigma$ ideals inspired by the LV-ideals (§1.7.3) was defined by K. Mazur ([**122**]. These ideals are of the form $\mathrm{Fin}(\sup_n \varphi_n)$ where $\varphi_n$ are submeasures that concentrate on disjoint finite sets and satisfy conditions (LV1) and (LV2) from Definition 1.7.7.

## 1.7. Analytic P-ideals

**1.7.1. Erdös–Ulam ideals (EU-ideals).** A function $f : \mathbb{N} \to \mathbb{R}_+$ is called an *EU-function* if it satisfies the following conditions (writing $\mu_f(A) = \sum_{n\in A} f(n)$):

(1) $\mu_f(\mathbb{N}) = \infty$.
(2) $\lim_n f(n)/\mu_f(n+1) = 0$.

If $f$ is an EU-function, the *upper $f$-density* on $\mathbb{N}$ is defined by

$$d_f(A) = \limsup_{n\to\infty} \frac{\sum_{i\in A\cap n} f(i)}{\sum_{i<n} f(i)}.$$

This function is subadditive (it is a submeasure, if necessary you can peek ahead at Definition 1.4.1) and the set

$$\mathcal{EU}_f = \{A : d_f(A) = 0\}$$

is an ideal that includes Fin. This is a P-ideal; the reader may prove this easily and we will return to this point in §2.7.2. Erdös–Ulam ideals were introduced by Just and Krawczyk in [**95**]. Important cases of EU-ideals are the ideal $\mathcal{Z}_0$ of *asymptotic density zero sets* (obtained with $f(n) = 1$):

$$\mathcal{Z}_0 = \left\{ A \subseteq \mathbb{N} : \lim_{n\to\infty} \frac{|A\cap n|}{n} = 0 \right\},$$

and the ideal $\mathcal{Z}_{\log}$ of *logarithmic density zero sets* (obtained when $f(n) = 1/n$):

$$\mathcal{Z}_{\log} = \left\{ A \subseteq \mathbb{N} : \lim_{n\to\infty} \frac{\sum_{i\in A\cap n} 1/i}{\sum_{i<n} 1/(i+1)} = 0 \right\}.$$

The class of EU-ideals is, unlike other classes of ideals considered here, not closed under isomorphisms. In Theorem 2.7.8 we will describe the class of ideals isomorphic to an EU-ideal.

**1.7.2. Density ideals and generalised density ideals.** The following definition is taken from [40].

**Definition 1.7.1** (Density ideals)**.** For a sequence $\mu = \{\mu_i\}$ of orthogonal measures on $\mathbb{N}$ concentrating on disjoint finite sets define a submeasure $\varphi_\mu$ by $\varphi_\mu = \sup_i \mu_i$. Then

$$\mathcal{Z}_\mu = \mathrm{Exh}(\varphi_\mu)$$

is a *density ideal generated by a sequence of orthogonal measures* (shortly, a *density ideal*). If $\|\nu_n\| = 1$ for all $n$ then the sequence of measures $\{\nu_n\}$ is called *normalised*, and the density ideal $\mathcal{Z}_\nu$ is *normalised*.

For a sequence $\psi_n$ of orthogonal submeasures concentrating on finite sets define a submeasure $\varphi_\psi$ by $\varphi_\psi = \sup_i \psi_i$. This is a lower semicontinuous submeasure and the ideal $\mathcal{Z}_\psi = \mathrm{Exh}(\varphi_\psi)$ is called a *generalised density ideal*.

**Remark 1.7.2.** As Jacek Tryba pointed out, the class of density ideals generated by a sequence of measures supported on disjoint *intervals* is strictly smaller than the class of density ideals as in Definition 1.7.1 ([157, Lemma 3.3]).

Every density ideal as per Definition 1.7.1 is isomorphic to one in which measures $\mu_n$ concentrate on disjoint *intervals*. In [46], [49], and [157, Definition 3.2] an ideal $\mathcal{Z}_\mu$ is called a density ideal if and only if the measures $\mu_n$ concentrate on disjoint *intervals*. Analogous remark applies to generalised density ideals, and in [157, Definition 4.1] an ideal is called generalised density ideal if it is of the form $\mathrm{Exh}(\sup_n \psi_n)$ where $\psi_n$ are orthogonal submeasures concentrating on disjoint finite intervals.

We'll stick to Definition 1.7.1, so that our classes of ideals (except for the EU-ideals) are closed under isomorphisms.

Since a generalised density ideal is not affected if $\psi_n$ is replaced with $\min(1, \psi_n)$ for all $n$, introducing normalised generalised density ideals makes little sense. As the supports of submeasures $\psi_n$ are assumed to be finite and disjoint, $A \in \mathrm{Exh}(\varphi_\psi)$ if and only $\limsup_n \psi_n(A) = 0$ and

$$\mathcal{Z}_\psi = \{A \subseteq \mathbb{N} : \limsup_n \psi_n(A) = 0\}. \tag{1.2}$$

Recall that an ideal $\mathcal{I}$ is dense if every infinite subset of $\mathbb{N}$ has an infinite subset in $\mathcal{I}$.

**Lemma 1.7.3.** *Every generalised density ideal is a P-ideal, and the condition* $\limsup_n \mathrm{at}^+(\varphi_n) = 0$ *is equivalent to asserting that* $\mathcal{Z}_\varphi$ *is dense.*

Proof. The first part is true for all ideals of the form $\mathrm{Exh}(\varphi)$ for a lowersemicontinuous submeasure $\varphi$. For the second part, note that since the supports of $\varphi_n$'s are finite we have $\lim_j \sup_n \varphi_n(\{j\}) = \limsup_n \mathrm{at}^+(\varphi_n)$ and apply Lemma 1.4.4. □

**Example 1.7.4.** The ideal $\emptyset \times \mathrm{Fin}$ (§1.3.5) is a density ideal. Let

$$s_m = \{(i,j) \in \mathbb{N}^2 : i + j = m\}$$
$$\mu_m(A) = \max\{3^{-i} : (i,j) \in A \cap s_m\}.$$

To see that $\emptyset \times \mathrm{Fin} = \mathcal{Z}_\mu$, fix $A \subseteq \mathbb{N}^2$.

Fix $m$. If $A \cap (\{m\} \times \mathbb{N}) \neq \emptyset$, fix $(m, j) \in A$ and note that $\mu_{m+j}(A) \geq 3^{-m}$. Thus if $A \notin \emptyset \times \mathrm{Fin}$ then $\limsup_n \mu_n(A) \geq 3^{-m}$ and $A \notin \mathcal{Z}_\mu$. If $A \cap (m \times \mathbb{N}) = \emptyset$, then we have $\mu_n(A) \leq \sum_{i\geq m} 3^{-i} \leq 2\cdot 3^{-m}$ for all $n$. Therefore if $A \in \emptyset \times \mathrm{Fin}$ then for every $\varepsilon > 0$ there is $F \Subset \mathbb{N}^2$ such that $\mu_n(A \setminus F) < \varepsilon$ and $A \in \mathcal{Z}_\mu$.

**Example 1.7.5** (Dense density ideal which is not isomorphic to an EU-ideal)**.** Let $J_n = [2^n, 2^{n+1})$, the ideal

$$\mathcal{Z}_\infty = \left\{ A : \limsup_n \frac{|A \cap J_n|}{n} = 0 \right\}$$

is a dense density ideal. Theorem 2.7.8, once proved, will imply that it is not isomorphic to an EU-ideal.

There is an example of a density ideal that is not an EU-ideal, but is isomorphic to one ([157]). By the following lemma, all dense density ideals that are not isomorphic to an EU-ideal look rather similar (see also Theorem 11.2.3). Note that $\mathcal{Z}_\mu$ is uniquely determined by $I_n$ and $\mu_n$, but not vice versa.

**Lemma 1.7.6.** *If $\mathcal{Z}_\mu$ is a dense density ideal, then it is either an EU-ideal or $\mu_n, I_n$ ($n \in \mathbb{N}$) can be chosen so that $\lim_n \mu_n(I_n) = \infty$.*

Proof. Assume $\mathcal{Z}_\mu$ is not an EU-ideal. If $\sup_n \mu_n(I_n) < \infty$, then Theorem 2.7.8 implies that $\mathcal{Z}_\mu$ is an EU-ideal. Hence there is an infinite $A \subseteq \mathbb{N}$ such that $\liminf_{n\in A} \mu_n(I_n) = \infty$. We may assume that $A$ is coinfinite, and by re-indexing that $A = \{2n : n \in \mathbb{N}\}$. Let $J_n = I_{2n-1} \cup I_{2n}$ and $\nu_n = \mu_{2n-1} + \mu_{2n}$. Then $\mathcal{Z}_\nu = \mathcal{Z}_\mu$ is as required. □

**1.7.3. LV-ideals.** A large class of generalized density ideals was introduced in [118], where it was proved that the quotients over these ideals are not Borel-isomorphic, even when considered with no algebraic structure. The salient property of these ideals is given by the following.

**Definition 1.7.7.** A generalized density ideal $\mathcal{Z}_\varphi$ given by submeasures $\varphi_n$ concentrating on disjoint finite sets $I_n$ satisfying the following two conditions is called an *LV-ideal*.

(LV1) $\varphi_i(I_i) \geq 1$ for all $i$, and
(LV2) $(\forall k)(\forall \varepsilon > 0)(\forall^\infty n)$

$$(\forall a_0, \dots, a_k \subseteq I_n)|\varphi_n(a_0 \Delta a_k) - \max_{i<k} \varphi_n(a_i \Delta a_{i+1})| < \varepsilon.$$

By Lemma 1.7.3, the following condition is equivalent to $\mathcal{Z}_\varphi$ being a dense ideal.

(LV3) $\lim_n \max_{j\in I_n} \varphi_n(\{j\}) = 0$,

Although the definition of LV-ideals in [49, §2.11] does not include ((LV3)). It was tacitly assumed because the conclusion [49, Theorem 5.5] fails without this assumption.

**Example 1.7.8** (An LV-ideal)**.** For an increasing sequence $\{n_i\}$ of natural numbers let $I_i$ be pairwise disjoint intervals such that $|I_i| = 2^{n_i}$ and let

$$\varphi_i(A) = \log_2(|A \cap I_i| + 1)/n_i.$$

Then $\mathcal{LV} = \mathcal{Z}_\varphi$ is a dense LV-ideal.

One could increase the logarithm base or even iterate log functions, but quotients associated with the resulting ideals would be similar (Theorem 11.2.6).

**Lemma 1.7.9.** *If $\mathcal{Z}_\varphi$ is an LV-ideal, then its restriction to any positive set $A$ is an LV-ideal.*

Proof. Write $I_n = \operatorname{supp}(\varphi_n) \cap A$. The proof is very similar to that of Lemma 2.7.4. If there is $\varepsilon > 0$ such that $X = \bigcup\{I_n : \varphi_n(I_n) < \varepsilon\}$ belongs to $\mathcal{Z}_\varphi$. By discarding $\bigcup_{n\in X} I_n$ and replacing the restriction of $\varphi_n$ to $I_n$ with $\psi_n = \varphi_n \varepsilon^{-1}$ for $n \notin X$, $\mathcal{Z}_\psi = \mathcal{Z}_\varphi \restriction A$ is as required.

We may therefore assume that that there is a decreasing sequence $\varepsilon_n$ such that $\lim_n \varepsilon_n = 0$ and each of the sets

$$Y_m = \{n : \varepsilon_m \le \|\varphi_n\| < \varepsilon_{m-1}\}$$

for $m \ge 1$ is infinite. By manipulating this sequence as in the proof of Lemma 2.7.4, one obtains submeasures $\psi_n$ as in Definition 1.7.7 and this shows that $\mathcal{Z}_\varphi \restriction A$ is isomorphic to $\mathcal{Z}_\psi$. □

**1.7.4. Between density and summable ideals.**

1.7.4.1. *The Hrušak–Zapletal ideal.* The following interesting analytic P-ideal was defined in [85, Example 3.12] . Fix a decreasing sequence $r_n$, for $n \in \mathbb{N}$, in $(0,1)$, let $f_n(k) = k^{-r_n}$, and let $\mathcal{I}_n$ be the corresponding summable ideal, $\mathcal{I}_{f_n}$. Then $r_n > r_{n+1}$ implies $\mathcal{I}_n \subseteq \mathcal{I}_{n+1}$, and a standard calculus argument shows that for every $\mathcal{I}_{n+1}$-positive set $A$ there is $B \subseteq A$ in $\mathcal{I}_{n+1} \setminus \mathcal{I}_n$. Then $\bigcap_n \mathcal{I}_n$ is an $F_{\sigma\delta}$ P-ideal (Lemma 1.3.3). Not only that it is not $F_\sigma$, but unlike any quotient $\mathcal{P}(\mathbb{N})/\mathcal{J}$ over an $F_\sigma$ ideal $\mathcal{J}$, $\mathcal{P}(\mathbb{N})/\mathcal{I}$ as a forcing notion collapses $\aleph_1$ if the CH holds ([85, Proposition 3.11]).

1.7.4.2. *Non-pathological analytic P-ideals different from both density and summable ideals.* Classes of density and summable ideals certainly do not exhaust the class of all non-pathological analytic P-ideals—for example, consider ideals of the form $\mathcal{I}_f \oplus \mathcal{Z}_\mu$. We give an example of a

**Example 1.7.10.** Let $\mathcal{I}$ be the ideal on the set $\{0,1\}^{<\mathbb{N}}$ of all finite sequences in $\{0,1\}$ defined as follows. A submeasure $\varphi$ with $\operatorname{supp}(\varphi) = \{0,1\}^{<\mathbb{N}}$ is defined by:

$$\varphi(A) = \sup_{x\in\{0,1\}^{\mathbb{N}}} \textstyle\sum_{x\restriction n\in A} \frac{1}{n}.$$

This is obviously a non-pathological submeasure. Let $\mathcal{J}_{\rm br}$ be the ideal on $\{0,1\}^{<\mathbb{N}}$ generated by its branches. For $t \in \{0,1\}^{<\mathbb{N}}$ let

$$[t]_- = \{s \in \{0,1\}^{<\mathbb{N}} t \sqsubseteq s\}.$$

**Claim 1.7.11.** *The following are equivalent for $A \subseteq \{0,1\}^{<\mathbb{N}}$.*

*(1) $A$ does not belong to $\langle \mathcal{J}_{\rm br}, \mathcal{I}\rangle$, the ideal generated by $\mathcal{I}$ and $\mathcal{J}_{\rm br}$.*
*(2) $\mathcal{I} \restriction A$ is not summable.*
*(3) There is a $B \in \mathcal{I}_+ \restriction A$ such that $\mathcal{I} \restriction B$ is a proper density ideal.*

Proof. For a set in $A \in \mathcal{J}_{\rm br}$ a function $f\colon A \to \mathbb{R}_+$ by $f(A) = \sum_{x\restriction n\in A} 1/n$ satisfies $\mathcal{I} \restriction A = \mathcal{I}_f \restriction A$. Therefore if (1) fails, (2) fails as well. Since (3) obviously implies (2), it remains only to prove (1) implies (3). So let $A$ be such that for every $B \in \mathcal{J}_{\rm br}$ set $A \setminus B$ is not in $\mathcal{I}$, and let

$$T(A) = \{t \in \{0,1\}^{<\mathbb{N}} : A \cap [t]_- \notin \langle \mathcal{J}_{\rm br}, \mathcal{I}\rangle\}$$

The tree $T(A)$ is infinite, so by König's lemma it has an infinite branch $C$. Then for some $\varepsilon > 0$ we have (let $\{0,1\}^{\le n}$ be the set of all $t \in \{0,1\}^{<\mathbb{N}}$ of length $\le n$):

$$\lim_{n\to\infty} \varphi((A \setminus C) \setminus \{0,1\}^{\le n}) \ge \varepsilon.$$

Now recursively pick a sequence $s_k$ of finite chains of $\{0,1\}^{<\mathbb{N}}$ included in $A \setminus C$ such that $\varepsilon/2 \leq \varphi(s_k) \leq \varepsilon$ for all $k$ and every $t \in s_k$ is incomparable with every $u \in s_l$ for $l \neq k$. This is done as follows: If $s_1, \dots, s_k$ are already chosen and satisfy the conditions, pick an $n$ such that $\bigcup_{i=1}^k s_i \subseteq \{0,1\}^{\leq n}$. Let $t \in C$ be of length $> n$ and such that $[t]_- \cap (A \setminus C) \notin \langle \mathcal{J}_{\mathrm{br}}, \mathcal{I} \rangle$. Find $s \subseteq [t]_- \cap (A \setminus C)$ such that $\varphi(s) > \varepsilon/2$, and let $s_k \subseteq s$ be a finite chain such that $\varphi(s_k) \geq \varepsilon/2$. This describes the construction.

If $s_k$ are as above, then $\bigcup_k s_k$ is not in $\mathcal{I}$, $\mu_k = \varphi \restriction s_k$ is a measure for every $k$, $\lim_k \mathrm{at}^-(\mu_k) = 0$, and $\varphi(D \cap \bigcup_k s_k) = \sup_k \mu_k(D)$, therefore $\mathcal{I} \restriction \bigcup_k s_k$ is a proper density ideal. □

By Claim 1.7.11, every $A$ such that $A \cap B$ is $\mathcal{I}$-positive for infinitely many branches $B$ of $\{0,1\}^{<\mathbb{N}}$ has subsets $A_0, A_1$ such that $\mathcal{I} \restriction A_0$ is a dense summable ideal and $\mathcal{I} \restriction A_1$ is a proper density ideal. □

**1.7.5. Analytic P-ideals from functional analysis and Borel equivalence relations.** For a while it was unclear whether there $F_\sigma$ P-ideals substantially different from summable ideals existed (see [106], [124]), and in particular whether some $F_\sigma$ P-ideal is not of the form $\mathrm{Exh}(\sum_n \varphi_n)$ for a sequence of submeasures with pairwise disjoint finite supports. The discovery of such ideals (see [145], [39], [43]) required importing ideas from the theory of infinite-dimensional Banach spaces

If $\mathsf{G}$ is a Polish group and $a_n$ is a sequence in $\mathsf{G}$, then

$$\{A \subseteq \mathbb{N} : \textstyle\sum_{n \in A} a_n \text{ is unconditionally convergent }\}$$

is an ideal. By slightly extending a result of [144] (Theorem 1.4.6), in [12] it was proved that $\mathcal{I}$ is an analytic P-ideal if and only if it is of this form.

If $\mathsf{X}$ is a Banach space and $x_n$, for $n \in \mathbb{N}$, is a sequence in $\mathsf{X}$, then

$$\{A \subseteq \mathbb{N} : \text{the series } \textstyle\sum_{n \in A} x_n \text{ is unconditionally convergent}\}$$

is an ideal on $\mathbb{N}$. By [12], an ideal is of this form if and only if it is a non-pathological analytic P-ideal. If $\mathsf{X}$ is an unusual Banach space, such as Tsirelson's space ([159]) or its dual, then the obtained ideals have unusual properties. In [39], [163], [43] ideals associated to Tsirelson's spaces were shown to lead to interesting Borel equivalence relations. For more on the relation between Borel equivalence relations and Borel ideals on $\mathbb{N}$ see [103] and [99].

## 1.8. More examples of $F_{\sigma\delta}$ ideals

**1.8.1. Matrix summability ideals and a generalisation.** These ideals, introduced in [64], form the well-studied class..

**Definition 1.8.1.** A 'density matrix' $\mathsf{A} = (a_{ij} : i, j \in \mathbb{N})$ of nonnegative reals is *regular* if the following conditions hold.

(1) $\lim_i a_{ij} = 0$ for all $j$.
(2) $\sum_j a_{ij} < \infty$ for all $i$.
(3) $\lim_i \sum_j a_{ij} = 1$ (in particular this limit exists).

The *upper* $\mathsf{A}$-*density* of $X \subseteq \mathbb{N}$ is $\overline{d^{\mathsf{A}}}(X) = \limsup_i \sum_{j \in B} a_{ij}$ and the *matrix summability ideal* associated with $\mathsf{A}$ is

$$\mathcal{I}(\mathsf{A}) = \{X \subseteq \mathbb{N} : \overline{d^{\mathsf{A}}}(X) = 0\}.$$

In other words, with $\mu_i(A) = \sum\{a_{ij} : j \in A\}$ we have

$$\mathcal{I}(\mathsf{A}) = \{X \subseteq \mathbb{N} : \limsup_i \mu_i(A) = 0\}.$$

By [63, Lemma 2.28], for every ideal of this form one may choose these measures so that the support of each $\mu_i$ is finite, and for every $n$ the set $\{i : \mathrm{supp}(\mu_i) \cap n \neq \emptyset\}$ is finite.

Every matrix summability ideal is $F_{\sigma\delta}$, as a routine counting of quantifiers shows. This also applies to the ideals $\mathcal{D}_\varphi$ defined in Lemma 1.8.3 below.

**Example 1.8.2.** Not every matrix summability ideal is a P-ideal. For example, let $\mathbb{N} = \bigsqcup_n A_n$ be a partition into infinite sets and let $\mu_n$ be a measure on $A_n$ such that $\mathrm{Exh}(\mu_n)$ is a proper summable ideal. Then the corresponding matrix summability ideal satisfies $A_n \in \mathcal{I}(\mathsf{A})$ for all $n$, but no set in $\mathcal{I}(\mathsf{A})$ includes all $A_n$ modulo finite.

See [158], [19, Example 4.16], [63, Section 4], see [6, Proposition 12], and [157, Proposition 4.5 and Proposition 4.6] for more on matrix summability ideals.

Taking $\varphi_n$, for $n \in \mathbb{N}$, in the following lemma to be measures and requiring that $\{n : \mathrm{supp}(\varphi_n) \cap m \neq \emptyset\}$ is finite for all $m$ results in a matrix summability ideal.

**Lemma 1.8.3.** *Suppose that $\varphi_n$, for $n \in \mathbb{N}$, are lower semicontinuous (not necessarily orthogonal) submeasures on $\mathbb{N}$ such that $\limsup_n \varphi_n(\mathbb{N}) > 0$. Then*

$$\mathcal{D}_\varphi = \{A \subseteq \mathbb{N} : \limsup_n \varphi_n(A) = 0\} \tag{1.3}$$

*is an $F_{\sigma\delta}$ ideal on $\mathbb{N}$.*

*If $\lim_n(\varphi_n(\{k\})) = 0$ for all $k \in \mathbb{N}$ and $\limsup_n \|\varphi_n\| > 0$, then $\mathcal{D}_\varphi$ is a proper ideal that includes* Fin

Proof. It is clear that $\mathcal{D}_\varphi$ is hereditary and closed under taking finite unions. Whether it includes Fin and whether $\mathbb{N} \in \mathcal{D}_\varphi$ depends on the choice of the submeasures. To see that it is $F_{\sigma\delta}$, fix $m$ and $n$ and let

$$\mathcal{X}_{m,n} = \{A \subseteq \mathbb{N}(\forall l \geq m)\varphi_l(A) \leq 1/n\}.$$

This is a closed set, and $\mathcal{D}_\varphi = \bigcap_m \bigcup_{n \geq m} \mathcal{X}_{n,n}$.

It is clear that $\mathbb{N} \in \mathcal{D}_\varphi$ if and only if $\limsup_n \varphi_n(\mathbb{N}) > 0$. If $\lim_n \varphi_n(\{k\}) = 0$ for every $k \in \mathbb{N}$, then every finite $F \subseteq \mathbb{N}$ satisfies $\varphi_n(F) = 0$ for all but finitely many $n$ and $\mathrm{Fin} \subseteq \mathcal{D}_\varphi$. □

**Lemma 1.8.4.** *Every matrix summability ideal is a proper ideal of the form $\mathcal{D}_\mu$ that includes* Fin *for a sequence $\mu_n$ of lower semicontinuous measures on $\mathbb{N}$.*

Proof. Suppose that $\mathsf{A} = (a_{ij} : i, j \in \mathbb{N})$ is a regular density matrix and define $\mu_i(X) = \sum_{j \in X} a_{ij}$. Then $\mu_i(\mathbb{N}) < \infty$ for all $\mathbb{N}$ and $\lim_i \mu_i(\mathbb{N}) = 1$, hence $\limsup_i \mu_i(\mathbb{N}) > 0$. Also, $\lim_i \mu_i(\{j\}) = \lim_i a_{i,j} = 0$ because $\mathsf{A}$ is regular.

Then $\overline{d^{\mathsf{A}}}(X) = \limsup_i \mu_i(X)$. Since $\lim_i a_{ij} = 0$ for all $j$, $\overline{d^{\mathsf{A}}}(s) = 0$ for every $s \in \mathrm{Fin}$. Therefore $\mathcal{D}_\mu = \{X : \overline{d^{\mathsf{A}}}(X) = 0\} = \mathcal{I}(\mathsf{A})$, as required. □

If $\mu_n$ are disjointly supported measures on $\mathbb{N}$, then under specific technical conditions the ideal $\mathcal{D}_\mu$ is a non-pathological analytic P-ideal ([158])

**Lemma 1.8.5.** *Suppose that $\mu_n$, for $n \in \mathbb{N}$, are disjointly supported measures on $\mathbb{N}$ such that $\limsup_n \|\mu_n\| > 0$. Then the following are equivalent*

*(1) $\mathcal{D}_\mu$ as defined in Lemma 1.8.3 is a P-ideal.*
*(2) For all but finitely many $n$, $\|\mu_n\| < \infty$.*

Proof. Let $X = \{n : \|\mu_n\| = \infty\}$ is infinite. Then each of the sets $\mathrm{supp}(\mu_n)$, for $n \in X$, belongs to $\mathcal{D}_\mu$. If $B \subseteq \mathbb{N}$ is such that $\mathrm{supp}(\mu_n) \setminus B$ is finite then $\mu_n(B) = \infty$ if and only if $\|\mu_n\| = \infty$, and therefore if $X$ is infinite then $\mathcal{D}_\mu$ is not a P-ideal.

Conversely, assume that $X$ is finite. If it is nonempty, then the restriction of $\mathcal{D}_\mu$ to $\bigcup_{n\in X} \mathrm{supp}(\mu_n)$ is a summable ideal given by the measure $\sum_{n\in X} \mu_n$. For every $n \notin X$ we can find $F_n \subseteq \mathrm{supp}(\mu_n)$ such that $\mathrm{supp}(\mu_n) \setminus F_n$ is finite and $\mu_n(F_n) < 1/n$. Then $\bigcup_{n\notin X} F_n$ belongs to $\mathcal{D}_\mu$ and the restriction of $\mathcal{D}_\mu$ to $\mathbb{N}\setminus(\bigcup_{n\notin X} F_n \cup \bigcup_{n\in X} \mathrm{supp}(\mu_n))$ is a density ideal (as in Definition 1.7.1 and therefore a P-ideal). $\square$

We will return to these ideals in §4.2.1.

1.8.1.1. *The Banach density (aka Weyl) ideal.*

**Definition 1.8.6.** Consider the following 'translation-invariant version' of $\mathcal{Z}_0$.

$$\mathcal{Z}_s = \left\{ A : \limsup_n \sup_k \frac{|A \cap [k, k+n)|}{n} = 0 \right\}.$$

Following [68, §2], we call $\mathcal{Z}_s$ the *Banach density ideal*.

In [101] and [50], $\mathcal{Z}_s$ was denoted $\mathcal{Z}_W$ and called the *Weyl ideal* (the terminology was influenced by [32] where this ideal appears implicitly). This is an $F_{\sigma\delta}$ ideal, since it is countably $\infty$-determined by closed approximations (Theorem 1.10.2). In [68, §2] Fremlin proved that this ideal and its quotient have remarkable properties.

### 1.8.2. Ideals associated to $\sigma$-ideals on the $\sigma$-algebra of Borel sets, I.

If $I$ is a $\sigma$-ideal on $\mathbb{R}$ (see [173], [174] for many examples and applications) then one can define

$$I(\mathbb{Q}) = \{A \subseteq \mathbb{Q} : \overline{A} \in I\}.$$

By [62, Proposition 6], if $I$ includes all singletons then the sequential topology on $\mathcal{P}(\mathbb{Q})/I(\mathbb{Q})$ is not metric and is therefore by Proposition 5.2.4 not isomorphic to the quotient over an analytic P-ideal. The following two special cases were considered in [62].

**Definition 1.8.7.** Consider the following $F_{\sigma\delta}$ ideals

$$\mathsf{nwd} = \{A \subseteq \mathbb{Q} \cap [0,1] : A \text{ is nowhere dense}\},$$
$$\mathsf{null} = \{A \subseteq \mathbb{Q} \cap [0,1] : \overline{A} \text{ has Lebesgue measure } 0\}.$$

The ideal $\mathsf{nwd}$ was denoted NWD($\mathbb{Q}$) in [62], but we adopt the simpler notation from [82] and [3].

It is not difficult to see that each one of these two ideals is $F_{\sigma\delta}$. We will prove a stronger statement, that they are all countably $\infty$-determined by closed approximations (Theorem 1.10.2).

In [95, Question C] Just and Krawczyk asked whether all homogeneous quotients over ideals which are $F_{\sigma\delta}$ but not $F_\sigma$ are pairwise isomorphic. By [62], the ideals $\mathsf{null}$ and $\mathsf{nwd}$ provide an example (Proposition 2.5.1, (Proposition 5.2.5).

**1.8.3. Ideals associated to $\sigma$-ideals on the $\sigma$-algebra of Borel sets, II: Trace ideals.** Suppose that $I$ is a $\sigma$-ideal of Borel subsets of a Polish space such as $[0,1]$ such that every set in $I$ is included in a $G_\delta$ set in $I$. For $A \subseteq \{0,1\}^{<\mathbb{N}}$ let

$$[A] = \{x \in \{0,1\}^{\mathbb{N}} : (\exists^\infty n) x \restriction n \in A\},$$
$$[A]_1 = \{x \in \{0,1\}^{\mathbb{N}} : (\exists n) x \restriction n \in A\}.$$

Clearly $[A]_1$ is open and $[A]$ is $G_\delta$. Let

$$\mathrm{tr}(I) = \{A \subseteq \{0,1\}^{<\mathbb{N}} : [A] \in I\}.$$

Note that the trace ideal associated with the Lebesgue measure is different from the ideal null, but nwd coincides with the trace ideal of the $\sigma$-ideal of meager sets.

These ideals were introduced in [146, §5], where the following was proved (see Definition 1.4.9 for continuous submeasures).

**Lemma 1.8.8.** *If $\mu$ is a continuous submeasure on the $\sigma$-algebra of Borel subsets of $\{0,1\}^{\mathbb{N}}$ and $\mathrm{Null}(\mu)$ denotes the $\sigma$-ideal of $\mu$-null sets, then $\mathrm{tr}(\mathrm{Null}(\mu))$ is an analytic P-ideal.*

Proof. We will define a lower semicontunuous submeasure $\varphi$ such that $\mathrm{tr}(\mathrm{Null}(\mu)) = \mathrm{Exh}(\varphi)$. Let $\varphi\colon \mathcal{P}(\{0,1\}^{<\mathbb{N}}) \to [0,\infty)$ be defined by

$$\varphi(A) = \mu([A]_1).$$

This is clearly a submeasure. Since $[A]_1 = \bigcup_{F \Subset \{0,1\}^{<\mathbb{N}}} [A \cap F]_1$ and $\mu$ is countably subadditive, $\varphi$ is lower semicontinuous. Since $\mu$ is a continuous submeasure, if $X_n \subseteq \{0,1\}^{\mathbb{N}}$ are Borel sets such that $\mu(\bigcap_n X_n) = 0$, then $\lim_n \mu(X_n) = 0$. We have that $[A] = \bigcap_{F \Subset \{0,1\}^{\mathbb{N}}} [A \setminus F]_1$, and therefore $A \in \mathrm{tr}(\mathrm{Null}(\mu))$ if and only if $A \in \mathrm{Exh}(\varphi)$. □

To the best of my knowledge, there has been no study of the ideal $\mathrm{tr}(\mathrm{Null}(\mu))$ in case when $\mu$ is a pathological Maharam submeasure ([151]).

## 1.9. Ideals of higher Borel complexity

*The ideal $\mathcal{I}_{CONV}$.* Let $\mathcal{I}_{\mathrm{CONV}}$ denote the ideal generated by convergent sequences in $\mathbb{Q} \cap [0,1]$. By counting quantifiers one shows that this ideal is $F_{\sigma\delta\sigma}$. An interesting property of this ideal is that every $\mathcal{I}_{\mathrm{CONV}}$-positive subset of $\mathbb{Q}$ has a subset $A$ such that $\mathcal{I}_{\mathrm{CONV}} \restriction A$ is isomorphic to $\mathrm{Fin} \times \mathrm{Fin}$.

**1.9.1. Ordinal ideals.** We will define a large class of RK-homogeneous Borel ideals on $\mathbb{N}$. An ordinal $\alpha$ is *additively indecomposable* (or simply *indecomposable*) if $\alpha$ cannot be represented as the sum of two strictly smaller ordinals.

An ordinal is indecomposable if and only if it is of the form $\omega^\alpha$ for an ordinal $\alpha$. The ideals $\mathcal{O}_\alpha$ were denoted $\mathcal{I}_\alpha$ in [40], but I decided that the letter $\mathcal{I}$ has been overused a bit.

**Definition 1.9.1.** For a countable ordinal $\alpha$ and a countable linear order $L$ let $\mathcal{O}_\alpha(L)$ be the ideal of all subsets of $L$ of order type strictly smaller than $\omega^{\alpha+1}$.

If $L = \omega^\alpha$, we write $\mathcal{O}_\alpha$ for $\mathcal{O}_\alpha(L)$.

In [40] the ideals $\mathcal{O}_\alpha$ were denoted $\mathcal{I}_\alpha$, but in the intervening time I decided to relax the heavy use of the letter $\mathcal{I}$ for ideals. We will prove that every ideal of the form $\mathcal{O}_{L,\alpha}$ where $\alpha$ is a countable ordinal and $L$ is a countable well-order has the

Fubini property (Corollary 4.2.8) and therefore has the Radon–Nikodym property. Each $\mathcal{O}_\alpha$ is a $\Sigma^0_{2\alpha}$-complete set ([**171**]; for definition see [**105**]).

**Lemma 1.9.2.** *The Frechét ideal* Fin *is isomorphic to* $\mathcal{I}_1$*.*
*The Fubini product* $\mathrm{Fin}\times\mathrm{Fin}$ *is isomorphic to* $\mathcal{I}_2$*.*

Proof. The order-preserving bijection between $\omega$ and $\mathbb{N}$ sends sets of order type $<\omega$ to finite sets, and is therefore an isomorphism between Fin and $\mathcal{I}_1$.

Let $f\colon \mathbb{N}^2\to\omega^2$ be defined by $f(m,n)=\omega^m{+}n$. This is a bijection. For $A\subseteq\mathbb{N}^2$ and $m\in\mathbb{N}$ we have that $A\setminus(m\times\mathbb{N})\in\emptyset\times\mathrm{Fin}$ if and only if $f[A]\setminus\omega^m$ has no accumulation points. This is equivalent to $\mathrm{otp}(f[A]\setminus\omega^m)\leq\omega$, thus $A\in\mathrm{Fin}\times\mathrm{Fin}$ if and only if $\mathrm{otp}(f[A])<\omega^2$. ☐

The ideals $\mathcal{O}_\alpha$ are RK-homogeneous (Proposition 2.5.1) and they have the Fubini and Radon–Nikodym properties (Theorem 4.1.2).

**1.9.2. CB-ideals.** A related class of (not necessarily homogeneous) topological ordinal ideals was suggested by W. Weiss. Let $\alpha$ be a countable ordinal. It is called multiplicatively indecomposable if it is not a product of two smaller ordinals.

**Definition 1.9.3.** If $\alpha$ is a multiplicatively indecomposable ordinal, then let $\mathcal{W}_\alpha$ be the family of all subsets of $\alpha$ which do not include a subset which is homeomorphic to $\alpha$ in the ordinal topology.

An application of Ramsey's theorem ([**170**]) shows that for a countable ordinal $\alpha$, $\mathcal{W}_\alpha$ is an ideal if and only if $\alpha$ is multiplicatively indecomposable. (This means that it is not a product of two smaller ordinals.) Ideals of the form $\mathcal{W}_\alpha$ have the Fubini property the Radon–Nikodym property ([**101**, Theorem 30]).

**Definition 1.9.4.** If $X$ is a countable topological space whose Cantor–Bendixson rank is at least $\alpha$ let then

$$\mathrm{CB}_\alpha(X)=\{Y\subseteq X : \text{Cantor–Bendixson rank of } Y \text{ is } <\alpha\}.$$

This is, again by [**170**], an ideal and $\mathcal{W}_{\omega^\alpha}=\mathrm{CB}_\alpha(\omega^\alpha)$. It is not difficult to see that $\mathcal{O}_\alpha(P)$ and $\mathrm{CB}_\alpha(X)$ are P-ideals only when $\alpha=\omega$.

## 1.10. Ideals countably $d$-determined by closed approximations

The main result of [**40**], OCA lifting theorem, was proved only for analytic P-ideals and $\mathrm{Fin}\times\emptyset$ assuming $\mathrm{OCA_T}$ and MA. In Definition 1.10.1 we introduce a larger class of ideals to which its conclusion applies, introduced in [**50**]. In Theorem 1.10.2 we prove that some common classes of ideals are strongly countably determined by closed approximations and prove some closure properties of this class in Theorem 1.10.5.

**Definition 1.10.1.** An ideal $\mathcal{I}$ is *countably determined* by closed (analytic, etc.) approximations if there are closed (analytic, etc.) hereditary sets $\mathcal{K}_n$ ($n\in\mathbb{N}$) such that

(1) $\mathcal{I}=\bigcap_{n=1}^\infty(\mathcal{K}_n \mathbin{\underline{\cup}} \mathrm{Fin})$.

If in addition for some $d\in\mathbb{N}$ we have

(2) $\mathcal{I}=\bigcap_{n=1}^\infty(\mathcal{K}_n^d \mathbin{\underline{\cup}} \mathrm{Fin})$

we say that $\mathcal{I}$ is *countably d-determined* by closed (analytic, etc.) approximations. If there are closed hereditary $\mathcal{K}_n$ such that (2) holds for all $d \in \mathbb{N}$, we say that $\mathcal{I}$ is *strongly countably determined by closed approximations*, or that it is *countably $\infty$-determined by closed approximations.*

Every countably 1-determined ideal is $F_{\sigma\delta}$ and includes Fin, and all known $F_{\sigma\delta}$-ideals that include Fin are countably $\infty$-determined by closed approximations. Our interest in this class comes from the fact that this is the largest known class of ideals to which the conclusion of Theorem 6.1.2 applies. Conjecturally, it applies to all Borel ideals.

Our main lifting theorem from forcing axioms, Theorem 7.1.1, applies to all ideals that are countably 80-determined by closed approximations. No care is taken to assure the optimality of this constant, in particular because it is possible that all $F_{\sigma\delta}$ ideals are countably $\infty$-determined (or at least 80-determined) by closed approximations.

The following theorem and its proof contain references to some classes of ideals that will be introduced later on, but it will not be involved in any circular reasoning.

**Theorem 1.10.2.** *Every ideal countably determined by closed approximations is $F_{\sigma\delta}$. Each of the following ideals is countably $\infty$-determined by closed approximations.*

1. *Every $F_\sigma$ ideal.*
2. *Every analytic P-ideal.*
3. *Every generalised density ideal.*
4. *Ideals* $\mathsf{n}$null, $\mathsf{n}$wd, *and $\mathcal{Z}_s$ (see Definition 1.8.7 and Definition 1.8.6).*
5. *Every matrix summability ideal (Definition 1.8.1)*
6. *All ideals of the form $\mathcal{D}_\varphi$ introduced in Lemma 1.8.3 such that* $\mathrm{Fin} \subseteq \mathcal{D}_\varphi$ *(see Lemma 1.10.4 below),*

Proof. If $\mathcal{K}_n$ is closed then $\mathcal{K}_n \cup \mathrm{Fin}$ is $F_\sigma$ hence $\bigcup_n(\mathcal{K}_n \cup \mathrm{Fin})$ is $F_{\sigma\delta}$.

(1) By Lemma 1.2.3 there is a closed hereditary set $\mathcal{K}$ such that $\mathcal{I} = \mathcal{K} \cup \mathrm{Fin}$, then let $\mathcal{K}_n = \mathcal{K}$ for all $n$. We have $\mathcal{K} \subseteq \mathcal{I}$ and therefore $\mathcal{K}^m \subseteq \mathcal{I}$, hence $\mathcal{I} = \mathcal{K}^m \cup \mathrm{Fin}$ for all $m$.

(2) If $\mathcal{I}$ is an analytic P-ideal then there is a lower semicontinuous submeasure $\varphi$ such that $\mathcal{I} = \mathrm{Exh}(\varphi)$ (Theorem 1.4.6). Then $\mathcal{K}_n = \{A : \varphi(A) \le 1/n\}$ is a closed hereditary set and $A \in \mathcal{I}$ if and only if $A \in \bigcap_n(\mathcal{K}_n \cup \mathrm{Fin})$

(3) This is a special case of Lemma 1.10.4.

(4) Enumerate $\mathbb{Q}$ as $\{q_i : i \in \mathbb{N}\}$. For $\mathsf{null}(\mathbb{Q})$. Let $\mathcal{F}_n$ be the family of all finite unions of open rational intervals of total measure at most $2^{-n}$, and enumerate $\mathcal{F}_n$ as $\{U_{ni} : i \in \mathbb{N}\}$. Let

$$\mathcal{K}_n = \bigcup_i \mathcal{P}((U_{ni} \cap \mathbb{Q}) \setminus \{q_j : j \le i\}).$$

By compactness, for every $n$, every compact null set $A$ such that $A \cap \mathbb{Q}$ is dense in it is covered by some $U_{ni}$. Therefore every set in $\mathsf{null}(\mathbb{Q})$ belongs to $\mathcal{K}_n \cup \mathrm{Fin}$ for all $n$. On the other hand, the closure of every set in $\mathcal{K}_n^d$ has measure at most $d2^{-n}$, and therefore $\bigcap_{n=1}^\infty(\mathcal{K}_n^d \cup \mathrm{Fin}) = \mathsf{null}(\mathbb{Q})$ for all $d$.

The proof that $\mathrm{NWD}(\mathbb{Q})$ has the same property will be more transparent if we consider the dyadic rationals in $\mathcal{P}(\mathbb{N})$ instead of $\mathbb{Q}$. The two spaces are homeomorphic hence the associated ideals of nowhere dense subsets are Rudin—Keisler isomorphic. For an additional harmless convenience, we will denote the dyadic rationals with $\mathbb{Q}$.

If $I \Subset \mathbb{N}$ and $s \subseteq I$, then we write

$$[I, s] = \{a \subseteq \mathbb{N} : a \cap I = s\}.$$

By Lemma 3.1.2, some $A \subseteq \mathbb{Q}$ is nowhere dense if and only if for every $n$ there are $s_j \subseteq I_j \Subset \mathbb{N}$ for $j < n$, such that $n < \min(I_0)$ and $\max(I_j) < \min(I_{j+1})$ for all $j < n-1$ and $A \cap \bigcup_{j<n}[I_j, s_j] = \emptyset$.

Let $\mathsf{S}_n$ be the family of all sets of the form $\bigcup_{j<n}[I_j, s_j]$ for $I_j \Subset \mathbb{N}$ that satisfy $\max(I_j) < \min(I_{j+1})$ for all $j < n-1$ and $s_j \subseteq I_j$.

**Claim 1.10.3.** *If $U_k$, for $k < n$, are elements of $\mathsf{S}_n$, then $\bigcap_{k<n} U_k$ is a nonempty open set.*

Proof. Let $U_k = \bigcup_{j<n}[I_j^k, s_j^k]$ as in the definition of $\mathsf{S}_n$. A simple argument (see [7, Lemma 2.3.5], also Lemma A.6.2 for a more involved variant) implies that there is a permutation $\pi$ of $n$ such that the intervals $I^k_{\pi(k)}$ for $k < n$ are disjoint. This implies that the open set $\bigcap_{k<n}[I^k_{\pi(k)}, s_k]$ is nonempty. □

Fix $n$ for a moment. Enumerate $\mathsf{S}_n$ as $U_{nj}$, for $j \in \mathbb{N}$ and let

$$\mathcal{K}_n = \bigcup_j \mathcal{P}(\mathbb{Q} \setminus (\{q_i : i < j\} \cup U_{nj})).$$

This is a closed hereditary set. Also, $A$ is nowhere dense if and only if so is $A \cup F$ for all $F \Subset \mathbb{Q}$. Therefore Lemma 3.1.2 implies that $A \in \mathsf{nwd}$ if and only if $A \in \mathcal{K}_n \mathbin{\underline{\cup}} \mathrm{Fin}$ for all $n$. This completes the proof.

For $\mathcal{Z}_s$, note that

$$\mathcal{Z}_s = \left\{A : (\forall \varepsilon > 0)(\exists m)(\forall l \geq m)(\forall k)\frac{|A \cap [k, k+l)|}{l} \leq \varepsilon\right\}.$$

The sets $X_{\varepsilon,m} = \{A : (\forall l \geq m)(\forall k)|A \cap [k, k+l)|/l \leq \varepsilon\}$ are closed and hereditary. The sets

$$X_\varepsilon = \{A : A \in X_{\varepsilon, \min(A)}\}$$

are closed and hereditary as well, and $\mathcal{Z}_s = \bigcap_n X_{1/n} \mathbin{\underline{\cup}} \mathrm{Fin}$.

Towards proving $\mathcal{Z}_s = \bigcap_n (X_{1/n})^d \mathbin{\underline{\cup}} \mathrm{Fin}$ for all $d \geq 1$, assume $A \in (X_\varepsilon)^d$. Then $A = \bigcup_{j<d} A_j$ for some $k_i$ and $A_i \in X_{\varepsilon,k_i}$, for $i \leq n$, therefore $A \in X_{d\varepsilon, \max(k_i)}$ and $A \in X_{d\varepsilon} \mathbin{\underline{\cup}} \mathrm{Fin}$. Thus we have $(X_\varepsilon)^d \subseteq X_{d\varepsilon} \mathbin{\underline{\cup}} \mathrm{Fin}$ and $\mathcal{Z}_s = \bigcap_n (X_{1/n})^d \mathbin{\underline{\cup}} \mathrm{Fin}$ for all $d$. This completes the proof.

(5) Lemma 1.8.4 implies that this is a special case of (6), and (6) is a consequence of Lemma 1.10.4 below. □

**Lemma 1.10.4.** *Suppose that $\varphi_n$, for $n \in \mathbb{N}$, are lower semicontinuous submeasures on $\mathbb{N}$ such that $\limsup_n \|\varphi_n\| > 0$ and $\mathrm{Fin} \subseteq \mathcal{D}_\varphi$. Then the ideal (as in Lemma 1.8.3)*

$$\mathcal{D}_\varphi = \{A \subseteq \mathbb{N} : \limsup_n \varphi_n(A) = 0\}$$

*is countably $\infty$-determined by closed approximations.*

Proof. For $m \geq 1$ and $k \in \mathbb{N}$ let

$$\mathcal{K}_{m,k} = \{A \subseteq \mathbb{N} : \min(A) \geq k \text{ and } (\forall j \geq k)\varphi_j(A) \leq 1/m\},$$
$$\mathcal{K}_m = \bigcup_k \mathcal{K}_{m,k}.$$

Clearly each $\mathcal{K}_{m,k}$ is closed and hereditary, and $\emptyset$ is the only accumulation point of these sets. Therefore $\mathcal{K}_m$ is closed and hereditary.

If $A \in \bigcap_m (\mathcal{K}_m \mathbin{\underline{\cup}} \mathrm{Fin})$ then for every $\varepsilon > 0$ there is $k$ large enough to have $\varphi_l(A \setminus k) < \varepsilon$ for all $j \geq k$. Since $\mathrm{Fin} \subseteq \mathcal{D}_\varphi$, $\limsup_j \varphi_j(k) = 0$ and therefore $\limsup_j \varphi_j(A) < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, $A \in \mathcal{D}_\varphi$ and $\bigcap_m (\mathcal{K}_m \mathbin{\underline{\cup}} \mathrm{Fin}) \subseteq \mathcal{D}_\varphi$ follows.

Conversely, assume $A \in \mathcal{D}_\varphi$ and fix $m$. If $k$ satisfies $\sup_{j \geq k} \varphi_j(A) \leq 1/m$, then $A \setminus k \in \mathcal{K}_{m,k}$ and therefore $A \in \mathcal{K}_m \mathbin{\underline{\cup}} \mathrm{Fin}$. Since $m$ was arbitrary, we have $\mathcal{D}_\varphi \subseteq \bigcap_m (\mathcal{K}_m \mathbin{\underline{\cup}} \mathrm{Fin})$.

Finally, suppose that $\mathcal{I}(\mathsf{A})$ is a matrix summability ideal. It is of the form $\mathcal{D}_\mu$ for a sequence of measures $\mu_n$, for $n \in \mathbb{N}$, such that $\lim_n \mu_n(\mathbb{N}) = 1$ and $\mathrm{Fin} \subseteq \mathcal{D}_\mu$ by Lemma 1.8.4. □

**Theorem 1.10.5.** *For $d \in \mathbb{N}$ or $d = \infty$, assume that $\mathcal{I}$ and $\mathcal{J}$ are ideals countably $d$-determined by closed approximations. Then each of the following ideals is countably $d$-determined by closed approximations.*

*(1) $\mathcal{I} \restriction X$, for any $X \in \mathcal{I}_+$.*
*(2) $\mathcal{I} \oplus \mathcal{J}$.*
*(3) Any $\mathcal{J} \leq_{\mathrm{RK}} \mathcal{I}$ such that $\mathcal{J} \supseteq \mathrm{Fin}$.*
*(4) For any $h\colon \mathbb{N} \to \mathbb{N}$, the ideal $h^{-1}(\mathcal{I}) = \{A \subseteq \mathbb{N} : h[A] \in \mathcal{I}\}$.*
*(5) The ideal $\mathcal{I} \times \emptyset$.*
*(6) The ideal $\emptyset \times \mathcal{I}$.*

Proof. Assume $d \in \mathbb{N}$ and $\mathcal{I} = \bigcap_n (\mathcal{K}_n^d \mathbin{\underline{\cup}} \mathrm{Fin})$ for a sequence of closed approximations $\mathcal{K}_n$, for $n \in \mathbb{N}$. In each of the cases below we will find a sequence of closed approximations $\mathcal{L}_n$ that depend only on $\mathcal{K}_j$, for $j \in \mathbb{N}$, such that the ideal in question is equal to $\bigcap_n (\mathcal{L}_n^d \mathbin{\underline{\cup}} \mathrm{Fin})$. Since the definition of $\mathcal{L}_n$ does not involve a reference to $d$, this will prove the theorem for all $d$. In some cases, the sets $\mathcal{L}_n$ will be indexed by $\mathbb{N} \times \mathbb{N}$ or other convenient countable set.

(1) Fix $X \in \mathcal{I}_+$ and let $\mathcal{L}_n = \mathcal{P}(X) \cap \mathcal{K}_n$. Then $\mathcal{L}_n^m = \mathcal{K}_n^m$ for all $m$ and for $A \subseteq X$ we have $A \in \bigcap_n (\mathcal{L}_n^m \mathbin{\underline{\cup}} \mathrm{Fin})$ if and only if $A \in \bigcap_n (\mathcal{K}_n^m \mathbin{\underline{\cup}} \mathrm{Fin})$, as required.

(2) Assume $\mathcal{J} = \bigcap_n (\mathcal{M}_n^d \mathbin{\underline{\cup}} \mathrm{Fin})$. Then $\mathcal{L}_{m,n} = \mathcal{K}_m \oplus \mathcal{M}_n$ for $m, n$ in $\mathbb{N}$ are as required.

(3) Fix $h\colon \mathbb{N} \to \mathbb{N}$ such that $\mathcal{J} = \{A : h^{-1}(A) \in \mathcal{I}\}$. We will first prove that $\mathcal{J}$ is countably $d$-determined by a sequence of closed approximations by using an additional assumption, that $h$ is finite-to-one (namely that $\mathcal{J} \leq_{\mathrm{RB}} \mathcal{I}$). For each $n$ let

$$\mathcal{L}_n = \{A \subseteq \mathbb{N} : h^{-1}(A) \in \mathcal{K}_n\}.$$

This set is hereditary because $\mathcal{K}_n$ is, and it is closed as a continuous preimage of a closed set. Fix $m$ and $A \subseteq \mathbb{N}$. Since $B \mapsto h^{-1}(B)$ is a Boolean algebra homomorphism that sends Fin to Fin, we have $A \in \mathcal{L}_n^m \mathbin{\underline{\cup}} \mathrm{Fin}$ if and only if $h^{-1}(A) \in \mathcal{K}_n^m \mathbin{\underline{\cup}} \mathrm{Fin}$. This implies $\mathcal{J} = \bigcap_n (\mathcal{L}_n \mathbin{\underline{\cup}} \mathrm{Fin})$.

It remains to prove the assertion in general case. If $X = \{n : h^{-1}(\{n\}) \text{ is finite}\}$ then $\mathcal{J} \cong \mathcal{J} \restriction X \oplus \mathcal{J} \restriction (\mathbb{N} \setminus X)$ and by the first part of the proof and (2) it suffices to prove that $\mathcal{J} \restriction (\mathbb{N} \setminus X)$ is countably $d$-determined by closed approximations. We can therefore assume $h^{-1}(\{n\})$ is infinite for all $n$. Also, $h^{-1}(\{n\}) \in \mathcal{I}$ for all $n$ because we are assuming $\mathcal{J} \supseteq \mathrm{Fin}$.

Fix $n$ and for each $j$ let

$$\mathcal{L}_{n,j} = \{B \setminus j : h^{-1}(B) \in \mathcal{K}_n \mathbin{\underline{\cup}} \mathcal{P}(j)\}.$$

This is a closed hereditary set and by Lemma 1.2.5 so is $\mathcal{L}_n = \bigcup_j \mathcal{L}_{n,j}$. Also, $A = h[B]$ for $B \in \mathcal{K}_n \underline{\cup} \mathrm{Fin}$ if and only if $A = h[B \underline{\cup} j]$ for some $j$, if and only if $A \in \mathcal{L}_{n,j} \underline{\cup} \mathcal{P}(j)$ for the same $j$. For every $m$ we have $h^{-1}(A) \in \mathcal{K}_n^m \underline{\cup} \mathrm{Fin}$ if and only if $h^{-1}(A) = \bigcup_{i<m} B_i$, where $B_i \in \mathcal{K}_n \underline{\cup} \mathrm{Fin}$ for all $i < m$. By the previous argument, this is equivalent to having $A = \bigcup_{i<m} h[B_i]$ with $h[B_i] \in \mathcal{L}_{n,j(i)} \underline{\cup} \mathrm{Fin}$ for some $j(i)$, for $i < m$. But this is equivalent to $A \in \mathcal{L}_n^m \underline{\cup} \mathrm{Fin}$.

(4) Fix $h \colon \mathbb{N} \to \mathbb{N}$, $\mathcal{J} = \{A : h[A] \in \mathcal{I}\}$. Although we will not using the following observation, note that if $h$ is $k$-to-one for some $k \in \mathbb{N}$ then $\mathcal{J}$ is isomorphic to the restriction of $\bigoplus_{i<k} \mathcal{I}$ to some positive set, and the conclusion follows by (2).

Fix $n$ and for each $j$ let

$$\mathcal{L}_{n,j} = \{h^{-1}(A) \setminus j : A \in \mathcal{K}_n \underline{\cup} \mathcal{P}(j)\}.$$

This set is clearly closed and hereditary and so is $\mathcal{L}_n = \bigcup_j \mathcal{L}_{n,j}$ by Lemma 1.2.5. Thus $A \in \mathcal{L}_{n,j}$ if and only if $A \cap j = \emptyset$ and $h[A] \in \mathcal{K}_n \cup \mathcal{P}(\mathrm{j})$.

We have $A \in \mathcal{L}_n \underline{\cup} \mathrm{Fin}$ if and only if there is $j$ such that $A \in \mathcal{L}_{n,j} \underline{\cup} \mathrm{Fin}$, if and only if $h[A] \in \mathcal{K}_n \underline{\cup} \mathrm{Fin}$. Similarly, $A \in \mathcal{L}_n^m \underline{\cup} \mathrm{Fin}$ if and only if $A = \bigcup_{i<m} A_i$ where $A_i \in \mathcal{L}_{n,j(i)} \underline{\cup} \mathrm{Fin}$ for some $j(i)$. This is equivalent to having $h[A] = \bigcup_{i<m} h[A_i]$ in $\mathcal{K}_n^m \underline{\cup} \mathrm{Fin}$.

(5) With $h \colon \mathbb{N}^2 \to \mathbb{N}$ defined by $h(m,n) = m$ we have $\mathcal{I} \otimes \emptyset = h^{-1}(\mathcal{I})$, and the conclusion follows by part (4).

(6) Recall that $\emptyset \times \mathcal{I}$ is $\{(m,n) : A_m \in \mathcal{I} \text{ for all } m\}$. For $A \subseteq \mathbb{N}^2$ write $A_m = \{n : (m,n) \in A\}$. For each $n$ let

$$\mathcal{L}_n = \{A \subseteq \mathbb{N}^2 : A_i \in \mathcal{K}_n \text{ for all } i \leq n\}.$$

This is a closed hereditary set and $\mathcal{L}_n^m = \{A \subseteq \mathbb{N}^2 : A_i \in \mathcal{K}_n^m \text{ for all } i \leq n\}$ for every $m$. Fix $m$ such that $\mathcal{I} = \bigcap_n \mathcal{K}_n^m \underline{\cup} \mathrm{Fin}$. We will prove that $\emptyset \otimes \mathcal{I} = \bigcap_n (\mathcal{L}_n^m \underline{\cup} \mathrm{Fin})$. Fix $A \subseteq \mathbb{N}^2$. Then

$$(\forall j)(\forall n)(\exists k(j,n)) A_j \in \mathcal{K}_n^m \setminus k(j,n)$$

if and only if (with $k'(n) = \max_{j \leq n} k(j,n)$)

$$(\forall n)(\forall j \leq n)(\exists k'(n)) A_j \in \mathcal{K}_n^m \setminus k'(n),$$

and this is equivalent to $(\forall n)(\exists k) A \in \mathcal{L}_n^m \setminus k$. Therefore $A \in \emptyset \otimes \mathcal{I}$ if and only if $A \in \bigcap_n \mathcal{L}_n^m \underline{\cup} \mathrm{Fin}$ as required. □

CHAPTER 2

# Orders and Morphisms

In this Chapter we introduce standard orders on ideals, starting with the simplest Rudin–Keisler and Rudin–Blass orders $\leq_{\text{RK}}$ and $\leq_{\text{RB}}$. A large part of this Chapter is devoted to computing these orders on ideals introduced in the first Chapter; that said, our results are incomplete and much remains to be done along these lines. Once proved, lifting theorems of Chapters 4 and 6 will imply that for many (conjecturally, all) analytic ideals $\mathcal{I}$, every embedding of a quotient over another analytic ideal into $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is given by an RK-reduction between them. We also briefly discuss Katětov order.

## 2.1. Rudin–Blass and Rudin–Keisler orders

**Definition 2.1.1.** If $\mathcal{I}$ and $\mathcal{J}$ are ideals on $\mathbb{N}$, we write $\mathcal{I} \leq_{\text{RK}} \mathcal{J}$ if there is $h\colon \mathbb{N} \to \mathbb{N}$ such that $A \in \mathcal{I}$ if and only if $h^{-1}(A) \in \mathcal{J}$. This is the *Rudin–Keisler order*. If $\mathcal{I} \leq_{\text{RK}} \mathcal{J}$, we say that $\mathcal{I}$ is *Rudin–Keisler reducible* to $\mathcal{J}$ (some authors say that $\mathcal{J}$ is Rudin–Keisler reducible to $\mathcal{I}$; regrettably, each one of these terminologies makes sense) or that $\mathcal{I}$ is *Rudin–Keisler below* $\mathcal{J}$ (to the best of my knowledge, nobody says that $\mathcal{J}$ is Rudin–Keisler below $\mathcal{I}$).

We write $\mathcal{I} <_{\text{RB}} \mathcal{J}$ if $\mathcal{I} \leq_{\text{RB}} \mathcal{J}$ and $\mathcal{I} \not\geq_{\text{RB}} \mathcal{J}$, and similarly define $<_{\text{RK}}$.

If the function $h$ is finite-to-one, then we write $\mathcal{I} \leq_{\text{RB}} \mathcal{J}$. This is the *Rudin–Blass order*, also called *finite-to-one reduction*, and denoted by $\leq_f$ in [144].

If there are $X \in \mathcal{I}^*$ and $Y \in \mathcal{J}^*$ and a bijection $h\colon X \to Y$ such that $A \in \mathcal{J}$ if and only if $h^{-1}(A) \in \mathcal{I}$ for every $A \subseteq Y$, then we say that $\mathcal{I}$ and $\mathcal{J}$ are *Rudin–Keisler-isomorphic*, or *RK-isomorphic*, or simply *isomorphic* and write $\mathcal{I} \sim_{\text{RK}} \mathcal{J}$.

Rudin–Keisler has been around for a while, and Rudin–Blass order was introduced in [116]. By Corollary 3.2.3, Fin is $\leq_{\text{RB}}$-reducible to every ideal that has the Property of Baire and includes Fin.

Let's get the easy facts out of our way.

**Lemma 2.1.2.** *(1) If $\mathcal{I} \leq_{\text{RB}} \mathcal{J}$ and $\mathcal{I}$ is dense, then $\mathcal{J}$ is dense.*

*(2) For every ideal $\mathcal{I}$ we have $\mathcal{I} \leq_{\text{RK}} \emptyset \times \mathcal{I}$.*

*(3) There are ideals $\mathcal{I}$ and $\mathcal{J}$ such that $\mathcal{I} \leq_{\text{RK}} \mathcal{J}$ and $\mathcal{I}$ is dense but $\mathcal{J}$ is not.*

*(4) If $\mathcal{J}$ is a P-ideal, then $\mathcal{I} \leq_{\text{RK}} \mathcal{J}$ implies $\mathcal{I} \leq_{\text{RB}} \mathcal{J}$.*

Proof. Assume that $\mathcal{I}$ dense and $h\colon \mathbb{N} \to \mathbb{N}$ is a finite-to-one reduction. Fix an infinite $A \subseteq \mathbb{N}$. We need to find infinite $B \subseteq A$ in $\mathcal{J}$. Since $h$ is finite-to-one, $h[A]$ is infinite and we can find an infinite $C \subseteq h[A]$ in $\mathcal{I}$. Then $h^{-1}(C) \cap A$ is an infinite subset of $A$ in $\mathcal{J}$, as required.

For the second part, $h\colon \mathbb{N}^2 \to \mathbb{N}$ defined by $h(m,n) = n$ is an RK-reduction of $\mathcal{I}$ to $\emptyset \times \mathcal{I}$. Taking $\mathcal{I}$ to be a dense ideal, the third part follows.

(4) Assume $h\colon \mathbb{N}\to\mathbb{N}$ is a $\leq_{\mathrm{RK}}$-reduction. Let $A$ be a set in $\mathcal{J}$ which includes modulo finite all $h^{-1}(\{n\})$. Then $h_1 = h \restriction (\mathbb{N}\setminus A)$ is a $\leq_{\mathrm{RB}}$-reduction, because $h_1^{-1}(B)\Delta h^{-1}(B)\subseteq A\in\mathcal{J}$ for all $B\subseteq\mathbb{N}$. □

It is well-known that the Rudin–Keisler order on ultrafilters satisfies the analog of the Schröder–Bernstein theorem: if $\mathcal{U}\leq_{\mathrm{RK}}\mathcal{V}$ and $\mathcal{V}\leq_{\mathrm{RK}}\mathcal{U}$ then $\mathcal{U}\sim_{\mathrm{RK}}\mathcal{V}$ (e.g., [**54**, Proposition 9.4.4]). The analogous fact immediately follows for the Rudin–Blass order, but it does not apply to either of these orders on ideals (Corollary 2.6.7).

**Definition 2.1.3.** Two ideals $\mathcal{I}$ and $\mathcal{J}$ are *Rudin–Keisler isomorphic*, or *isomorphic* if and only if there are $A\in\mathcal{I}_*$, $B\in\mathcal{J}_*$, and a bijection $f\colon A\to B$ such that for every $C\subseteq A$ we have $C\in\mathcal{I}$ if and only if $f[C]\in\mathcal{J}$.

Note that Rudin–Keisler coincides with the 'Rudin–Blass isomorphism', hence we will not use the latter.

**Proposition 2.1.4.** *Every countably generated ideal $\mathcal{I}$ that includes* Fin *is isomorphic to* Fin *or to* $\mathrm{Fin}\times\emptyset$.

Proof. Let $A_n$ ($n\in\mathbb{N}$) be the generating sequence of $\mathcal{I}$. We can assume that $A_n\subseteq A_{n+1}$ for all $n$. If the set $A_{n+1}\setminus A_n$ is infinite for infinitely many $n$, then we can assume (by going to a subsequence of $\{A_n\}$) that it is infinite for all $n$. Then a bijection $h\colon\mathbb{N}\to\mathbb{N}^2$ defined so that $h''A_n=(n+1)\times\mathbb{N}$ gives an isomorphism between the ideals $\mathcal{I}$ and $\mathrm{Fin}\times\emptyset$.

If the set $A_{n+1}\setminus A_n$ is finite for all but finitely many $n$, then the ideal is clearly isomorphic to Fin. □

The only pair of isomorphic ideals $\mathcal{I}$, $\mathcal{J}$ such that the isomorphism between them is not implemented by a permutation of $\mathbb{N}$ is Fin and $\mathrm{Fin}\oplus\mathbb{N}$. This is because in any other case one can choose $A$ and $B$ as in Definition 2.1.3 so that $\mathbb{N}\setminus A$ and $\mathbb{N}\setminus B$ are both infinite and extend $f$ to a permutation by a bijection between these two sets. Some authors define ideals to be isomorphic if and only if some apermutation of $\mathbb{N}$ witnesses the isomorphism, and therefore consider Fin and $\mathrm{Fin}\oplus\mathcal{P}(\mathbb{N})$ as nonisomorphic. As our considerations are driven by the structure of quotients, we do not distinguish between these ideals.

## 2.2. Katětov order

Another prominent order on ideals ought to be mentioned, although its relation to quotient rigidity is somewhat tangential. If $\mathcal{I}$ and $\mathcal{J}$ are ideals on $\mathbb{N}$, we say that $\mathcal{I}$ is *Katětov below* $\mathcal{J}$, $\mathcal{I}\leq_{\mathrm{K}}\mathcal{J}$, if there is a function $h\colon\mathbb{N}\to\mathbb{N}$ such that $h^{-1}[X]\in\mathcal{J}$ for all $X\in\mathcal{I}$. The Katětov–Blass order is defined like the Katětov order, but the function $h$ is required to be finite-to-one: We say that $\mathcal{I}$ is *Katětov–Blass below* $\mathcal{J}$, $\mathcal{I}\leq_{\mathrm{KB}}\mathcal{J}$, if there is a finite-to-one function $h\colon\mathbb{N}\to\mathbb{N}$ such that $h^{-1}[X]\in\mathcal{J}$ for all $X\in\mathcal{I}$. Equivalently, $\mathcal{I}\leq_{\mathrm{K}}\mathcal{J}$ if $\mathcal{I}\leq_{\mathrm{RK}}\mathcal{J}'$ for some ideal $\mathcal{J}'$ included in $\mathcal{J}$.

A canonical, albeit aging, reference for the Katětov order is [**86**].

The Category Dichotomy ([**82**, Theorem 5.20]) asserts that every Borel ideal $\mathcal{I}$ satisfies (see Definition 1.8.7 and Lemma 1.6.3) $\mathcal{I}\leq_{\mathrm{K}}\mathsf{nwd}$ if and only if $\mathcal{ED}\not\leq_{\mathrm{K}}\mathcal{I}\restriction X$ for every $\mathcal{I}$-positive $X$. This is not true for ideals that are not necessarily Borel, since the ideal of nowhere dense subsets of a certain countable topological space is a counterexample ([**29**]).

The Measure Dichotomy from [82] will be used in §4.2.1.

It is not difficult to see that no dense Borel ideal is $\leq_{\mathrm{RK}}$-minimal, and by [76] there is no $\leq_{\mathrm{K}}$-minimal dense Borel ideal. Note that every maximal ideal is $\leq_{\mathrm{K}}$-minimal and that, as is well-known, it is $\leq_{\mathrm{RK}}$-minimal if and only if it is maximal and its dual is a selective ultrafilter. In [137] it was also proved that there is a $\leq_{\mathrm{K}}$-maximal analytic P-ideal, and that it is also $\leq_{\mathrm{K}}$-above every $F_\sigma$ ideal.

The study of the Katětov order is closely connected to the study of inclusions between ideals. By [137], every analytic ideal is included in a Borel ideal. By [77], every dense analytic ideal contains an analytic $F_\sigma$ ideal, and $\mathcal{Z}_0$ is not contained in an $F_\sigma$ ideal

There is no known characterisation of analytic ideals not included in an $F_\sigma$ ideal. In [82] it was conjectured that an analytic ideal $\mathcal{I}$ is not included in an $F_\sigma$ ideal if and only if $\mathcal{I}_{\mathrm{CONV}} \leq_{\mathrm{K}} \mathcal{I}$ (see §1.9). The latter is equivalent to $\mathcal{Z}_0 \leq_{\mathrm{K}} \mathcal{I}$.)

Cardinal invariants of quotients are very relevant to understanding the Katětov order; see for example section 5 of [84].

Katětov order is particularly important in the study of the behaviour of ideals in forcing extensions. An ideal $\mathcal{I}$ is *diagonalised* by forcing $\mathbb{P}$ if $\mathbb{P}$ adds an infinite subset of $\mathbb{N}$ which does not include any infinite ground-model set belonging to $\mathcal{I}$. By [115], every $F_\sigma$ ideal can be diagonalised without adding an unbounded real. It is not known whether every $F_{\sigma\delta}$ ideal can be diagonalised without adding a dominating real. In particular, it is not known whether this holds for $\mathcal{Z}_0$.

### 2.3. Quasi-orders on analytic quotients

A *lifting* of a homomorphism $\Phi\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})/\mathcal{I}$ is any $\Phi_*\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ such that the diagram on Fig. (1) commutes ($\pi_{\mathcal{I}}$ is the quotient map associated to $\mathcal{I}$).[1] This is a set-theoretic lifting whose existence follows from the Axiom of Choice. We do not assume that a lifting has any algebraic properties. As a matter of fact, our main objective is to show that an algebraic lifting exists under fairly general assumptions on $\mathcal{I}$ and $\Phi$ (Theorem 4.3.1) and when the assumptions on $\Phi$ are replaced by forcing axioms (Theorem 6.1.2 and Theorem 6.1.3).

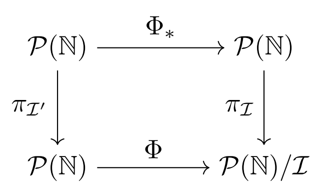



FIGURE 1. A lifting $\Phi_*$ of $\Phi$.

**Definition 2.3.1.** A set in a topological space) is called *meagre* (or of *first category*) if it can be covered by countably many nowhere dense sets, and *nonmeagre* (of *second category*) otherwise. It has the *Property of Baire* (is *Baire measurable*) if it is equal to an open set modulo some meagre set. A function is *Baire measurable* (or simply *Baire*) if the preimage of every open set has the property of Baire.

[1] Not assuming that $\ker(\Phi)$ includes Fin leads to some complications but the results are well worth the trouble.

The following orders on ideals are defined in terms of their quotients.

**Definition 2.3.2.** Suppose that $\mathcal{I}$ and $\mathcal{J}$ are ideals on $\mathbb{N}$. Consider the following relations.

(1) *Baire embeddability*: $\mathcal{I} \leq_{\rm BE} \mathcal{J}$ if there is a Baire measurable $F\colon P(\mathbb{N}) \to P(\mathbb{N})$ which is a lifting of an injective homomorphism $\Phi$ of the quotient $\mathcal{P}(\mathbb{N})/\mathcal{I}$ into $\mathcal{P}(\mathbb{N})/\mathcal{J}$ as in Fig. (1)
(2) $\mathcal{I} \leq^+_{\rm BE} \mathcal{J}$ if $\mathcal{I} \leq_{\rm BE} \mathcal{J} \restriction A$ for some $\mathcal{J}$-positive $A$.

The main lifting result of §4 shows that the Baire-embeddability order in many cases reduces to the Rudin–Keisler order. The analogous fact is not true for the order $\leq^+_{\rm BE}$, which will play a role in Chapter 6 (see Corollary 7.2.2). Perhaps the earliest result about preorders on analytic ideals is Corollary 3.2.3. Before we continue, let us make some easy observations that will be frequently used below.

**Definition 2.3.3.** An *amalgamation* of homomorphisms $\Phi\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})/\mathcal{I}$ and $\Psi\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})/\mathcal{J}$ is the homomorphism

$$\Phi \oplus \Psi\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N} \times \{0,1\})/(\mathcal{I} \oplus \mathcal{J})$$

defined by its lifting, $A \mapsto \Phi_*(A) \times \{0\} \cup \Psi_*(A) \times \{1\}$. In other words, $\Phi \oplus \Psi$ is a homomorphism which makes the diagram in Fig. (2) commute, where $i_1\colon \mathcal{P}(\mathbb{N})/\mathcal{I} \to \mathcal{P}(\mathbb{N} \oplus \mathbb{N})/\mathcal{I} \oplus \mathcal{J}$ is given by its lifting $A \mapsto A \times \{0\}$ and $i_2\colon \mathcal{P}(\mathbb{N})/\mathcal{J} \to \mathcal{P}(\mathbb{N} \oplus \mathbb{N})/\mathcal{I} \oplus \mathcal{J}$ is given by its lifting $A \mapsto A \times \{1\}$.

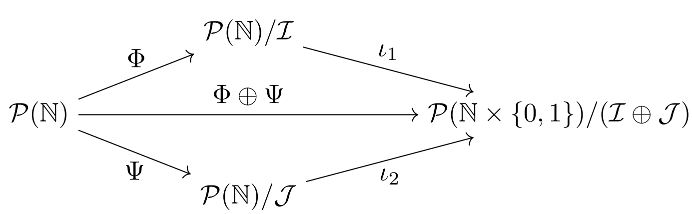



FIGURE 2. An amalgamation of homomorphisms.

Note that $\ker(\Phi \oplus \Psi) = \ker(\Phi) \cap \ker(\Psi)$. Amalgamations will play an important role in Chapter 6.

**Proposition 2.3.4.** *Suppose that $\mathcal{I}$ and $\mathcal{J}$ are analytic ideals on $\mathbb{N}$.*

*(1) $\mathcal{I} \leq_{\rm RB} \mathcal{J}$ implies $\mathcal{I} \leq_{\rm RK} \mathcal{J}$, $\mathcal{I} \leq_{\rm RK} \mathcal{J}$ implies $\mathcal{I} \leq_{\rm BE} \mathcal{J}$, and $\mathcal{I} \leq_{\rm BE} \mathcal{J}$ implies $\mathcal{I} \leq^+_{\rm BE} \mathcal{J}$.*
*(2) $\mathcal{I} \leq^+_{\rm BE} \mathcal{J}$ if and only if $\mathcal{I} \leq_{\rm BE} \mathcal{J}$ or $\mathcal{I} \oplus \mathrm{Fin} \leq_{\rm BE} \mathcal{J}$.*

PROOF. (1) This holds for arbitrary ideals $\mathcal{I}$ and $\mathcal{J}$. At most the implication from $\mathcal{I} \leq_{\rm RK} \mathcal{J}$ to $\mathcal{I} \leq_{\rm BE} \mathcal{J}$ requires a proof. Assume $h\colon \mathbb{N} \to \mathbb{N}$ is a reduction of $\mathcal{I}$ to $\mathcal{J}$, and define the mapping $\Phi_h\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ by

$$\Phi_h(A) = \bigcup_{n \in A} h^{-1}(\{n\}) = h^{-1}(A).$$

This is a homomorphism of $\mathcal{P}(\mathbb{N})$ into $\mathcal{P}(\mathbb{N})$. Since the preimage of $\mathcal{J}$ is equal to $\mathcal{I}$ by the assumption on $h$, $\Phi_h$ is a lifting of an injective homomorphism of $\mathcal{P}(\mathbb{N})/\mathcal{I}$ into $\mathcal{P}(\mathbb{N})/\mathcal{J}$

(2) If $\Phi\colon \mathcal{P}(\mathbb{N}\times\{0,1\})/\mathcal{I}\oplus\mathrm{Fin}\to\mathcal{P}(\mathbb{N})/\mathcal{J}$, then $B=\Phi(\mathbb{N}\times\{0\})$ satisfies $\mathcal{I}\leq_{\mathrm{BE}}\mathcal{J}\restriction B$, and therefore $\mathcal{I}\leq^+_{\mathrm{BE}}\mathcal{J}$. On the other hand, assume $\mathcal{I}\leq^+_{\mathrm{BE}}\mathcal{J}$ holds, say $\Phi_1$ witnesses $\mathcal{I}\leq_{\mathrm{BE}}\mathcal{J}\restriction B$ for some $B$. If $\mathbb{N}\setminus B$ is in $\mathcal{J}$ there is nothing to prove. Otherwise, Corollary 3.2.3 implies that $\Phi_2$ witnesses $\mathrm{Fin}\leq_{\mathrm{BE}}\mathcal{J}\restriction(\mathbb{N}\setminus B)$. Then mapping $\Phi_1\oplus\Phi_2$ witnesses $\mathcal{I}\oplus\mathrm{Fin}\leq_{\mathrm{BE}}\mathcal{J}$. □

### 2.4. Orthogonals and separation

This section is all that remains from [40, §5].

**Definition 2.4.1.** An *orthogonal* of an ideal $\mathcal{I}$ is the ideal

$$\mathcal{I}^\perp=\{A : A\cap B \text{ is finite for all } B\in\mathcal{I}\}.$$

Two ideals $\mathcal{I}$ and $\mathcal{J}$ are *separated* if there is $C\subseteq\mathbb{N}$ such that $A\setminus C$ is finite for all $A\in\mathcal{I}$ and $B\cap C$ is finite for all $B\in\mathcal{J}$. They are *countably separated* if there are sets $C_n$, for $n\in\mathbb{N}$, such that for every pair $(A,B)$ in $\mathcal{I}\times\mathcal{J}$ some $C_n$ separates $A$ from $B$, in the sense that both $A\setminus C_n$ and $B\cap C_n$ are finite.

The ideal $\emptyset\times\mathrm{Fin}$ cannot be separated from its orthogonal, $\mathrm{Fin}\times\emptyset$, but $\emptyset\times\mathrm{Fin}$ and $\mathrm{Fin}\times\emptyset$ can be *countably separated*: there is a sequence $\{c_n\}$ of sets of integers such that for every $a\in\emptyset\times\mathrm{Fin}$ and every $b\in\mathrm{Fin}\times\emptyset$ there is $c_n$ such that $a\subseteq^* c_n$ and $b\cap c_n$ is finite.

**Lemma 2.4.2.** *If $\mathcal{I}$ is an analytic P-ideal, then $\mathcal{I}^\perp$ is countably generated.*

Proof. This is [154, Theorem 2]. It can also be deduced from Theorem 1.4.6 as follows. Let $\varphi$ be a lower semicontinuous submeasure such that $\mathcal{I}=\mathrm{Exh}(\varphi)$, and let

$$C_n=\{i : \varphi(\{i\})\geq 1/n\}.$$

We claim that these sets generate $\mathcal{I}^\perp$. Since $\lim_{i\to\infty}\varphi(C_n\setminus i)\geq 1/n$, each $C_n$ is in $\mathcal{I}^\perp$, so it will suffice to show that every $A\in\mathcal{I}^\perp$ is included in some $C_n$. Assume that $A\not\subseteq C_n$ for every $n$, and that $A$ is infinite. Then we can find an infinite $B=\{m_i : i\in\mathbb{N}\}$ included in $A$ such that $\varphi(\{m_i\})\leq 1/i^2$, and this implies that $B\in\mathcal{I}$, and therefore $A$ is not in $\mathcal{I}^\perp$. This completes the proof. □

It is clear that every ideal $\mathcal{I}$ satisfies $(\mathcal{I}^\perp)^\perp\supseteq\mathcal{I}$. An ideal $\mathcal{I}$ is said to have the *Fréchet property* if $(\mathcal{I}^\perp)^\perp=\mathcal{I}$.

**Corollary 2.4.3.** *An analytic P-ideal has the Fréchet property if and only if it is RK-isomorphic to* Fin *or to* $\emptyset\times\mathrm{Fin}$.

Proof. For the direct implication, let $C_n$ be a family which separates $\mathcal{I}$ from $\mathcal{I}^\perp$. We may assume that every $C_n$ is orthogonal to $\mathcal{I}$ (either by the proof of Lemma 2.4.2, or by using the fact that $\mathcal{I}$ is $\sigma$-directed under $\subseteq^*$). Therefore, $\mathcal{I}^\perp$ is generated by $\{C_n : n\in\mathbb{N}\}$. By Proposition 2.1.4, it is RK-isomorphic to one of Fin or $\mathrm{Fin}\times\emptyset$, and therefore $\mathcal{I}=(\mathcal{I}^\perp)^\perp$ is isomorphic to one of Fin or $\emptyset\times\mathrm{Fin}$.

The converse is obvious, since each of Fin and $\emptyset\times\mathrm{Fin}$ is clearly countably separated from its orthogonal. □

## 2.5. RK-homogeneity

An ideal $\mathcal{I}$ is said to be *RK-homogeneous* if it is Rudin–Keisler isomorphic to its restriction to every $\mathcal{I}$-positive set. It is *weakly RK-homogeneous* if every positive $A$ has a subset such that the restriction of $\mathcal{I}$ to it is RK-isomorphic to $\mathcal{I}$. Clearly Fin is RK-homogeneous, and every homogeneous ideal on $\mathbb{N}$ not isomorphic to Fin is dense. No dense analytic P-ideal is homogeneous (Proposition 2.5.2).

**Proposition 2.5.1.** *The following ideals are homogeneous.*

*(1) Ideals* $\mathsf{nwd}$ *and* $\mathsf{null}$ *(Definition 1.8.7).*
*(2) Ideals* $\mathcal{I}_\alpha$ *for an indecomposable countable ordinal* $\alpha$ *(Definition 1.9.1).*

Proof. For $\mathsf{nwd}$, fix $A \subseteq \mathbb{Q}$ whose closure $\overline{A}$ is not nowhere dense and let $B$ be the intersection of $A$ with the interior of $\overline{A}$. Then $B$ is homeomorphic to $\mathbb{Q}$, and the homeomorphism is an RK-isomorphism.

For $\mathsf{null}$, we prove homogeneity in two stages. Fix $A \subseteq \mathbb{Q}$ whose closure has positive Lebesgue measure (denoted $\lambda$) and no isolated points. Then $g\colon \overline{A} \to [0,1]$ defined by

$$g(x) = \lambda([0,x] \cap \overline{A})/\lambda(\overline{A})$$

has the property that $g[\overline{A}] = [0,1]$, $g^{-1}(\{x\})$ has at most two points for every $x \in [0,1]$ (otherwise the middle point would be isolated in $A$), and $X \subseteq \overline{A}$ is null if and only if $g[X]$ is null. Therefore, the restriction of $\mathsf{null}$ to a positive set $A$ is RK-isomorphic to its restriction to a set dense in $[0,1]$.

Let $A$ and $B$ be two dense subsets of $[0,1]$. By the Cantor–Bendixson analysis of $\overline{A}$ and $\overline{B}$, we may remove countable scattered sets from $A$ and $B$ whose closures are countable (and therefore in $\mathsf{null}$) and assure that neither $A$ nor $B$ has isolated points. Fix enumerations $A = \{x_n : n \in \mathbb{N}\}$ and $B = \{y_n : n \in \mathbb{N}\}$. Recursively define a bijection $f\colon A \to B$ such that $d(x_n, y_{f(n)}) < 1/n$. Then for every $C \subseteq A$ we have that $\overline{f[C]}\Delta\overline{C} \subseteq f[C] \cup C$, and therefore belongs to $\mathsf{null}$. This implies that $f$ is an RK-isomorphism between the restrictions of $\mathsf{null}$ to $A$ and to $B$.

Fix a countable indecomposable ordinal $\alpha$ To prove that $\mathcal{O}_\alpha$ is homogeneous, note that $A \subseteq \omega^\alpha$ is $\mathcal{O}_\alpha$-positive if and only if $\operatorname{otp}(A) = \omega^\alpha$, hence the order-preserving bijection between $A$ and $\omega^\alpha$ is an isomorphism between the restriction of $\mathcal{O}_\alpha$ to $A$ and $\mathcal{O}_\alpha$. □

**Proposition 2.5.2.** *The only RK-homogeneous, nonpathological analytic P-ideal is* Fin*.*

Proof. Let $\mathcal{I}$ be an RK-homogeneous non-pathological analytic P-ideal not RK-isomorphic to Fin. Using Theorem 1.4.6 fix a lower semicontinuous submeasure $\varphi$ such that $\mathcal{I} = \operatorname{Exh}(\varphi)$. The ideal $\mathcal{I}$ is dense, since otherwise the ideal $\mathcal{I} \restriction A$ would be isomorphic to Fin for some positive set $A$, contradicting the RK-homogeneity. Therefore

(1) $\lim_i \varphi(\{i\}) = 0$.

Using (1) we can recursively find sequences $u_i, v_i$ ($i \in \mathbb{N}$) and $f\colon \mathbb{N} \to \mathbb{N}$ such that for all $i$ the following conditions hold.

(2) $u_1 < u_2 < \dots$ are finite subsets of $\mathbb{N}$.
(3) $v_1 < v_2 < \dots$ are finite subsets of $\mathbb{N}$.
(4) $|\varphi(u_i) - 1| < 2^{-i-1}$.
(5) $|\varphi(v_i) - 1| < 2^{-i-1}$.

(6) $f(i)=\sum_{j=1}^{i}|u_j|$.
(7) $\varphi(\{k\})<1/(2f(i))$, for all $k\in v_i$.
(8) $\varphi(\{l\})<1/(2^i|v_i|)$ for all $l\in u_{i+1}$.

Let $A=\bigcup_i u_i$ and $B=\bigcup_i v_i$. By (4) and (5), each one of these sets is positive. Assume towards contradiction that there is an RK-isomorphism between $\mathcal{I}\restriction A$ and $\mathcal{I}\restriction B$ implemented by a function $h$. Since the restriction of $\mathcal{I}$ to each of these sets is dense, this isomorphism is implemented by a bijection $h\colon A\to B$. We claim that with

$$U_i=\bigcup_{j=1}^{i}u_i$$

for every $i$ there is $s_i\subseteq v_i$ such that $h[s_i]\cap U_i=\emptyset$ and $\varphi(s_i)\geq 1/4$. To prove this, let $t=h^{-1}(U_i)$. Since $h$ is a bijection, (7) implies

$$\varphi(t)\leq\frac{1}{2f(i)}|U_i|=\frac{1}{2},$$

and $s_i=v_i\setminus t$ is as required. Next, we claim that for every $i$,

$$\varphi(h[v_i]\setminus U_i)<2^{-i}.$$

To prove this, we may assume $h[v_i]\cap U_i=\emptyset$. By (8),

$$\varphi(h''v_i)\leq\sum_{k\in v_i}\varphi(\{h(k)\})<\tfrac{1}{2^i|v_i|}|v_i|=2^{-i},$$

and the claim follows.

Therefore, $\bigcup_i s_i$ is a positive set but $h^{-1}[\bigcup_i s_i]$ belongs to $\mathcal{I}$; contradiction. □

## 2.6. Summable ideals, II

With an eye to the lifting theorems of §4 and §6, we study the structure of the set of all summable ideals with respect to the Rudin–Blass order. In Lemma 2.6.4 we show that every summable ideal belongs to one of the disjoint classes as in Definition 2.6.3. Proposition 2.6.6 gives a characterisation of when some $h$ is an RB-reduction between given summable ideals. In Corollary 2.6.7 we prove that $\mathcal{I}_{1/n}$ and $\mathcal{I}_{1/\sqrt{n}}$ are not RK-isomorphic and that the direct sums of Fin with these ideals violate the analog of the Schröder–Bernstein theorem for $\leq_{\mathrm{RB}}$. Theorem 2.6.10 (3) shows that the set of all dense summable ideals with respect to $\leq_{\mathrm{RB}}$ is a dense partial order with no minimal or maximal elements and that $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ embeds into it.

We start with some straightforward properties of summable ideals. The following is obvious.

**Lemma 2.6.1.** *The class of summable ideals is closed under taking restrictions to positive sets and isomorphisms. If $\mathcal{I}_f$ is summable and $\mathcal{I}\leq_{\mathrm{RK}}\mathcal{I}_f$, then $\mathcal{I}$ is summable.*

Proof. At most the last sentence requires a proof. Suppose that $h$ is an RK-reduction of $\mathcal{I}$ to $\mathcal{I}_f$ and let $g(n)=\mu_f(h^{-1}(\{n\}))$. Then $\mu_g(A)=\mu_f(h^{-1}(A))$ for all $A\subseteq\mathbb{N}$, hence $\mathcal{I}=\mathcal{I}_g$. □

It will be useful to have $f(n)>0$ for all $n$ when studying $\mathcal{I}_f$.

**Lemma 2.6.2.** *For every summable ideal $\mathcal{I}_f$ there is a strictly positive $g$ such that $\mathcal{I}_f=\mathcal{I}_g$. If $\mathcal{I}_f$ is dense then there is a summable ideal $\mathcal{I}_g$ isomorphic to $\mathcal{I}_f$ such that $g$ is decreasing.*

Proof. For the first part, let $g(n) = f(n) + 2^{-n}$.

For the second part, since $\lim_n f(n) = 0$ and $f(n) > 0$ for all $n$ we can compose $f$ with a permutation of $\mathbb{N}$ to obtain a nonincreasing $g$ such that $\mathcal{I}_f = \mathcal{I}_g$. By a minor modification of $g$ as in the first part, we can assure that $g$ is decreasing. □

### 2.6.1. Four classes of summable ideals.

**Definition 2.6.3.** Suppose that $\mathcal{I}_f$ is a summable ideal and for $\varepsilon > 0$ let

$$A_{f\varepsilon^+} = \{n : f(n) \geq \varepsilon\}.$$

Consider the following classes of summable ideals.

(S1) There is $\varepsilon > 0$ such that $A_{f\varepsilon^+}$ is infinite and $\mu_f(\mathbb{N} \setminus A_{f\varepsilon^+}) < \infty$.
(S2) There is $\varepsilon > 0$ such that $A_{f\varepsilon^+}$ is infinite, $\mu_f(\mathbb{N} \setminus A_{f\varepsilon^+}) = \infty$, and $\lim_{n \notin A_{f\varepsilon^+}} f(n) = 0$.
(S3) There is a decreasing sequence $\varepsilon_n$, for $n \in \mathbb{N}$, such that $\lim_n \varepsilon_n = 0$ and $A_{f\varepsilon_n^+} \setminus A_{f\varepsilon_{n+1}^+}$ is infinite for all $n$.
(S4) $\lim_{n\to\infty} f(n) = 0$.

In [40] the ideals in class ((S1)) were called atomic.

**Lemma 2.6.4.** *Suppose that $\mathcal{I}_f$ is a summable ideal.*

*(1) $\mathcal{I}_f \sim_{\mathrm{RK}} \mathrm{Fin}$ if and only if $\mathcal{I}_f$ belongs to the class (S1).*
*(2) $\mathcal{I}_f^\perp \sim_{\mathrm{RK}} \mathrm{Fin}$ if and only if $\mathcal{I}_f$ belongs to the class (S2).*
*(3) $\mathcal{I}_f^\perp \sim_{\mathrm{RK}} \mathrm{Fin} \times \emptyset$ if and only if $\mathcal{I}_f$ belongs to the class (S3).*
*(4) $\mathcal{I}_f$ is a dense ideal if and only if $\mathcal{I}_f$ belongs to the class (S4).*

*In particular, membership to one of the classes (S1)–(S4) does not depend on the choice of $f$, and every summable ideal belongs to exactly one of the classes defined in Definition 2.6.3.*

Proof. All four classes are nonempty: Fin belongs to (S1), $\mathrm{Fin} \oplus \mathcal{I}_{1/n}$ belongs to (S2), $\mathcal{I}_{1/n}$ belongs to (S4). Finally, if $f\colon \mathbb{N} \to \mathbb{R}_+$ is given by

$$f(2^n(2m-1)) = \frac{1}{m} \qquad \text{for } m, n \in \mathbb{N},$$

then $\mathcal{I}_f$ belongs to (S3). These classes are also clearly disjoint.

It remains to prove that each summable ideal $\mathcal{I}_f$ belongs to at least one of the classes (S1)–(S4). If $A_{f\varepsilon^+}$ is finite for all $\varepsilon > 0$ then $\lim f(n) = 0$ and $\mathcal{I}_f$ is dense. Otherwise, for some $\varepsilon > 0$ the set $A_{f\varepsilon^+}$ is infinite. If there is a small enough $\varepsilon > 0$ such that $\lim_{n \notin A_{f\varepsilon^+}} f(n) = 0$ then the ideal belongs either to (S1) or to (S2). Otherwise, we can recursively choose a sequence $\{\varepsilon_n\}$ as in (S3).

The 'in particular' part follows immediately. □

Proof of the following is given at the end of §2.6.2.

**Proposition 2.6.5.** *Let $\mathcal{I}_i$ be an ideal one the classes (S1)–(S4). Then the following holds.*

*(1) $\mathcal{I}_1 \leq_{\mathrm{RB}} \mathcal{I}_j$ and $\mathcal{I}_j \leq_{\mathrm{RB}} \mathcal{I}_4$ for all $j$.*
*(2) $\mathcal{I}_2 \leq_{\mathrm{RB}} \mathcal{I}_3$ and $\mathcal{I}_3 \leq_{\mathrm{RB}} \mathcal{I}_2$.*
*(3) $\mathcal{I}_j \not\leq_{\mathrm{RB}} \mathcal{I}_1$ for $j \geq 2$.*
*(4) $\mathcal{I}_4 \not\leq_{\mathrm{RB}} \mathcal{I}_j$ for $j \leq 3$.*

**2.6.2. RB-reductions between summable ideals.** The following proposition gives simple characterisation of $\leq_{\rm RB}$-comparability on dense summable ideals.

**Proposition 2.6.6.** *If $\mathcal{I}_f$ and $\mathcal{I}_g$ are dense summable ideals then the following two conditions are equivalent:*

*(1) For every $M \in \mathbb{N}$ there are $k_0, k_1$, and $k_2$ such that the following conditions hold.*
   *(A1) $\mu_g([k_1, k_2]) > M \cdot \mu_f(k_0)$.*
   *(A2) $g(k_2) > M \cdot f(k_0)$.*
   *(A3) $k_0 > M$ and $k_2 > k_1 > M$*

*(2) $\mathcal{I}_f \not\leq_{\rm RB} \mathcal{I}_g$.*

*In particular, the set $\{(f,g) : f, g \in [0,\infty)^{\mathbb{N}}, \mathcal{I}_f \leq_{\rm RB} \mathcal{I}_g\}$ is $F_{\sigma\delta}$.*

We will prove this proposition, as well as the following corollary, at the end of this subsection.

**Corollary 2.6.7.** *The ideals $\mathcal{I}_{1/n}$ and $\mathcal{I}_{1/\sqrt{n}} = \{A : \sum_{n\in A} 1/\sqrt{n+1} < \infty\}$ are not Rudin–Keisler isomorphic. The ideals $\mathcal{I} = \mathrm{Fin} \oplus \mathcal{I}_{1/n}$ and $\mathcal{J} = \mathrm{Fin} \oplus \mathcal{I}_{1/\sqrt{n}}$ satisfy $\mathcal{I} \leq_{\rm RB} \mathcal{J}$ and $\mathcal{J} \leq_{\rm RB} \mathcal{I}$, but not $\mathcal{I} \sim_{\rm RK} \mathcal{J}$.*

By Lemma 2.6.2 we may assume that for every summable ideal $\mathcal{I}_f$ we have $f(n) > 0$ for all $n$. Then the expression in the following lemma is well-defined.

**Lemma 2.6.8.** *Assume $\mathcal{I}_f$ and $\mathcal{I}_g$ are dense summable ideals. A finite-to-one function $h\colon \mathbb{N} \to \mathbb{N}$ is a reduction of $\mathcal{I}_g$ to $\mathcal{I}_f$ if and only if there are $A \in \mathcal{I}_f$ and $B \in \mathcal{I}_g$ such that*

$$\inf_{n\in\mathbb{N}\setminus A} \frac{\mu_g(h^{-1}(\{n\}) \setminus B}{f(n)} > 0, \tag{2.1}$$

$$\sup_{n\in\mathbb{N}\setminus A} \frac{\mu_g(h^{-1}(\{n\}) \setminus B}{f(n)} < \infty. \tag{2.2}$$

Proof. Suppose that $h$ satisfies (2.1) and (2.2) and let $c$ and $C$ denote the infimum and the supremum of the displayed expressions. By the additivity of $\mu_f$ and $\mu_g$, if $X \subseteq \mathbb{N}$ then $c \leq \frac{\mu_g(h^{-1}(X\cap A)\setminus B)}{\mu_f(X\setminus A) \leq \mu_g(h^{-1})} \leq C$. Thus $X \setminus A \in \mathrm{Fin}(\mu_f)$ if and only if $h^{-1}(X) \setminus B \in \mathrm{Fin}(\mu_g)$, as required.

For the direct implication, recall that $\mathcal{I}^*$ is the filter dual to $\mathcal{I}$ and suppose that $h$ is an RB-reduction. For $0 \leq c < C \leq \infty$ let

$$A[c, C] = \{n : c \leq \mu_g(h^{-1}(\{n\}))/\mu_f(\{n\}) \leq C\}. \tag{2.3}$$

We will find $0 < c < C < \infty$ such that $A[c, C] \in \mathcal{I}_f^*$ and $h^{-1}(A[c, C]) \in \mathcal{I}_g^*$.

We claim that $A[c, \infty] \in \mathcal{I}_f^*$ for some $c > 0$. Otherwise, some disjoint finite sets $p_m \subseteq \mathbb{N} \setminus A[1/m^2, \cdot)$ satisfy $1 \leq \mu_f(p_m) \leq 2$ for all $m$. The set $B = \bigcup_{m=1}^\infty p_m$ is not in $\mathcal{I}_f$, but $\mu_g(h^{-1}(B)) \leq 2\sum_{m=1}^\infty 1/m^2$ and therefore $h^{-1}(B)$ is in $\mathcal{I}_g$, contradicting the assumption on $h$. Fix such $c$.

We claim that $A[0, C] \in \mathcal{I}_f^*$ for some $C < \infty$. Otherwise, recursively choose disjoint finite sets $p_m \subseteq \mathbb{N} \setminus A[0, m]$ satisfying the following for all $m$.

(1) $1 \leq \mu_f(p_m) \leq 2$.
(2) $\min(p_m)$ is large enough to have $\max_{j\in p_m} f(j) < 1/m^2$.
(3) There is a partition $p_m = \bigsqcup_{i=1}^{m^2} p_m^i$ satisfying $\mu_f(p_m^i) \leq 2/m^2$ for all $i$.

The recursive construction of these objects is straightforward.

Since $\mu_g(h^{-1}(p_m)) \geq m$, some $i = i(m) \leq m^2$ satisfies $\mu_g(h^{-1}(p^i_m)) \geq 1/m$. Let $D = \bigcup_{m=1}^{\infty} p_m^{i(m)}$. Then $\mu_f(D) \leq \sum_{m=1}^{\infty} 2/m^2$ but $\mu_g(h^{-1}(D)) \geq \sum_{m=1}^{\infty} 1/m$, and therefore $D$ belongs to $\mathcal{I}_f$ but $h^{-1}(D)$ does not belong to $\mathcal{I}_g$, contradicting the assumption on $h$. We can therefore fix $C$ such that $A[0,C] \in \mathcal{I}^*_f$.

With the chosen $c$ and $C$ we have $A[c,C] \in \mathcal{I}^*_f$. We claim that $h^{-1}(A[c,C])$ belongs to $\mathcal{I}^*_g$. Otherwise, $\mathbb{N} \setminus A[c,C]$ is in $\mathcal{I}_f$ but $h^{-1}(\mathbb{N} \setminus A)$ is not in $\mathcal{I}_g$; contradiction. Since every $B \subseteq A[c,C]$ satisfies

$$c \leq \frac{\mu_g(h^{-1}(B))}{\mu_f(B)} \leq C,$$

we have that $B \mapsto h^{-1}(B)$ is an RB-reduction as required. □

**Lemma 2.6.9.** *Suppose that $\mathcal{I}_f$ and $\mathcal{I}_g$ are summable ideals such that $\mathcal{I}_g$ is dense. Then there is $A \in (\mathcal{I}_g)_+$ such that $\mathcal{I}_f \leq_{\mathrm{RB}} \mathcal{I}_g \restriction A$ and $\mathrm{Fin} \oplus \mathcal{I}_f \leq_{\mathrm{RB}} \mathcal{J} \oplus \mathcal{I}_g$ for every analytic ideal $\mathcal{J} \supseteq \mathrm{Fin}$.*

Proof. Find an increasing sequence

$$n_1^1 < n_1^2 < \cdots < n_1^{k(1)} < n_2^1 < \cdots < n_2^{k(2)} < n_3^1 < \dots$$

such that $|\mu_g(\{n_i^1, n_i^2, \dots, n_i^{k(i)}\}) - f(i)| < \frac{1}{2^{-i}}$ for all $i$. This is possible because $\lim_n g(n) = 0$. Let $A = \{n_i^j : i \in \mathbb{N},\, j \leq k(i)\}$ and define $h \colon A \to \mathbb{N}$ by $h(n_i^j) = i$. Then $h$ witnesses $\mathcal{I}_f \leq_{RB} \mathcal{I}_g \restriction A$.

For the second part, suppose that $\mathcal{J} \supseteq \mathrm{Fin}$ is an analytic ideal. Corollary 3.2.3 implies $\mathrm{Fin} \leq_{\mathrm{RB}} \mathcal{J}$, and $\mathrm{Fin} \oplus \mathcal{I}_f \leq_{\mathrm{RB}} \mathcal{J} \oplus \mathcal{I}_g$ follows. □

Proof of Corollary 2.6.7. In order to prove that $\mathcal{I}_{1/n}$ and $\mathcal{I}_{1/\sqrt{n}}$ are not RK-isomorphic, it suffices to prove there are no $A, B$ and a bijection $h \colon A \to B$ such that $\mathbb{N} \setminus A \in \mathcal{I}_{1/\sqrt{n}}$, $\mathbb{N} \setminus B \in \mathcal{I}_{1/n}$, and $C \in \mathcal{I}_{1/\sqrt{n}}$ if and only if $h[C] \in \mathcal{I}_{1/n}$. Assume otherwise. By Lemma 2.6.8, we may assume that there are $0 < p \leq q < \infty$ such that

$$p \leq \frac{h(n)}{\sqrt{n}} \leq q$$

for all $n \in A$. This implies that $h(n) \leq q\sqrt{n}$ for all $n \in A$. Since $h$ is a bijection, $\mathbb{N} \setminus A$ does not belong to $\mathcal{I}_{1/\sqrt{n}}$; contradiction.

It remains to prove that $\mathcal{I} = \mathrm{Fin} \oplus \mathcal{I}_{1/n}$ and $\mathcal{J} = \mathrm{Fin} \oplus \mathcal{I}_{1/\sqrt{n}}$ violate the Schröder–Bernstein property for $\leq_{\mathrm{RB}}$. Because Fin is not RK-isomorphic to a dense ideal, since $\mathcal{I}_{1/n}$ and $\mathcal{I}_{1/\sqrt{n}}$ are dense and not RK-isomorphic, then the ideals $\mathcal{I}$ and $\mathcal{J}$ are not RK-isomorphic either. On the other hand, Lemma 2.6.9 implies $\mathcal{I} \leq_{\mathrm{RB}} \mathcal{J}$ and $\mathcal{J} \leq_{\mathrm{RB}} \mathcal{I}$. □

Proof of Proposition 2.6.6. Suppose that $\mathcal{I}_f$ and $\mathcal{I}_g$ are summable ideals. By Lemma 2.6.2 we may assume that both $f$ and $g$ are monotonic.

1 ⇒ 2 Towards contradiction, suppose that for every $M \in \mathbb{N}$ there are $k_0, k_1$, and $k_2$ such that conditions ((A1))–((A3)) hold but $\mathcal{I}_f \leq_{\mathrm{RB}} \mathcal{I}_g$. Since $\mathcal{I}_f \leq_{\mathrm{RB}} \mathcal{I}_g$, let $h \colon \mathbb{N} \to \mathbb{N}$ and $0 < c < C < \infty$ be as guaranteed by Lemma 2.6.8. Let

$$B = \{n : c > \mu_g(h^{-1}(\{n\}))/\mu_f(\{n\}) \text{ or } \mu_g(h^{-1}(\{n\}))/\mu_f(\{n\}) > C\}$$

and fix $M \geq C+1$ large enough to have $\mu_g(B \setminus M) < 1$ and $\mu_f(M) > 1$. Fix $k_0, k_1, k_2$ that satisfy (A1)–(A3) and let

$$t = [k_1, k_2] \setminus h^{-1}(k_0).$$

Then $\mu_g(t) \geq \mu_g([k_1, k_2]) - C\mu_f(k_0) \geq (M-C)\mu_f(k_0) > 1$. If $l \in t \setminus B$ is such that $k = h(l) \geq k_0$, then ((A2)) implies

$$C \geq \frac{\mu_g(h^{-1}(\{m\}))}{f(k)} \geq \frac{g(l)}{f(k)} \geq \frac{g(k_2)}{f(k_0)} > M,$$

contradiction. Therefore $h^{-1}(k)$ is disjoint from $t$ for all $k \geq k_0$, and $t$ is included in $B$. This contradicts $\mu_g(B \setminus M) < 1$, $\mu_g(t) > 1$, and $\min(t) \geq M$.

2 $\Rightarrow$ 1 Assume that 1 fails. Then for some $M$ and all $k_0 > M$ and $k_2 > k_1 > M$ we have that

$$\mu_g([k_1, k_2]) > M \cdot \mu_f(k_0) \text{ implies } g(k_2) \leq M \cdot f(k_0). \tag{2.4}$$

Choose $N_0, N_1 > M$ such that

$$\mu_g([M+1, N_0]) > M \cdot \mu_f(N_1).$$

Recursively choose a sequence $k_1 \leq k_2 \leq k_3 \leq \dots$ such that $k_1 = N_0$ and $k_{i+1}$ is the minimal integer which satisfies

$$\mu_g([k_i, k_{i+1})) > Mf(N_1 + i) \tag{2.5}$$

for every $i$. Then

$$\mu_g([M+1, k_{i+1})) > M\mu_f(N_1) + M\textstyle\sum_{j<i} f(N_1 + j) = M\mu_f(N_1 + i)$$

and therefore (2.4) implies $g(k_{i+1} - 1) \leq Mf(N_1 + i)$. Therefore

$$\mu_g([k_i, k_{i+1})) = \mu_g([k_i, k_{i+1} - 1)) + g(k_{i+1} - 1) \leq 2Mf(N_1 + i). \tag{2.6}$$

Let $h\colon [N_0, \infty) \to [N_1 + 1, \infty)$ be a map which collapses the interval $[k_i, k_{i+1})$ to $N_1 + i$. As both the domain and the range of $h$ are cofinite, (2.5) and (2.6) hold, $h$ satisfies the assumptions of Lemma 2.6.8 with $c = M$ and $C = 2M$. It therefore witnesses $\mathcal{I}_g \leq_{\mathrm{RB}} \mathcal{I}_f$. □

Proof of Proposition 2.6.5. Suppose that $\mathcal{I}_j$, for $1 \leq j \leq n$, belongs to the corresponding class (S1)–(S4) from Definition 2.6.3.

(1) Clearly $\mathcal{I}_1$ is RB-equivalent to Fin.

(2) By Corollary 3.2.3 we have Fin $\leq_{\mathrm{RB}} \mathcal{J}$ for every analytic ideal $\mathcal{J}$ that includes Fin, in particular $\mathcal{I}_1 \leq_{\mathrm{RB}} \mathcal{I}_j$ for $1 \leq j \leq 4$. If $i \geq 2$ then $\mathcal{I}_i = \mathcal{J} \oplus \mathcal{I}_f$ for a dense summable ideal $\mathcal{I}_f$ and $\mathcal{I}_j \oplus \mathrm{Fin} \leq_{\mathrm{RB}} \mathcal{I}_i$ by Lemma 2.6.9.

(3) Assume $h\colon \mathbb{N} \to \mathbb{N}$ is finite-to-one and such that $A \in \mathcal{I}$ if and only if $h^{-1}(A) \in \mathrm{Fin}$. This implies $\mathcal{I} = \mathrm{Fin}$, and therefore $\mathcal{I}$ is not equal to $\mathcal{I}_i$ for $i \geq 2$.

(4) Assume $h\colon \mathbb{N} \to \mathbb{N}$ is an RB-reduction of $\mathcal{I}_4 \leq_{\mathrm{RB}} \mathcal{I}_j$ and $j \leq 3$. Then $\mathcal{I}_j \cong \mathrm{Fin} \oplus \mathcal{I}$ for a summable ideal $\mathcal{I}$. If $\mathbb{N} = X \sqcup Y$ is the partition of $\mathbb{N}$ such that Fin lives on $X$ and $\mathcal{I}$ lives on $Y$, then as in (3) the ideal $\mathcal{I}_4 \restriction h[X]$ is isomorphic to Fin. However $\mathcal{I}_4$ is a dense ideal; contradiction. □

**2.6.3. The quasi-order $\leq_{\mathrm{RB}}$ of summable ideals.** The following theorem, together with our main lifting results (Theorem 4.1.2 and Theorem 7.1.1) has strong consequence to the rigidity of quotients of summable ideals.

**Theorem 2.6.10.** *The quasi-ordered set of all dense summable ideals with respect to $\leq_{\mathrm{RB}}$ has the following properties.*

*(1) It has no maximal elements.*
*(2) It has no minimal elements.*
*(3) It includes an isomorphic copy of $\langle \mathcal{P}(\mathbb{N})/\operatorname{Fin}, \subseteq^* \rangle$.*
*(4) If $\mathcal{I}_f <_{\mathrm{RB}} \mathcal{I}_g$ then some $h$ satisfies $\mathcal{I}_f <_{\mathrm{RB}} \mathcal{I}_h <_{\mathrm{RB}} \mathcal{I}_g$.*

A moment of reflection shows that (4) is not a formal consequence of (3). A proof of Theorem 2.6.10 is given after a few lemmas.

If $\mathcal{I}_f$ is a summable ideal, consider the sequence of partial sums

$$a_k^f = \textstyle\sum_{i \leq k} f(i). \tag{2.7}$$

We write $a_k$ when $f$ is clear from the context.

**Lemma 2.6.11.** *If $\mathcal{I}_f$ and $\mathcal{I}_g$ are summable ideals, then the following holds.*

*(1) $\mathcal{I}_f = \{A \subseteq \mathbb{N} :$ the set $\bigcup_{k \in A} [a_{k-1}^f, a_k^f)$ has finite Lebesgue measure$\}$.*
*(2) Every unbounded increasing sequence $\{a_k\}$ in $\mathbb{R}_+$ determines function $f = f_{\bar{a}}$ by $f(k) = a_k - a_{k-1}$ and a proper summable ideal $\mathcal{I}_f$.*
*(3) If $f$ and $g$ are such that $\{a_k^g\}_{k \geq m}$ is a subsequence of $\{a_k^f\}$ for some positive $m > 0$, then $\mathcal{I}_g \leq_{\mathrm{RB}} \mathcal{I}_f$.*

Proof. Only (3) requires a proof. Since the ideals corresponding to $\{a_k^g\}_{k \geq m}$ and $\{a_k^g\}_{k \geq 1}$ are isomorphic, we can assume $m = 1$. Let $n(k)$ be an increasing sequence such that $a_k^g = a_{n(k)}^f$ for all $k$. Define $h \colon \mathbb{N} \to \mathbb{N}$ by (let $n(0) = 0$)

$$h^{-1}(k) = [n(k-1), n(k)).$$

Then $\mu_f(h^{-1}(k)) = g(k) = a_k^g - a_{k-1}^g$ (where $a_0^g = 0$), so $h$ is as required. □

**Lemma 2.6.12.** *Assume $f, g$ are nonincreasing functions from $\mathbb{N}$ into $(0, \infty)$ such that $\{a_k^g\}$ is a subsequence of $\{a_k^f\}$. Let $\{n(k)\}$ be the increasing sequence determined by $a_k^g = a_{n(k)}^f$. Suppose that for arbitrarily large positive integer $N$ there are $N < k < k'$ such that:*

*(B1) $a_{k'}^g = a_{n(k')}^f > N \cdot a_{n(k)}^f$, and*
*(B2) $g(k') > N \cdot f(n(k))$.*

*Then $\mathcal{I}_g \leq_{\mathrm{RB}} \mathcal{I}_f$ and $\mathcal{I}_f \not\leq_{\mathrm{RB}} \mathcal{I}_g$.*

Proof. Lemma 2.6.11 (3) implies $\mathcal{I}_g \leq_{\mathrm{RB}} \mathcal{I}_f$. To prove $\mathcal{I}_f \not\leq_{\mathrm{RB}} \mathcal{I}_g$, we have to verify 1 of Proposition 2.6.6. For $M < \infty$ pick $N > 2M$ such that $a_N^g > 2a_M^g$. If $N < k < k'$ are such that (B1) and (B2) are satisfied, then let $k_0 = n(k)$, $k_1 = M+1$ and $k_2 = k'$. We verify the conditions of Proposition 2.6.6 (A1). Condition (A1) holds because

$$\mu_g([k_1, k_2)) = \mu_g(k_2) - \mu_g(k_1) > \mu_g(k_2)/2 = a_{k_2}^g/2 > N \cdot a_{n(k)}^f/2 > M \cdot \mu_f(k_0).$$

For (A2), $g(k_2) = g(k') > N \cdot f(n(k)) > M \cdot f(k_0)$, and (A3) is obvious. Therefore, Proposition 2.6.6 implies $\mathcal{I}_f \not\leq_{\mathrm{RB}} \mathcal{I}_g$. □

Proof of Theorem 2.6.10. (1) To prove that there are no minimal elements in the structure of dense summable ideals with respect to the quasi-order $\leq_{\mathrm{RB}}$, fix a dense summable ideal $\mathcal{I}_f$. Since $\lim_n f(n) = 0$, we can assume $f$ is monotonic, possibly by composing it with a suitable permutation of the integers. By Lemma 2.6.12 it suffices to construct a subsequence $\{a_i^g\}$ of $\{a_i^f\}$ which satisfies (B1) and (B2) and such that $a_i^g - a_{i-1}^g$ nonincreasingly converges to zero. We recursively find increasing sequences of positive integers $n(i)$ and $k(i)$ such that the sequence $a_i^g = a_{n(i)}^f$ satisfies (B1) and (B2) for $N = j$ with $k = k(j)$ and $k' = k(j+1)$ for all $j \in \mathbb{N}$. We will also arrange the following holds for all $i$.

$$a_{k(i)+1}^f - a_{k(i)}^f < 1/i^3, \tag{2.8}$$

$$a_{k(i+1)}^f > 2i \cdot a_{k(i)}^f. \tag{2.9}$$

The recursive construction of $\{k(i)\}$ is as follows: If $k(1), k(2), \dots, k(i)$ are chosen, pick $k(i+1)$ large enough so that (2.9) holds and

$$f(k(i+1)) < \frac{1}{(i+1)^3}.$$

A sequence $\{k(i)\}$ constructed in this manner satisfies (2.8) and (2.9). For every $i$ find integers $l_i$ and $k(i) = n(i,1) < n(i,2) < \cdots < n(i,l_i) = k(i+1)$ such that

$$\frac{1}{i^2} < a_{n(i,j+1))}^f - a_{n(i,j)}^f < \frac{2}{i^2} \tag{2.10}$$

for all $j = 1, \dots, l-1$. Note that (2.8) implies this is possible. Let $n(k)$ be the increasing enumeration of $\{n(i,j) : i \in \mathbb{N},\ j \leq l_i\}$ and let $g$ be defined by sequence $a_k^g = a_{n(k)}^f$. Then (2.10) implies that $a_{k+1}^g - a_k^g$ converges to zero, and by the construction this sequence is monotonic. Given $N > 0$, let $k = k(N)$, $k' = k(N+1)$; then ((B1)) follows from (2.9) and ((B2)) is satisfied because (2.10) and (2.9) together imply

$$a_k^g - a_{k-1}^g > \frac{1}{N^2} = N\frac{1}{N^3} > a_{n(k)+1}^f - a_{n(k)}^f,$$

therefore Lemma 2.6.12 implies the desired conclusion.

(2) To prove that there are no maximal elements in the structure of dense summable ideals with respect to the preorder $\leq_{\mathrm{RB}}$, fix a dense summable ideal $\mathcal{I}_g$. Since $\lim_n g(n) = 0$, we can assume $g$ is monotonic, possibly by composing it with a suitable permutation of the integers. We will find a sequence $\{a_i^f\}$ including $\{a_i^g\}$ so that $f$ is decreasing and Lemma 2.6.12 applies to prove $\mathcal{I}_g$ is strictly below $\mathcal{I}_f$. First pick an increasing sequence of positive integers $\{k(i)\}$ so that

$$a_{k(i+1)}^g > 2i \cdot a_{k(i)}^g \tag{2.11}$$

for all $i$. Define $a_n^f$ recursively: assume $a_1^f, \dots, a_{m(i)}^f$ are defined so that $a_{m(i)}^f = a_{k(i)}^g$. Then for $j \in [k(i), k(i+1))$ partition interval $[a_j^g, a_{j+1}^g)$ into the pieces of equal length less than both

$$\frac{1}{i}(a_{k(i+1)}^g - a_{k(i+1)-1}^g) \ \ \& \ \ \frac{1}{2}(a_{m(i)+1}^f - a_{m(i)-1}^f). \tag{2.12}$$

Let $a_{m(i)+1}^f, \dots, a_{m(i+1)}^f$ be an increasing enumeration of endpoints of these intervals; this describes the construction.

Then function $f = f\{a^f_n\}$ is nonincreasing and $\lim_n(a^f_{n+1} - a^f_n) = 0$. To see that the conditions of Lemma 2.6.12 are satisfied, fix $N > 0$ and consider $k = k(N)$, $k' = k(N+1)$. Then (2.11) implies (B1) and (2.12) reads as

$$\frac{1}{N}(a^g_k - a^g_{k-1}) > a^f_{n(k)+1} - a^f_{n(k)}$$

which is equivalent to (B2). An application of Lemma 2.6.12 ends the proof.

(3) For $A \subseteq \mathbb{N}$ define $f_A \colon \mathbb{N} \to \mathbb{R}_+$ as follows: Let $0 = n^A_0 < n^A_1 < n^A_2 < \dots$ be a sequence of integers recursively defined by

$$n^A_{k+1} - n^A_k = \begin{cases} ((2k)!)^2, & k \notin A, \\ 2k((2k)!)^2, & k \in A, \end{cases}$$

and for $i \in \mathbb{N}$ let $k^A(i)$ be the unique $k$ such that $i \in [n^A_k, n^A_{k+1})$. Let

$$f^A(i) = \begin{cases} 1/(2k)!, & k^A(i) \notin A \\ 1/(2k(2k)!), & k^A(i) \in A. \end{cases}$$

In particular we have $\mu_{f^A}([n^A_k, n^A_{k+1})) = (2k)!$. If $A, B$ are such that $A \subseteq^* B$, then $\mathcal{I}_{f^A} \leq_{\mathrm{RB}} \mathcal{I}_{f^B}$ because the sequence $\{a^{f^A}_i\}$ is almost included in the sequence $\{a^{f^B}_i\}$, hence Lemma 2.6.12 applies. If $k \in B \setminus A$ is large enough, then $k_1 = n^A_k$, $k_2 = n^A_{k+1}$ and $k_0 = n^B_k$ satisfy (A1)–(A3) of Proposition 2.6.6 for $g = f^A$, $f = f^B$ and $M = k$, as follows. For the first condition,

$$\mu_f([n^A_k, n^A_{k+1})) = (2k)! > k^2(2k-2)! > k\sum_{i=0}^{k-1}(2i)! + 1 = k\sum_{i=1}^{n^B_k} f^B(i) + 1.$$

For the second, $f^A(n^A_{k+1}) = \dfrac{1}{(2k)!} > k\dfrac{1}{2k(2k)!} = f^B(n^B_k)$, and the third is obvious. Therefore, if the set $B \setminus A$ is infinite then Proposition 2.6.6 implies $\mathcal{P}(\mathbb{N})/\mathcal{I}_{f^A} \not\leq_{\mathrm{BE}} \mathcal{P}(\mathbb{N})/\mathcal{I}_{f^B}$, and $A \mapsto \mathcal{I}_{f^A}$ is an embedding of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ into the class of dense summable ideals ordered by $\leq_{\mathrm{RB}}$, or equivalently, by $\leq_{\mathrm{BE}}$.

(4) We have to prove that if $\mathcal{I}_f \leq_{\mathrm{RB}} \mathcal{I}_g$ and not vice versa, then there is an $f'$ such that $\mathcal{I}_f \leq_{\mathrm{RB}} \mathcal{I}_{f'} \leq_{\mathrm{RB}} \mathcal{I}_g$ and both relations are irreversible. Fix a finite-to-one mapping $h \colon \mathbb{N} \to \mathbb{N}$ witnessing $\mathcal{I}_f \leq_{\mathrm{RB}} \mathcal{I}_g$. Define $f' \colon \mathbb{N} \to \mathbb{R}_+$ by

$$f'(n) = \mu_g(h^{-1}(n)).$$

Then $\mathcal{I}_{f'} = \mathcal{I}_f$, because $h$ is a Rudin–Blass reduction and $\mu_{f'}(A) = \mu_g(h^{-1}(A))$ for all $A$. Without a loss of generality we may assume that $h^{-1}(k) = [n_k, n_{k+1})$ for some sequence

$$0 = n_0 < n_1 < n_2 < \dots.$$

For $B \subseteq \mathbb{N}$ let $f^B$ be a mapping defined by

$$\operatorname{dom}(f^B) = (\mathbb{N} \setminus B) \times \{0\} \cup \bigcup_{k \in B} [n_k, n_{k+1}) \times \{1\}$$

$$f^B(i,j) = \begin{cases} f'(i), & i \notin B \quad \text{and } j = 0, \\ g(i), & i \in \bigcup_{k\in B}[n_k, n_{k+1}) \quad \text{and } j = 0. \end{cases}$$

Let $\mathcal{I}_{f^B}$ be the summable ideal on the index-set $\operatorname{dom}(f^B)$ determined by $f^B$. Then

$$\mathcal{I}_f \leq_{\mathrm{RB}} \mathcal{I}_{f^A} \leq_{\mathrm{RB}} \mathcal{I}_{f^B} \leq_{\mathrm{RB}} \mathcal{I}_g, \qquad \text{for } A \subseteq B \subseteq \mathbb{N}.$$

Let $\mathcal{J} = \{B : \mathcal{I}_{f^B} \leq_{\mathrm{RB}} \mathcal{I}_f\}$ and $\mathcal{F} = \{B : \mathcal{I}_g \leq_{\mathrm{RB}} \mathcal{I}_{f^B}\}$. Then $\mathcal{J}$ and $\mathcal{F}$ are analytic sets (the characterisation of orders $\leq_{\mathrm{RB}}$ and $\leq_{\mathrm{BE}}$ from Proposition 2.6.6 is Borel, and in fact $F_\sigma$) which are downwards (respectively. upwards) closed, closed under finite changes, and not equal to $\mathcal{P}(\mathbb{N})$. Therefore these two sets must be meagre (by [87] or [149]; see also Theorem 3.2.2 below) hence there is a set $B \in \mathcal{P}(\mathbb{N}) \setminus (\mathcal{F} \cup \mathcal{J})$. Then $\mathcal{I}_{f^B}$ is the required summable ideal. □

**Corollary 2.6.13.** *There are summable ideals $\mathcal{I}_f$ and $\mathcal{I}_g$ such that $\mathcal{I}_f \not\leq_{\mathrm{RB}} \mathcal{I}_g$ but $\mathcal{I}_f \leq_{\mathrm{RB}} \mathcal{I}_g \restriction A$ for some $\mathcal{I}_g$-positive set $A$.*

Proof. Theorem 2.6.10 guarantees that there are dense summable ideals $\mathcal{I}_f$ and $\mathcal{I}_g$ such that $\mathcal{I}_f \not\leq_{\mathrm{RB}} \mathcal{I}_g$. Lemma 2.6.9 implies that $\mathcal{I}_f \leq_{\mathrm{RB}} \mathcal{I}_g \restriction A$ for some $\mathcal{I}_g$-positive set $A$. □

## 2.7. Density ideals

In §1.7.2 density ideals were defined as the ideals of the form $\mathrm{Exh}(\varphi_\mu)$ where $\varphi_\mu = \sup_i \mu_i$ and $\mu_i$ are measures concentrating on disjoint finite sets $I_i$.

Lemma 2.7.2 gives a rough classification of density ideals in four classes. In Lemma 2.7.5 we prove that a direct sum of density ideals is isomorphic to a density ideal. In Theorem 2.7.8 we characterise density ideals that are RK-isomorphic to an EU-ideal. By Theorem 2.7.12, every nonpathological density ideal is RB-reducible to every dense density ideal. In Theorem 2.7.16 we give a criterion for when two density ideals are not RK-isomorphic, used to prove that $\mathcal{Z}_0$ and $\mathcal{Z}_{\log}$ are not isomorphic (Corollary 2.7.17).

### 2.7.1. Four classes of density ideals.

Recall that for a submeasure $\varphi$ we write $\mathrm{at}^+(\varphi) = \sup_{k \in \mathrm{supp}(\varphi)} \varphi(\{k\})$ and $\mathrm{at}^-(\varphi) = \inf_{k \in \mathrm{supp}(\varphi)} \varphi(\{k\})$.

**Definition 2.7.1.** Consider the following classes of density ideals.

($\mathcal{Z}1$) Ideals RK-isomorphic to Fin.[2]
($\mathcal{Z}2$) Density ideals $\mathcal{Z}_\mu$ that are neither RK-isomorphic to Fin nor dense.
($\mathcal{Z}3$) Density ideals RK-isomorphic to some EU-ideal.
($\mathcal{Z}4$) Dense density ideals not RK-isomorphic to a EU-ideal.

**Lemma 2.7.2.** *Suppose that $\mathcal{Z}_\mu$ is a density ideal.*

*(1) $\mathcal{Z}_\mu$ is RK-isomorphic to* Fin *if and only if* $\inf_n \mathrm{at}^-(\mu_n) > 0$
*(2) $\mathcal{Z}_\mu$ is neither RK-isomorphic to* Fin *nor dense if and only if*

$$\inf_i \mathrm{at}^-(\mu_i) = 0 \text{ and } \limsup_i \mathrm{at}^+(\mu_i) > 0.$$

*(3) $\mathcal{Z}_\mu$ is RK-isomorphic to an EU-ideal if and only if*

$$\lim_i \mathrm{at}^+(\mu_i) = 0 \text{ and } \sup_i \|\mu_i\| < \infty.$$

*(4) $\mathcal{Z}_\mu$ is dense and not RK-isomorphic to an EU-ideal if and only if*

$$\lim_i \mathrm{at}^+(\mu_i) = 0 \text{ and } \sup_i \|\mu_i\| = \infty.$$

*Moreover, each of these classes is nonempty and every density ideal belongs to exactly one of these classes.*

The proof of Lemma 2.7.2 is given after Theorem 2.7.8, at the end of §2.7.2.

The following is slightly less obvious than the analogous fact for summable ideals (Lemma 2.6.1).

[2] In [40] such ideals were called atomic.

**Lemma 2.7.3.** *Each of the classes of density ideals, generalised density ideals, dense density ideals, and dense generalised densty ideals is closed under RK-isomorphisms and taking restrictions to positive sets.*

Proof. Closure under RK-isomorphisms is obvious from the definitions.

To prove closure under restrictions to positive sets, fix finite sets $I_n$ and submesures $\varphi_n$ on $I_n$ and assume that $A \in (\mathcal{Z}_\varphi)_+$. Then with $J_n = A \cap I_n$ and $\psi_n(X) = \varphi_n(X \cap A)$ we have that $\mathcal{Z}_\psi$ is isomorphic to $\mathcal{Z}_\varphi \restriction A$, hence a generalised density ideal. If all $\varphi_n$ are measures, then so are all $\psi_n$, and $\mathcal{Z}_\varphi \restriction A$ is a density ideal. The restriction of a dense ideal to a positive set is clearly dense. □

It is not true that every ideal RK-reducible to an EU-ideal is an EU-ideal, or that every ideal RK-reducible to a density ideal is a density ideal. This is because Theorem 2.7.12 implies that every nonpathological generalised density ideal is RK-reducible to every EU-ideal. One can however add the adjective 'nonpathological' to Lemma 2.7.3 at various places so that it remains true.

Example 1.7.4 and the following lemma were extracted from a proof in [40].

**Lemma 2.7.4.** *Every proper density ideal $\mathcal{Z}_\mu$ is isomorphic to a density ideal $\mathcal{Z}_\nu$ such that $\|\nu_n\| \geq 1$ for all $n$. Moreover, the following conditions hold.*

(1) $\sup_n \|\nu_n\| = \infty$ *if and only if* $\sup_m \|\mu_m\| = \infty$.

(2) $\limsup_n \max_j \nu_n(\{j\}) = 0$ *if and only if* $\limsup_n \max_j \mu_n(\{j\}) = 0$.

*If* $\sup_n \|\nu_n\| < \infty$ *then* $\mathcal{Z}_\nu$ *is isomorphic to a normalised density ideal.*

Proof. Write $I_n = \mathrm{supp}(\mu_n)$. Suppose for a moment that there is $\varepsilon > 0$ such that $X = \bigcup\{I_n : \mu_n(I_n) < \varepsilon\}$ belongs to $\mathcal{Z}_\mu$. Then (by replacing $\mathcal{Z}_\mu$ with an isomorphic ideal) we may assume $\|\mu_n\| \geq \varepsilon$ for all $n$. With $\nu_n = \varepsilon^{-1}\mu_n$ we have $\mathcal{Z}_\nu = \mathcal{Z}_\mu$ as required.

We may therefore assume that that there is a decreasing sequence $\varepsilon_n$ such that $\lim_n \varepsilon_n = 0$ and each of the sets

$$Y_m = \{n : \varepsilon_m \leq \|\mu_n\| < \varepsilon_{m-1}\}$$

for $m \geq 1$ is infinite. By passing to a subsequence we may assume $\sum_m \varepsilon_m < \infty$. Since replacing $\mu_n$ with $\varepsilon_1^{-1}\mu_n$ for all $n \in \mathbb{N}$ does not change the ideal, we may assume $\|\mu_n\| \geq 1$ for all $n \in Y_1$.

Since the submeasures $\mu_n$ for $n$ in the set $Y_0 = \{n : \|\mu_n\| \geq \varepsilon_0\}$ will remain unchanged, we will ignore this set, but only after pointing out that after replacing $\mathbb{N}$ with $\mathbb{N} \setminus \bigcup\{I_n : n \in Y_0\}$ we have $\sup_n \|\mu_n\| < \infty$.

Let $A_m = \bigcup_{n \in Y_m} I_n$.

We claim that $B \in \mathcal{Z}_\mu$ if and only if $B \cap A_m \in \mathcal{Z}_\mu$ for all $m$. The direct implication is trivial. For the converse, assume $B \notin \mathcal{Z}_\mu$. Fix $m$ such that $\liminf_n \mu_n(B \cap I_n) \geq \varepsilon_m$. Since $\mu_n(B \cap I_n) > \varepsilon_m$ implies $n \in \bigcup_{j \leq m} A_j$, we have that $B \cap \bigcup_{j \leq m} A_j$ is not in $\mathcal{Z}_\mu$, and therefore $B \cap A_j \notin \mathcal{Z}_\mu$ for some $j \leq m$. This proves the claim.

Fix an enumeration $Y_m = \{k(m,n) : n \in \mathbb{N}\}$. Let, for $n \in \mathbb{N}$,

$$J_n = \bigcup_{m \leq n} I_{k(m,n)},$$
$$\nu_n(B) = \sum_{m \leq n} \mu_{k(m,n)}(B).$$

Then $\mathrm{supp}(\nu_n) = J_n$ and $\varepsilon_1 = 1 \leq \|\nu_n\| < \sum_m \varepsilon_m < \infty$ for all $n$.

Note that for every $n$ we have $\max_j \nu_n(\{j\}) = \max_{m\leq n}\max_j \mu_{k(m,n)}(\{j\})$. This implies $\limsup_n \max_j \nu_n(\{j\}) = 0$ if and only if $\limsup_n \max_j \mu_n(\{j\}) = 0$.

It remains to prove $\mathcal{Z}_\mu = \mathcal{Z}_\nu$. Since by construction for every $B \subseteq \mathbb{N}$ we have $\sup_n \mu_n(X) \leq \sup_n \nu_n(B)$, we have $\mathcal{Z}_\mu \supseteq \mathcal{Z}_\nu$.

For the converse, fix $B \notin \mathcal{Z}_\nu$ and fix $\varepsilon > 0$ such that $\limsup_n \nu_n(B) > \delta$. Fix $l$ such that $\sum_{m>l} \varepsilon_{m-1} < \delta$ and let $\varepsilon' = \delta - \sum_{m>l}\varepsilon_{m-1}$. For each $n$ let $J'_n = \bigcup_{m\leq l} I_m$. Then $\nu_n(J_n \setminus J'_n) \leq \sum_{m>l}\|\nu_m\| \leq \sum_{m>l}\varepsilon_{m-1} < \delta$. Therefore $\nu_n(B \cap J'_n) > \varepsilon'$ for infinitely many $n$. This implies that there is $m \leq l$ such that $\varepsilon'/l < \nu_n(B \cap I_{k(m,n)}) = \mu_{k(m,n)}(B)$ for infinitely many $n$, and therefore $\limsup_m \mu_m(B) > 0$ and $B \notin \mathcal{Z}_\mu$ as required.

Since $\inf_n \|\mu_n\| > 0$, if $\sup_n \|\mu_n\| < \infty$ then replacing $\mu_n$ with $\|\mu_n\|^{-1}\mu_n$ does not change the ideal. The resulting ideal is clearly normalised. □

The following closure property of normalised density ideals will be used in the proof of Theorem 2.7.8; compare it with (3) of Corollary 1.4.8.

**Lemma 2.7.5.** *Suppose that $\mathcal{J}_n$, for $n \in \mathbb{N}$, are normalised density ideals. Then the ideal*

$$\mathcal{I} = \{A \subseteq \mathbb{N}^2 : (\forall m)\{n : (m,n) \in A\} \in \mathcal{J}_n\}$$

*is isomorphic to a normalised density ideal $\mathcal{Z}_\mu$ such that $\|\mu_n\| = 1$ for all $n$.*

Proof. For each let $\mu^n_j$ for $j \in \mathbb{N}$ be a normalised sequence of measures concentrating on disjoint sets such that $\mathcal{J}_n = \mathcal{Z}_{\mu^n}$. Let $\nu_{2^n(2j+1)} = 2^{-n}\mu^n_j$, so that $\|\nu_{2^n(2j+1)}\| = 2^{-n}$. We claim that $\mathcal{I} = \mathcal{Z}_\nu$. Some $A \subseteq \mathbb{N}$ is $\mathcal{Z}_\nu$-positive if and only if there is $\varepsilon > 0$ such that $\nu_k(A) \geq \varepsilon$. This is equivalent to the existence of $n \leq 1/\varepsilon$ such that $\nu_{2^n(2j+1)}(A) \geq \varepsilon$ for infinitely many $j$, which is equivalent to $A$ being $\mathcal{I}$-positive. By Lemma 2.7.4, $\mathcal{I}$ is a normalised density ideal. □

The analogous fact for generalised density ideals has analogous proof.

**Lemma 2.7.6.** *Suppose that $\mathcal{J}_n$, for $n \in \mathbb{N}$, are generalised density ideals. Then the ideal $\mathcal{I} = \{A \subseteq \mathbb{N}^2 : (\forall m)\{n : (m,n) \in A\} \in \mathcal{J}_n\}$ is isomorphic to a generalised density ideal.* □

**Lemma 2.7.7.** *If $\mathcal{Z}_\nu$ is a normalised dense density ideal and $\mathcal{Z}_\mu$ is a density ideal such that $\sup_n \|\mu_n\| = \infty$, then $\mathcal{Z}_\mu$ and $\mathcal{Z}_\nu$ are not isomorphic.*

Proof. Assume the contrary and let

$$S_i = \{j : \operatorname{supp}(\nu_j) \cap \operatorname{supp}(\mu_i) \neq \emptyset\}.$$

We claim that for every $m \geq 1$, there are infinitely many $i$ for which some $t_i \subseteq \operatorname{supp}(\mu_i)$ satisfies $\mu_i(t_i) \geq 1$ but $\sup_j \nu_j(t_i) \leq 1/m$.

We will first prove that for infinitely many $i$ and all $j \in S_i$ there are $a_j \subset \operatorname{supp}(\nu_j) \cap \operatorname{supp}(\mu_i)$ such that $\nu_j(a_j) < 1/m$ and $\mu_i(\bigcup_{j\in S_i} a_j) \geq 1$. Take $\delta = 1/(m+1)^2$, so that $(m+1)(1/m - \delta) > 1$. Consider $i$ large enough to have $\mu_i(\operatorname{supp}(\mu_i)) \geq m+1$ and $\sup_{j\in S_i} \operatorname{at}^+(\nu_j) < \delta$. For $j \in S_i$, partition $\operatorname{supp}(\nu_j) \cap \operatorname{supp}(\mu_i)$ into pieces of measure $< 1/m$, each so that all but (at most) one of the pieces have measure in the interval $[1/m - \delta, 1/m)$.) Let $a_j$ be the piece of the largest $\mu_i$ measure. Then

$$\mu_i(a_j) \geq \frac{1}{m+1}\mu_i(\operatorname{supp}(\nu_j) \cap \operatorname{supp}(\mu_i)),$$

and therefore $\mu_i(t_i) \geq \mu_i(\operatorname{supp}(\mu_i))/(m+1) \geq 1$.

We can now choose an increasing sequence of $i(1) < i(2) < \dots$ such that the sets $S_{i(m)}$ $(m = 1, 2, \dots)$ are pairwise disjoint, and find $t_m \subseteq \operatorname{supp}(\mu_{i(m)})$ which satisfies $\mu_{i(m)}(t_m) \geq 1$ but $\sup_j \nu_j(t_m) \leq 1/m$. Then $\bigcup_i t_i$ is in $Z_\nu \setminus Z_\mu$. □

An alternative proof of Lemma 2.7.7, showing that even the quotients associated with these ideals are nonisomorphic, is given in §5.1.

**2.7.2. EU-ideals as density ideals.** In Theorem 2.7.8 below we will prove that all EU-ideals are density ideals. Recall from §1.7.1 that afunction $f\colon \mathbb{N} \to \mathbb{R}_+$ is called an *EU-function* if it satisfies the following (writing $\mu_f(A) = \sum_{n\in A} f(n)$):

(EU1) $\mu_f(\mathbb{N}) = \infty$.
(EU2) $\lim_n f(n)/\mu_f(n+1) = 0$.

Since for reals $a \geq 0$ and $b > 0$ and $\delta > 0$ we have that $\frac{a}{a+b} < \delta$ implies $\frac{a}{b} < 2\delta$, condition ((EU2)) is equivalent to

(EU3) $\lim_n f(n)/\mu_f(n) = 0$.

If $f$ is an EU-function, the *upper $f$-density* on $\mathbb{N}$ and the EU-ideal associated with $f$ are defined by

$$d_f(A) = \limsup_{n\to\infty} \frac{\sum_{i\in A\cap n} f(i)}{\sum_{i<n} f(i)}.$$
$$\mathcal{EU}_f = \{A : d_f(A) = 0\}.$$

The function

$$\varphi_f(A) = \sup_n \mu_f(A \cap n)/\mu_f(n)$$

is a lower semicontinuous submeasure and $\operatorname{Exh}(\varphi_f) = \mathcal{EU}_f$.

Note that $\emptyset \times \mathrm{Fin}$ is an example of a density ideal which is not an EU-ideal (all EU-ideals are dense). No EU-ideal is $F_\sigma$ (see Proposition 2.8.4). In [40, Theorem 1.13.3] I proved that all EU-ideals are density ideals and characterised those density ideals that are *isomorphic* to EU-ideals, while claiming that I characterised those density ideals that *are* EU-ideals. The latter statement is incorrect, as pointed by Jacek Tryba (the class of EU-ideals is not closed under isomorphisms, see [157, Corollary 3.9]).

Seeds of the proof of (1) in Theorem 2.7.8 below can be found in [130] (see also Theorem 2.7.12). The proof of (2) given below is a simplification of my original proof due to Max Burke, included in [40] with his kind permission. Recall that for a lower semicontinuous submeasure $\psi$ we consider a submeasure

$$\psi_\infty(A) = \lim_n \psi(A \setminus n).$$

This submeasure vanishes on Fin, moreover $\operatorname{Exh}(\psi) = \mathsf{null}(\psi_\infty)$, and $\psi_\infty$ defines a complete metric on $\mathcal{P}(\mathbb{N})/\operatorname{Exh}(\psi)$ by Lemma 5.2.2.

**Theorem 2.7.8.** *(1) Every Erdös–Ulam ideal is equal to a normalised density ideal $\mathcal{Z}_\nu$. Moreover, $(\varphi_\nu)_\infty \leq (\varphi_f)_\infty \leq 2(\varphi_\nu)_\infty$.*
*(2) A density ideal $\mathcal{Z}_\mu$ is isomorphic to an EU-ideal if and only if it is dense and $\sup_n \|\nu_n\| < \infty$.*

Proof. (1) Fix an EU-function $f$ and consider an ideal $\mathcal{EU}_f$. Let $n_0 = 0$. By (EU3) we can choose $n_1$ so that $\mu_f(n_1) \geq 1$ and $f(k)/\mu_f(k) < 1$ for all $k \geq n_1$. Find $n_i$, for $i \geq 1$, so that

$$\mu_f([n_i, n_{i+1})) \leq \mu_f(n_i) < \mu_f([n_i, n_{i+1} + 1)). \tag{2.13}$$

Since $f(k)/\mu_f(k) < 1$ for all $k \geq n_i$, sequence $n_i$, for $i \in \mathbb{N}$, is strictly increasing.

Let $J_i = [n_i, n_{i+1})$. Then (2.13) implies

$$2\mu_f(J_i) \leq \mu_f(n_{i+1}) < 2\mu_f(J_i) + f(n_{i+1})$$

and therefore

$$\left|\frac{2\mu_f(J_i)}{\mu_f(n_{i+1})} - 1\right| < \frac{f(n_{i+1})}{\mu_f(n_{i+1})}. \tag{2.14}$$

Let

$$\nu_i(A) = \frac{\mu_f(A \cap J_i)}{\mu_f(n_{i+1})}.$$

This is a measure whose support is included in $J_i$ and (2.14) implies $\lim_i \|\nu_i\| = 1/2$. Therefore with $\tilde{\nu}_i = 2\|\nu_i\|^{-1}\nu_i$ we have that $\mathcal{Z}_\nu = \mathcal{Z}_{\tilde{\nu}}$, and for the equality of the ideals it will suffice to prove $\mathcal{EU}_f = \mathcal{Z}_\nu$.

This equality will follow from the inequality of submeasures from the 'moreover' clause of (1) once it is proven. Since $\varphi_{\tilde{\nu}} = \varphi_\nu$, we need to prove that

$$(\varphi_\nu)_\infty \leq (\varphi_f)_\infty \leq 2(\varphi_\nu)_\infty.$$

Since $\nu_i(A) \leq \mu_f(A \cap n_{i+1})/\mu_f(n_{i+1})$ for all $i$ and $(\varphi_\nu)_\infty = \limsup_i \nu_i$, the left hand-side inequality is immediate.

Towards proving the other inequality, fix $A \subseteq \mathcal{P}(\mathbb{N})$ and $r > 0$ and assume $\varphi_\nu(A) < r$. Since removing a finite set from $A$ does not change the submeasures involved in this inequality, we may assume that $\nu_i(A) < r$ for all $i$. Fix $\delta > 0$ and choose $m$ large enough to have that for $i \geq m$, $A \cap J_i \neq \emptyset$ implies $\frac{1}{2} - \frac{\mu_f(J_i)}{\mu_f(n_{i+1})} < \delta$ and therefore $\frac{1}{2} - \frac{\mu_f(n_i)}{\mu_f(n_{i+1})} < \delta$ and

$$\nu_f(n_i) > \left(\frac{1}{2} - \delta\right)\nu_f(n_{i+1}).$$

Then all $i \geq m$ satisfy

$$\frac{\mu_f(A \cap n_{i+1})}{\mu_f(n_{i+1})} = \frac{\sum_{j \leq i} \mu_f(A \cap J_j)}{\sum_{j \leq i} \mu_f(J_j)} < r.$$

Now fix an arbitrary $k \geq n_m$ and fix$i$ such that $n_i \leq k < n_{i+1}$. Then

$$\frac{\mu_f(A \cap k)}{\mu_f(k)} \leq \frac{\mu_f(A \cap n_{i+1})}{\mu_f(n_i)} < \frac{2}{1-2\delta}\frac{\mu_f(A \cap n_{i+1})}{\mu_f(n_{i+1})} < \frac{2r}{1-2\delta}.$$

Since $\delta > 0$ was arbitrary, this proves $(\varphi_f)_\infty \leq 2(\varphi_\nu)_\infty$, as claimed.

(2) By Lemma 1.7.3, $\mathcal{Z}_\mu$ is dense if and only if $\limsup_n \max_j \mu_n(\{j\}) = 0$. We first prove the direct implication. Fix a density ideal $\mathcal{Z}_\nu$ such that its measures satisfy $\sup_n \|\nu_n\| < \infty$ and $\lim_n \mathrm{at}^+(\mu_n) = 0$. We need to prove that $\mathcal{Z}_\nu$ is isomorphic to an EU-ideal. By Lemma 2.7.4, we may assume $\|\nu_n\| = 1$ for all $n$ (this will not affect the condition that $\limsup_n \max_j \nu_n(\{j\}) = 0$).

By replacing $\mathcal{Z}_\nu$ with an isomorphic ideal, we may assume that there is a sequence $0 = n_0 < n_1 < \dots$ such that $\mathrm{supp}(\mu_i) = [n_i, n_{i+1})$. Define $f\colon \mathbb{N} \to \mathbb{R}^+$ by

$$\begin{aligned} f(k) &= 2^i \mu_i(\{k\}), \qquad \text{if } n_i < k \leq n_{i+1}, \\ f(0) &= 1. \end{aligned}$$

We claim that $f$ is an EU-function. Since $\mu_f([n_i, n_{i+1})) = \mu_f(n_i) = 2^i$ for all $i$, all $n \in [n_i, n_{i+1})$ satisfy

$$\frac{f(n)}{\mu_f(n+1)} \leq \mu_i(\{n\}).$$

Since $\limsup_i \mu_i(\{n\}) = 0$, $f$ is an EU-function.

For every $i$, $n_{i+1}$ is the minimal $n > n_i$ such that $\mu_f([n_i, n)) \geq \mu_f(n_i)$. Also, every $A \subseteq \mathbb{N}$ satisfies

$$\mu_i(A) = \frac{\mu_f(A \cap [n_i, n_{i+1})}{\mu_f(n_{i+1})},$$

and therefore $\mu_i$ is equal to the measure $\nu_i$ constructed in the proof of part (1) of this theorem as applied to the ideal $\mathcal{EU}_f$. Hence (1) implies that the ideals $\mathcal{Z}_\mu$ and $\mathcal{EU}_f$ coincide.

It remains to prove the converse implication. Assume $\mathcal{Z}_\mu$ is a density ideal such that $\sup_n \|\nu_n\| = \infty$ or $\limsup_i \mathrm{at}^+(\mu_i) > 0$. In the latter case, let $\varepsilon > 0$ be such that the set $A$ of all $i$ for which $\mu_i(\{k_i\}) \geq \varepsilon$ for some $k_i$ is infinite. Then the set $\{k_i : i \in A\}$ does not have an infinite subset in $\mathcal{Z}_\mu$. Since EU-ideals are dense, $\mathcal{Z}_\mu$ is not an EU-ideal.

It remains to treat the case when $\mathcal{Z}_\mu$ is dense and $\sup_n \|\mu_n\| = \infty$. By Lemma 2.7.7, $\mathcal{Z}_\mu$ is not isomorphic to a normalised density ideal. By (1), $\mathcal{Z}_\mu$ is not isomorphic to an EU-ideal. □

**Corollary 2.7.9.** *If $\mathbb{N} = \bigsqcup_n A_n$ is a partition of $\mathbb{N}$ into infinite sets and $\mathcal{Z}_n$ is an ideal on $A_n$ isomorphic to an Erdös–Ulam ideal, then*

$$\mathcal{I} = \{B : B \cap A_n \in \mathcal{Z}_n \textit{ for all } n\}$$

*is isomorphic to an Erdös–Ulam ideal.*

Proof. By Theorem 2.7.8, each $\mathcal{Z}_n$ is isomorphic to a normalised density ideal, and Lemma 2.7.5 implies the same for $\mathcal{I}$. Theorem 2.7.8 that $\mathcal{I}$ is an EU-deal. □

Proof of Lemma 2.7.2. Let $\mu_i$, $I_i$, be the measures and disjoint finite sets such that with $\varphi_\mu = \sup_I \mu_i$ we have $\mathcal{Z}_\mu = \mathrm{Exh}(\varphi_\mu)$. By Lemma 1.7.3, $\mathcal{Z}_\mu$ is dense if and only if it belongs to ($\mathcal{Z}3$) or ($\mathcal{Z}4$). By Theorem 2.7.8, these two classes are disjoint. Clearly every ideal in ($\mathcal{Z}1$) is isomorphic to Fin and this is not the case with ideals in ($\mathcal{Z}2$).

All four of these classes are nonempty—e.g., the ideal $\emptyset \times \mathrm{Fin}$ belongs to ($\mathcal{Z}2$). It therefore suffices to prove that every density ideal $\mathcal{Z}_\mu$ belongs to one of these four classes. If $\mathcal{Z}_\mu$ is not dense, then it belongs to either ($\mathcal{Z}1$) or ($\mathcal{Z}2$), so let us assume it is dense. By Lemma 2.7.4 , we may assume $\|\mu_i\| \geq 1$ for all $i$, and therefore $\mathcal{Z}_\mu$ is in ($\mathcal{Z}3$) or ($\mathcal{Z}4$), depending on whether $\sup_i \|\mu_i\|$ is finite or not. □

**2.7.3. RK-reductions between density ideals.** While the quasiorder $\leq_{\mathrm{RB}}$ on the class of dense summable ideals includes an isomorphic copy of $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ (Theorem 2.6.10), Theorem 2.7.12 below implies that $\leq_{\mathrm{RB}}$ is trivial on the class of dense density ideals. Since not all EU-ideals are RK-isomorphic (Theorem 2.7.16 below) the Schröder–Bernstein property fails in the realm of quotients over EU-ideals.

**Proposition 2.7.10.** *If $\mathcal{Z}_\mu$ is a density ideal such that $\|\mu_i\| = 1$ for all $i$ then $\mathcal{Z}_\mu$ is RB-reducible to every EU-ideal. In particular, every two EU-ideals $\mathcal{EU}_f$ and $\mathcal{EU}_g$ satisfy $\mathcal{EU}_f \leq_{\mathrm{RB}} \mathcal{EU}_g$.*

We will prove a stronger result in Theoren 2.7.12 below. The special case, $\mathcal{Z}_0 \leq_{\mathrm{RB}} \mathcal{Z}_{\log}$, of Proposition 2.7.10 was known to Just (see [92, p. 904]) and the proof of the general case is very similar.

Once the following trivial key fact is isolated, the proof of Proposition 2.7.10 is obvious.[3]

**Lemma 2.7.11.** *Suppose that $\mu$ and $\nu$ are measures on finite sets $I$ and $J$ such that $\|\mu\| = \|\nu\| = 1$ and $\varepsilon > 0$ satisfies $\varepsilon \operatorname{at}^-(\mu) > \operatorname{at}^+(\nu)$. Then some $B \subseteq J$ and $h\colon B \to I$ satisfy $\nu(B) > 1 - \varepsilon$ and $\mu(a) - \varepsilon < \nu(h^{-1}(a)) \leq \mu(a)$ for all $a \subseteq I$.*

Proof. Of course the easiest way to prove this is by using an ultraproduct, but we'll do it the hard way. Since $\|\mu\| = 1$ and $\operatorname{at}^-(I)|I| \leq 1$, we have

$$\operatorname{at}^+(\nu)|I| \leq \frac{\operatorname{at}^+(\nu)}{\operatorname{at}^-(\mu)} < \varepsilon.$$

A greedy algorithm produces disjoint $F_i \subseteq J$, for $i \in I$, such that

$$\mu(\{i\}) - \operatorname{at}^+(\nu) < \nu(F_i) \leq \mu(\{i\})$$

for all $i$. Let $B = \bigcup_i F_i$ and let $h\colon B \to I$ be such that $h^{-1}(\{i\}) = F_i$. Then $\nu(B) > 1 - |I| \operatorname{at}^+(\nu)$ and $\mu(a) - |a| \operatorname{at}^+(\nu) < \nu(h^{-1}(a)) \leq \mu(a)$ for all $a \subseteq I$. Since $|a| \operatorname{at}^+(\nu) \leq |I| \operatorname{at}^+(\nu) < \varepsilon$, the conclusion follows. □

Proof of Proposition 2.7.10. Fix parameters $\mu_i, I_i$ determining $\mathcal{Z}_\mu$, By Theorem 2.7.8 there are $\nu_i, J_i$, for $i \in \mathbb{N}$, such that $\mathcal{EU}_g = \mathcal{Z}_\nu$. Since $\lim_i \operatorname{at}^+(\nu_i) = 0$ (by e.g., (2) of the same theorem), there is an increasing sequence $m_j$, for $j \in \mathbb{N}$, such that all $i \geq 1$ and $m_i \leq j < m_{i+1}$ satisfy

$$\frac{1}{i} \operatorname{at}^-(\mu_i) \geq \operatorname{at}^+(\nu_j).$$

For such $i$ and $j$, Lemma 2.7.11 implies that some $B_j \subseteq J_j$ and $h_j\colon B_j \to I_i$ satisfy $\nu_j(J_j \setminus B_j) < 1/i$ and $|\nu_j(h_j^{-1}(a)) - \mu_i(a)| < 1/i$. With $B = \bigcup_j B_j$, $\mathbb{N} \setminus B$ belongs to $\mathcal{Z}_\nu$ and the function $h\colon B \to \mathbb{N}$ that agrees with $h_j$ on $B_j$ for all $j$ is an RB-reduction as required.

For the second part, apply Theorem 2.7.8 (1) to $\mathcal{EU}_f$ and use the first part. □

The following was inspired by [130].

**Theorem 2.7.12.** *For a generalised density ideal $\mathcal{Z}_\varphi$ the following are equivalent.*

*(1) $\mathcal{Z}_\varphi$ is RB-reducible to $\mathcal{Z}_0$.*
*(2) $\mathcal{Z}_\varphi$ is RB-reducible to every EU-ideal.*
*(3) $\mathcal{Z}_\varphi$ is nonpathological.*

The proof of Theorem 2.7.12 is given after two lemmas. Recall that (somewhat nonstandardly) a submeasure is pathological if it is the equal to the supremum of measures majorised by it.

**Lemma 2.7.13.** *Suppose that $\varphi$ is a nonpathological submeasure on a finite set $I$. Then there are $n$, finite sets $J_i$, for $i < n$, measures $\mu_i$ on $J_i$, $X \subseteq \prod_{i<j} J_i$, and $h\colon X \to I$ such that every $a \subseteq I$ satisfies*

$$\varphi(a) = \max_{i<n} \mu_i(h^{-1}(a)).$$

[3] This is like Littlewood's 'two trivialities omitted can add up to an impasse', except that we are not omitting the trivialities and the conclusion is not exactly an impasse.

*Moreover, we can assure that* $\max_{i<n}\|\mu_i\| = \|\varphi\|$.

Proof. Since $I$ is finite, for every $a \subseteq I$ there is a measure $\mu_a$ on $I$ such that $\mu_a(a) = \varphi(a)$. Let $n = |\mathcal{P}(I)|$, fix an enumeration $a(i)$, for $i < n$ of $\mathcal{P}(I)$, let $J_i = I$ and let $\mu_i = \mu_{a(i)}$. Let $X \subseteq \prod_{i<n} J_i$ be the diagonal, $X = \{(x, x, \dots, x) : x \in I\}$ and define $h\colon X \to I$ by $h((x, x, \dots, x)) = x$. Then for every $a \subseteq I$ we have that $h^{-1}(a) = \{(x, \dots, x) : x \in a\}$, and $\sup_{i<n}\mu_i(h^{-1}(a)) = \sup_{b\subseteq I}\mu_b(a) = \varphi(a)$. By the construction, $\|\mu_i\| \leq \|\varphi\|$ for all $i$. □

Proof of Theorem 2.7.12. (1) ⇒ (2) by Proposition 2.7.10.

(2) ⇒ (3) because any ideal RK-reducible to a nonpathological analytic P-ideal is nonpathological.

(3) ⇒ (1) Suppose that $\mathcal{Z}_\varphi$ is a nonpathological density ideal with parameters $I_i, \varphi_i$. By Proposition 2.7.10, it suffices to prove that $\mathcal{Z}_\varphi$ is RB-reducible to some density ideal. By replacing each $\varphi_i$ with $\min(1, \varphi_1)$ we may assume that $\limsup_i \|\varphi_i\| \leq 1$. Apply Lemma 2.7.13 to each $i$, and find intervals and meaasures on them $J_{ij}$, $\mu_{ij}$, for $j < n(i)$, $X_i \subseteq \prod_{j<n(i)} J_{ij}$ and $h_i\colon X_i \to I_i$ such that $\varphi_i(a) = \max_{j<n(i)}\mu_{ij}(h^{-1}(a))$ and $\max_{j<n(i)}\|\mu_{ij}\| \leq 1$.

Since $\sum_i \sum_{j<n(i)} |J_{ij}|$ is countable, we may assume that the intervals $J_{ij}$, for $i \in \mathbb{N}$ and $j < n(i)$, form a partition of $\mathbb{N}$. The set $X = \bigcup_i X_i$ is positive with respect to the density ideal $\mathcal{Z}_\mu$ with parameters $J_{ij}$, $\mu_{ij}$. The function whose graph is the union of graphs of $h_i$, for $i \in \mathbb{N}$, is an RB-reduction of $\mathcal{Z}_\varphi$ to $\mathcal{Z}_\mu \restriction X$, and Lemma 2.7.3 implies that $\mathcal{Z}_\mu \restriction X$ is isomorphic to a density ideal. □

A related result that motivated Theorem 2.7.12 was proved in [132]. While no pathological density ideal is $\leq_{\mathrm{RB}}$-reducible to $\mathcal{Z}_0$, the associated orbit equivalence relations are all Borel-reducible to the relation associated with $\mathcal{Z}_0$. The ideal $\mathcal{Z}_\infty$ (Example 1.7.5) is, by Theorem 2.7.8 (2), not RK-isomorphic to any EU-ideal. We can do better.

**Proposition 2.7.14.** *If* $\mathcal{Z}_\nu$ *is a dense density ideal such that* $\sup_n \|\nu_n\| = \infty$ *then no EU-ideal is RK-reducible to* $\mathcal{Z}_\nu$.

Proof. Assume otherwise, fix a EU-ideal RK-reducible to $\mathcal{Z}_\nu$. By Theorem 2.7.8 (1) this ideal is a density ideal $\mathcal{Z}_\mu$ such that $I_i, \mu_i$ satisfy $\|\mu_i\| = 1$ for all $i$. Let $J_n, \nu_n$ be parameters determining $\mathcal{Z}_\nu$, so that $\sup_n \|\nu_n\| = \infty$ and $\limsup \mathrm{at}^+(\nu_n) = 0$ because $\mathcal{Z}_\nu$ is dense. Since $\mathcal{Z}_\nu$ is a P-ideal, we can choose the reduction $h$ to be a finite-to-one function. Assume for a moment that

$$\sup_{m,n} \nu_n(h^{-1}(I_m)) = \infty.$$

Since all $I_m$ and all $J_n$ are finite, we can choose disjoint $I_{m(j)}$ and disjoint $J_{n(j)}$, for $j \in \mathbb{N}$, such that $\nu_{n(j)}(h^{-1}(I_{m(j)})) > j$, and $\mathrm{at}^+(\mu_{m(j)}) < 1/^2 j$. By the latter, we can partition $h[J_{n(j)}] \cap (I_{m(j)})$ into $j$ pieces each one of them of $\mu_{m(j)}$-measure not greater than $1/j$. At least one of these pieces, call it $a_j$, satisfies $\mu_{m(j)}(h^{-1}(a_j)) \geq 1$. Since $m(j)$, for $j \in \mathbb{N}$, are distinct $a = \bigcup_j a_j$ belongs to $\mathcal{Z}_\mu$, but $h^{-1}(a) \notin \mathcal{Z}_\nu$.

We may therefore assume that

$$K = \sup_{m,n} \nu_n(h^{-1}(I_m)) < \infty.$$

Recursively choose $J_{n(j)}$, $k(j)$, and $I_{m(j,l)}$, for $l < k(j)$, such that the following holds for all $j$.

(1) $\nu_{n(j)}(J_{n(j)}) \geq Kj$.
(2) $h[J_{n(j)}] \subseteq \bigcup_{l<k(j)} I_{m(j,l)}$.
(3) $I_{m(j,l)} \neq I_{m(j',l')}$ if and only if $j = j'$ and $l = l'$.
(4) $\mathrm{at}^+(\mu_{m(j,l)}) \leq 1/j^2$ for all $l < k(j)$.

Since $\mu_{m(j,l)}(I_{m(j,l)}) \leq 1$, for every $j$ and $l < k(j)$ we can choose $a_{j,l} \subseteq I_{m(j,l)}$ so that $\mu_j(a_{j,l}) \leq 1/j$ and $\nu_{n(j)}(h^{-1}(a_{j,l}) \geq \nu_{n(j)}(h^{-1}(I_{m(j,l)}))/j$. Then $a_j = \bigcup_{l<k(j)} a_{j,l}$ satisfies $\nu_{n(j)}(h^{-1}(a_j)) \geq K$, hence $a = \bigcup_j a_j$ satisfies $\limsup_n \nu_n(h^{-1}(a)) \geq K$ and $h^{-1}(a) \notin \mathcal{Z}_\nu$. On the other hand, since the intervals $I_{m(j,l)}$ are disjoint, since for every $j$ only finitely many $n$ satisfy $\mu_n(a) \geq 1/j$, we have $\limsup_n \mu_n(a) = 0$, hence $a \in \mathcal{Z}_\mu$. □

The proof shows that no EU-ideal is Katětov reducible to $\mathcal{Z}_\nu$.

As mentioned earlier, the introduction of Erdös–Ulam ideals was motivated by the question whether quotients over $\mathcal{Z}_0$ and $\mathcal{Z}_{\log}$ were isomorphic. In [95], Just and Krawczyk used CH to prove that all quotients over Erdös–Ulam ideals are isomorphic. Just ([92], [94]) proved that under a different set-theoretic axiom the quotients over $\mathcal{Z}_0$ and $\mathcal{Z}_{\log}$ are not isomorphic. By Theorem 2.7.12 this is quite optimal, since $\mathcal{Z}_0 \leq_{\mathrm{RB}} \mathcal{Z}_{\log}$ and $\mathcal{Z}_{\log} \leq_{\mathrm{RB}} \mathcal{Z}_0$. Our methods provide many pairs of nonisomorphic quotients over density ideals, in particular $\mathcal{Z}_0$ and $\mathcal{Z}_{\log}$. In connection with methods of Chapter 6, this gives another proof of Just's result (see Corollary 7.1.2).

### 2.7.4. RK-isomorphisms of EU-ideals.

**Definition 2.7.15.** For a density ideal $\mathcal{Z}_\mu$ such that $\lim_i \mathrm{at}^+(\mu_i) = 0$ and $\delta > 0$ define functions $F_\mu, G_{\mu\delta} \colon \mathbb{N} \to \mathbb{N}$ by

$$F_\mu(n) = |\operatorname{supp}(\mu_n)|,$$
$$G_{\mu\delta}(n) = \max\{j : \mu_j(s) \geq \delta \text{ for some } s \text{ of cardinality} \leq n\}.$$

The function $G_{\mu\delta}$ is well-defined because $\lim_i \mathrm{at}^+(\mu_i) = 0$.

**Theorem 2.7.16.** *If the density ideals $\mathcal{Z}_\mu$ and $\mathcal{Z}_\nu$ satisfy*

$$\lim_i \mathrm{at}^+(\mu_i) = \lim_i \mathrm{at}^+(\nu_i) = 0 \ \ \& \ \ G_{\nu\delta} \circ F_\mu = o(n)$$

*for all $\delta > 0$, then $\mathcal{Z}_\mu$ and $\mathcal{Z}_\nu$ are not RK-isomorphic.*

Proof. We will prove the contrapositive. Let $h$ be an injection from a subset of $\mathbb{N}$ into $\mathbb{N}$ witnessing the RK-isomorphism, so that $A \in \mathcal{Z}_\mu$ if and only if $h''A \in \mathcal{Z}_\nu$. Let $D_i = \operatorname{supp}(\mu_i)$.

We claim that there exists $\delta > 0$ such that for all but finitely many $i$ some $j = J(i)$ satisfies $\nu_j(h''D_i) > \delta$. Assume otherwise, that for every $m$ there are infinitely many $i$ such that for all $j$ we have $\nu_j(h''D_i) < 1/m$. Since $\operatorname{supp}(\nu_j)$ is finite for every $j$, we can find a sparse enough subsequence of $\{D_i\}$ whose union $A$ is such that $\limsup_j \nu_j(h''A) = 0$, which contradicts the choice of $h$.

Fix such $\delta$. We claim that there exists $m$ such that the function $J$ is at most $m$-to-1. In particular, $J(n) \neq o(n)$. Assume otherwise, that for every $m$ there is a $k$ for which $|J^{-1}(k)| \geq m$. Since $\lim_k \mathrm{at}^+(\nu_k) = 0$, we can find such $k$ and $s_k \subseteq \operatorname{supp}(\nu_k)$ so that $\nu_k(s_k) \geq \delta$ and $\varphi_\mu(h^{-1}(s_k)) \leq 2/m$. Now it is easy to find an infinite set $A$ such that the set $\bigcup_{k\in A} s_k$ is not in $\mathcal{Z}_\nu$ but its $h$-preimage is in $\mathcal{Z}_\mu$.

Since $J(n) \leq G_{\nu\delta} \circ F_\mu(n)$, the function $G_{\nu\delta} \circ F_\mu$ cannot be $o(n)$. □

**Corollary 2.7.17.** *Ideals $\mathcal{Z}_0$ and $\mathcal{Z}_{\log}$ are not RK-isomorphic.*

Proof. Let $\{\mu_i\}$ and $\{\nu_i\}$ be sequences of measures induced by EU-functions such that $\mathcal{Z}_0 = \mathcal{Z}_\mu$ and $\mathcal{Z}_{\log} = \mathcal{Z}_\nu$, as in Theorem 2.7.8 (1). Then every $\delta > 0$ satisfies $F_\mu(n) = 2^n$, $G_{\mu\delta}(n) = O(\log(n))$, $F_\nu(n) = O(a^{2^i})$ (for some $a > 1$) and $G_{\nu\delta}(n) \leq K\log(n)$, for a fixed $K > 0$. Therefore

$$G_{\nu\delta} \circ F_\mu(n) \leq K \log(2^n) = o(n)$$

for all $\delta > 0$, and Theorem 2.7.16 implies the desired conclusion. □

**2.7.5. RK-minimality of LV-ideals.** The second part of the following is a baby version of [44, Proposition 5.3] (see Definition 1.7.7 for LV-ideals and recall that Fin is an LV-ideal since we do not require LV-ideals to be dense).

Compare the following with the last part of Lemma 2.7.3.

**Proposition 2.7.18.** *(1) If $\mathcal{Z}_\varphi$ is a generalised density ideal and $\mathcal{I} \leq_{\mathrm{RK}} \mathcal{Z}_\varphi$, then $\mathcal{I}$ is a generalised density ideal.*
*(2) Every ideal RK-reducible to an LV-ideal is an LV-ideal.*

Proof. (1) Since $\mathcal{Z}_\varphi$ is a P-ideal, there is an RB-reduction $h\colon \mathbb{N} \to \mathbb{N}$. Let $I_m, \varphi_m$, for $m \in \mathbb{N}$, be the finite sets and submeasures on them determining $\mathcal{Z}_\varphi$. Since each $I_m$ is finite, we can recursively find an increasing sequence $n_i$, for $i \in \mathbb{N}$, such that for every $m$ there is $i$ such that set $h[I_m] \subseteq [n_i, n_{i+2})$. With $J_j = [n_j, n_{j+1})$, for $A \subseteq J_j$ let

$$\psi_j(A) = \max_i \varphi_i(h^{-1}(A) \cap I_i)$$

(the maximum is attained since only finitely many of the values are nonzero).

Since for every $j$ the set $h^{-1}(J_j) \cap I_i$ is nonempty for at most two values of $i$, every $A \subseteq \mathbb{N}$ satisfies

$$\limsup_j \psi_j(A) \leq \limsup_i \varphi_i(h^{-1}(A)) \leq 2 \limsup_j \psi_j(A) \tag{2.15}$$

and therefore $\mathcal{I} = \mathcal{Z}_\psi$, a generalised density ideal. (2) Assume $\mathcal{Z}_\varphi$ is an LV-ideal, hence the submeasures $\varphi_n$ satisfy

$$(\forall k)(\forall \varepsilon > 0)(\forall^\infty n)(\forall a_0, \dots, a_k \subseteq I_n)|\varphi_n(a_0 \Delta a_k) - \max_{i<k} \varphi_n(a_i \Delta a_{i+1})| < \varepsilon.$$

Then (2.15) implies that the submeasures $\psi_n$ defined in the first part of the proof inherit this property, and $\mathcal{I}$ is an LV-ideal. □

## 2.8. Summable ideals vs. density ideals

We first define an ideal that is neither summable nor a density ideal but locally looks like a summable or a density ideal. Then we prove that summable ideals are RB-incomparable to density ideals except in trivial cases (Proposition 2.8.4). Proof of the following is an easy exercise.

**Lemma 2.8.1.** *For $A \subseteq \mathbb{N}^2$ and $m \in \mathbb{N}$ let $\mu_m(A) = \sum_{(m,n)\in A} 1/mn$, and let*

$$\mathcal{I}_\infty = \mathrm{Exh}(\sup_m \mu_m).$$

*Then the restriction of $\mathcal{I}_\infty$ to $\{n\} \times \mathbb{N}$ is summable, but if $A \in \emptyset \times \mathrm{Fin}$ is $\mathcal{I}_\infty$-positive then $\mathcal{I}_\infty \restriction A$ is isomorphic to a dense density ideal.* □

**Lemma 2.8.2.** *The ideals $\mathcal{I}_\infty$, $\mathcal{I}_\infty \oplus \mathcal{I}_{1/n}$, $\mathcal{I}_\infty \oplus \mathcal{Z}_0$, and $\mathcal{I}_\infty \oplus \mathcal{Z}_\infty$ are RK-isomorphic.*

Proof. The proof is very similar to the proof of Lemma 1.7.6. □

The following consequence [44, Theorem 4.2] may have been overlooked since it was stated in terms of Borel equivalence relations.

**Theorem 2.8.3.** *Suppose that $\mathcal{Z}_\mu$ is a density ideal not RK-isomorphic to* Fin *and $\mathcal{I}$ is a dense $F_\sigma$ ideal. Then $\mathcal{I}$ and $\mathcal{Z}_\mu$ are $\leq_{\mathrm{RB}}$-incomparable.*

A proof is provided at the end of this section, after the following special case and its self-contained proof.

**Proposition 2.8.4.** *If $\mathcal{I}_f$ and $\mathcal{Z}_\mu$ are a summable ideal and a density ideal, and none of them is RK-isomorphic to* Fin*, then $\mathcal{I}_f$ and $\mathcal{Z}_\mu$ are $\leq_{\mathrm{RB}}$-incomparable.*

Proof. Since RB-reductions are continuous and density ideals are not $F_\sigma$, it suffices to prove $\mathcal{I}_f \not\leq_{\mathrm{RB}} \mathcal{Z}_\mu$. This is a consequence of Theorem 2.8.3 above, but we provide a self-contained proof.

Assume the contrary, and let $h$ be such that $A \in \mathcal{I}_f$ if and only if $h^{-1}(A) \in \mathcal{Z}_\mu$. Find a sequence $1 = n_1 < n_2 < n_3 < \dots$ and $t_i \subseteq \{n_i, \dots, n_{i+1} - 1\}$ such that for all $i, j$ (let $D_k = \mathrm{supp}(\mu_k)$):

(1) $1/i \leq \mu_f(t_i) < 2/i$
(2) $h^{-1}(t_i) \cap D_k \neq \emptyset$ and $h^{-1}(t_j) \cap D_k \neq \emptyset$ for some $k$ implies $i = j$.

The recursive construction of these sequences proceeds as follows:

Assume $n_1, \dots, n_{k-1}$ and $t_1, \dots, t_{k-1}$ are already chosen. Since the set

$$E = \bigcup_{l=1}^{k-1} t_l \cup h^{-1}(t_l)$$

is finite, there are only finitely many $j$ such that $D_j \cap E \neq \emptyset$, therefore we can pick an $n_k$ so that $\{1, \dots, n_k - 1\}$ includes $E$ and all such $D_j$'s. Now choose $t_k$ satisfying (1) (this is possible because $\mathcal{Z}_\mu$ is a proper density ideal) and such that $\min(t_k) \geq n_k$. This describes the recursive construction, and clearly (1) and (2) hold.

Assume that $t_i$ ($i \in \mathbb{N}$) are chosen to satisfy (1) and (2). Then $\bigcup_i t_i$ is, by (1), not in $\mathcal{I}_f$, and therefore $\lim_j \sup_i \mu_j(h^{-1}(t_j)) = \varepsilon > 0$. By (2)

$$\limsup_j \mu_j(\bigcup_i t_i) = \lim_j \sup_i(\mu_j(t_i)),$$

and there are subsequences $t_i'$ of $t_i$ and $\mu_i'$ of $\mu_i$ such that $\mu_i'(t_i') > \varepsilon/2$ or all $i$, in particular $\bigcup_{i \in A} t_i'$ is $\mathcal{I}_f$-positive whenever $A$ is infinite. But by (1) there is an infinite set $A$ such that $\bigcup_{i \in A} t_i'$ is in $\mathcal{I}_f$; contradiction. Since $\mathcal{I}_f \restriction A$ is summable and $\mathcal{Z}_\mu \restriction B$ is a density ideal, these ideals are $\leq_{\mathrm{RB}}$-incomparable. □

Proof of Theorem 2.8.3. Let $I_n$ and $\mu_n$ be finite sets and measures associated with $\mathcal{Z}_\mu$. We may assume that $\mu_n(\{j\}) > 0$ for all $j \in I_n$, by a negligible modification of $\mu_n$. Then $X_n = \mathcal{P}(I_n)$, with the metric $d_n(s, t) = \mu_n(s \Delta t)$, is a finite metric space and the equivalence relation on $\mathcal{P}(\mathbb{N})$ defined by $A \sim_\mu B$ if $A \Delta B \in \mathcal{Z}_\mu$ is a $c_0$-equality, as per [44, Definition 3.1].

Suppose that $\mathcal{I}$ is an $F_\sigma$-ideal such that $\mathcal{I} \leq_{\mathrm{BA}} \mathcal{Z}_\mu$, and let $\mathrm{E}_{\mathcal{I}}$ be the Borel equivalence relation on $\mathcal{P}(\mathbb{N})$ defined by $A \, \mathrm{E}_{\mathcal{I}} \, B$ if $A \Delta B \in \mathcal{I}$. Then [44, Theorem 4.2 (2)] applied to $\mathrm{E}_{\mathcal{I}}$ implies that $\mathrm{E}_{\mathcal{I}}$ is essentially countable, meaning that it is Borel-reducible to $\mathrm{E}_{\mathrm{Fin}}$ (usually denoted $\mathrm{E}_0$). However, $\mathcal{I}$ was assumed to be a dense ideal, and therefore $E_{\mathcal{I}}$ is a turbulent equivalence relation ([81]) and not reducible to $E_{\mathrm{Fin}}$. □

**Corollary 2.8.5.** *Suppose that $\mathcal{I}$ is an ideal that is RK-reducible to an $F_\sigma$ ideal and to a density ideal. Then $\mathcal{I}$ is RK-isomorphic to* Fin*.*

Proof. Since $\mathcal{I}$ is RK-reducible to an $F_\sigma$ ideal, it is $F_\sigma$. Assume that it is not RK-isomorphic to Fin. Then the restriction of $\mathcal{I}$ to some positive set $A$ is a dense ideal. Since $\mathcal{I}$ is RK-reducible to some density ideal $\mathcal{Z}_\mu$, $\mathcal{I} \restriction A$ is RK-reducible to $\mathcal{Z}_\mu \restriction B$ for some positive set $B$. However $\mathcal{Z}_\mu \restriction B$ is a density ideal by Lemma 2.7.3, hence this contradicts Theorem 2.8.3. □

## 2.9. RK-automorphisms

In this section we define a study RK-automorphisms of an ideal. The group of RK-automorphisms of $\mathcal{I}$ isomorphically embeds into the automorphism group of the quotient $\mathcal{P}(\mathbb{N})/\mathcal{I}$. Our lifting theorems imply that for a large class of ideals (all ideals 80-determined by closed approximations with the Radon–Nikodym property), $\mathrm{OCA_T}$ and MA($\sigma$-linked) imply that every automorphism of $\mathcal{P}(\mathbb{N})/\mathcal{I}$ corresponds to an RK-automorphisms of $\mathcal{I}$. The RK-automorphism group of every dense summable ideal contains both the free abelian group with continuum many generators and the free group with continuum many generators as subgroups (Proposition 2.9.4). By Theorem 2.9.6 there is a summable ideal $\mathcal{I}_f$ such that every RK-reduction of $\mathcal{I}_f$ to itself is automatically an RK-automorphism.

Recall that ideals $\mathcal{I}$ and $\mathcal{J}$ on $\mathbb{N}$ are RK-isomorphic if there are $X \in \mathcal{I}^*$, $Y \in \mathcal{J}^*$, and a bijection $f\colon X \to Y$ such that $A \in \mathcal{J}$ if and only if $h^{-1}(A) \in \mathcal{I}$, for all $A \subseteq Y$. If $\mathcal{I} = \mathcal{J}$, such $f$ is called a *pre-RK-automorphism* of $\mathcal{I}$.[4]

**Definition 2.9.1.** Let $\mathbb{F}_{\mathcal{I}}$ denote the set of all pre-RK-automorphisms of $\mathcal{I}$. For $f$ and $g$ in $\mathbb{F}_{\mathcal{I}}$ let

$$f \sim_{\mathcal{I}} g \Leftrightarrow \{n \in \mathrm{dom}(f) \cap \mathrm{dom}(g) : f(n) \neq g(n)\} \in \mathcal{I}.$$

This is an equivalence relation. Its equivalence classes are *RK-automorphisms* of $\mathcal{I}$.

Clearly $f \in \mathbb{F}_{\mathcal{I}}$ implies $f^{-1} \in \mathbb{F}_{\mathcal{I}}$, while $f \sim_{\mathcal{I}} g$ and $f' \sim_{\mathcal{I}} g'$ together imply $f \circ f' \sim_{\mathcal{I}} g'$. Therefore,

$$\mathrm{Aut}_{\mathrm{RK}}(\mathcal{I}) = \mathbb{F}_{\mathcal{I}}/\sim_{\mathcal{I}}$$

is a group. The other natural definition of $\sim_{\mathcal{I}}$ is equivalent to this one.

**Lemma 2.9.2.** *Suppose that $\mathcal{I}$ is an ideal on $\mathbb{N}$ and $f, g$ are pre-RK-automorphisms of $\mathcal{I}$. Then the following are equivalent.*

(1) $f \sim_{\mathcal{I}} \mathrm{id}_{\mathbb{N}}$.
(2) *For every $A \subseteq \mathbb{N}$, $A \in \mathcal{I}$ if and only if $f[A] \in \mathcal{I}$.*
(3) *For every $A \subseteq \mathbb{N}$, $A \in \mathcal{I}$ if and only if $f^{-1}(A) \in \mathcal{I}$,*

*Also, the following are equivalent.*

(4) $f \sim_{\mathcal{I}} g$.
(5) *For every $A \subseteq \mathbb{N}$, $f^{-1}(A)\Delta g^{-1}(A) \in \mathcal{I}$.*
(6) *For every $A \subseteq \mathbb{N}$, $f[A]\Delta g[A] \in \mathcal{I}$.*

[4] The only case when we cannot assume $X = Y = \mathbb{N}$ is when $\mathcal{I}$ is Fin, when there is an index obstruction.

Proof. Clearly (1) implies the other two conditions, and it suffices to prove that if (1) fails then so do both (2) and (3).

Suppose that the set $A = \{n \in \mathrm{dom}(f) : f(n) \neq n\}$ is $\mathcal{I}$-positive. Then there is a partition $A = \bigsqcup_{i<3} A_i$ such that $f[A_i] \cap A_i = \emptyset$ for all $i$. (By recursion on $n \in A$ choose which piece to assign it to; at every stage there are at most two 'forbidden' pieces, the one to which $f(n)$ belongs and the one to which $f^{-1}(n)$, if defined, belongs; Lemma 9.1 of [18].) Then at least one of the $A_i$ is $\mathcal{I}$-positive. It is not equal to $f^{-1}[A_i]$ or to $f[A_i]$ modulo $\mathcal{I}$, as required.

For the second part, note that $f \circ g^{-1}$ is a pre-RK-automorphism of $\mathcal{I}$ and apply the first part. □

Recall that an *almost permutation* of $\mathbb{N}$ is a bijection between its cofinite subsets. The following complexity estimate may be unoptimal, and we will prove that for sufficiently well-understood ideals this is indeed the case.

**Theorem 2.9.3.** *For a Borel ideal $\mathcal{I} \neq \mathrm{Fin}$, $\mathrm{Aut}_{\mathrm{RK}}(\mathcal{I})$ is a quotient of a coanalytic subgroup of $S_\infty$ modulo a relatively Borel subgroup.* $\mathrm{Aut}_{\mathrm{RK}}(\mathrm{Fin})$ *is a quotient of the (Borel) semigroup of all almost permutations of $\mathbb{N}$ modulo the ideal of finitely supported permutations of $\mathbb{N}$.*

Proof. The proof is by counting quantifiers. Since $\mathcal{I} \neq \mathrm{Fin}$, it contains an infinite set and every $\sim_{\mathcal{I}}$-equivalence class in $\mathbb{F}_{\mathcal{I}}$ contains a permutation. Then $\mathcal{X} = \mathbb{F}_{\mathcal{I}} \cap S_\infty$ is a coanalytic set because $f \in \mathcal{X}$ if and only if $(\forall A) A \in \mathcal{I} \Leftrightarrow f^{-1}[A] \in \mathcal{I}$. Finally, some $f \in \mathcal{X}$ satisfies $f \sim_{\mathcal{I}} \mathrm{id}_{\mathbb{N}}$ if and only if $\{n : f(n) = n\} \in \mathcal{I}$. □

**Proposition 2.9.4.** *Suppose that $\mathcal{I}_f$ is a summable ideal not RK-isomorphic to* Fin*. Then the following holds.*

1. *(1)* $\mathrm{Aut}_{\mathrm{RK}}(\mathcal{I})$ *has a subgroup isomorphic to the free Abelian group with continuum many generators.*
2. *(2)* $\mathrm{Aut}_{\mathrm{RK}}(\mathcal{I})$ *has a subgroup isomorphic to the free group with continuum many generators.*

Proof. Since $\mathcal{I}_f$ is not RK-isomorphic to Fin, its restriction to some positive set $A$ is a dense ideal. Since the automorphism group of $\mathcal{P}(A)/\mathcal{I}_f \restriction A$ is a subgroup of the automorphism group of $\mathcal{P}(\mathbb{N})/\mathcal{I}$, we may assume that $f$ is nonincreasing. Find a sequence $1 = n_1 < n_2 < \dots$ such that for all $i$ we have the following.

- (a) $\mu_f([n_i, n_{i+1})) \geq 1$, and
- (b) $\mu_f(\{n_{i+1} - 1 : i \in \mathbb{N}\}) < \infty$.

For $B \subseteq \mathbb{N}$ define $h_B \colon \mathbb{N} \to \mathbb{N}$ by

$$h_B(k) = \begin{cases} k+1, & \text{if } k \in [n_i, n_{i+1}) \text{ for } i \in B, \\ k, & \text{if } k \in [n_i, n_{i+1}) \text{ for } i \notin B. \end{cases}$$

We claim that $A \mapsto h_B^{-1}(A)$ is a lifting of an automorphism $\Phi_B^*$ of $\mathcal{P}(\mathbb{N})/\mathcal{I}_f$.

To prove this, for $\varepsilon > 0$ let $A_\varepsilon = \left\{n : \frac{f(n)}{f(n-1)} < \varepsilon\right\}$. If $\mu_f(A_\varepsilon) = \infty$ for every $\varepsilon > 0$, then we can find a sequence of finite sets of integers $s_i$ such that for all $i$

- (c) $s_i \subseteq A_{1/i}$,
- (d) $\max s_i < \min s_{i+1}$, and
- (e) $\mu_f(s_i) > 1$.

Let $B = \bigcup_i s_i$, and enumerate $B$ increasingly as $\{k_j\}$. Then by (c), (d) and the monotonicity of $f$ we have $\lim_j f(k_j)/f(k_{j-1}) = 0$ which implies $\mu_f(B) < \infty$, contradicting (e). Therefore we can find an $\varepsilon > 0$ such that $\mu_f(A_\varepsilon) < \infty$. Then for all $k \in \mathbb{N} \setminus A_\varepsilon$ we have $0 < \varepsilon \leq f(k)/f(k-1) \leq 1$. Since $h_B^{-1}(k) \subseteq \{k, k-1\}$ for all $k$, this implies that for $k \in A_\varepsilon$ we have

$$1 \leq \frac{f(h_B^{-1}(k))}{f(k)} \leq 1 + \frac{1}{\varepsilon}.$$

Therefore Lemma 2.6.8 implies that $\Phi_B^*$ is a homomorphism of $\mathcal{P}(\mathbb{N})/\mathcal{I}_f$. Moreover, $h_B$ is injective on the set $\mathbb{N} \setminus \{n_{i+1} - 1 : i \in \mathbb{N}\}$, by (b) this set is in the dual filter of $\mathcal{I}_f$, and $\Phi_B^*$ is an automorphism of $\mathcal{P}(\mathbb{N})/\mathcal{I}_f$, as required.

Let $F$ be the free Abelian group with generators $g_\xi$, for $\xi \in \mathbb{R}$, and let $B(\xi)$, for $\xi \in \mathbb{R}$, be a family of infinite almost disjoint subsets of $\mathbb{N}$. The subgroup $G$ of $\operatorname{Aut}(\mathcal{P}(\mathbb{N})/\mathcal{I}_f)$ generated by $\Phi_{B(\xi)}$, for $\xi \in \mathbb{R}$, is Abelian since for $\xi \neq \eta$ the set $B(\xi) \cap B(\eta)$ is finite and $\Phi_{B(\xi)} \circ \Phi_{B(\eta)} = \Phi_{B(\xi) \cup B(\eta)}$.

We claim that the group homomorphism $\Omega \colon F \to G$ defined by

$$\Omega(g_\xi) = \Phi_{B_\xi}$$

for $\xi \in \mathbb{R}$ is an isomorphism. It is clearly surjective. To see that it is injective, fix a reduced word $a = g_1^{l_1} g_2^{l_2} \dots g_n^{l_n}$ in $F$. We need to verify that $\Phi = \Omega(a)$ is not the identity on $\mathcal{P}(\mathbb{N})/\mathcal{I}_f$. Let $h = h_n^{l_n} \circ \dots h_2^{l_2} \circ h_1^{l_1}$. Then $\Phi_h$ is a lifting of $\Phi$. Since $a$ is a reduced word, $l_1 \neq 0$ and $g_i \neq g_{i+1}$ for $i < n-1$. Therefore $h$ has no fixed points in the set $C = \bigcup_{j \in B_1} [n_j, n_{j+1})$. By the argument as in the proof of Lemma 2.9.2, there is a partition $C = \bigsqcup_{i<3} C_i$ such that $h[C_i]$ and $C_i$ are disjoint for all $i < 3$. At least one of these sets is $\mathcal{I}_f$-positive. Since it is moved by $\Phi$, $\Phi$ is not the identity, as required.

If $\mathcal{I}_f$ is not a proper summable ideal, then $\operatorname{Aut}(\mathcal{P}(\mathbb{N})/\mathcal{I}_f)$ has a subgroup isomorphic to $\operatorname{Aut}(\mathcal{P}(\mathbb{N})/\operatorname{Fin})$, and it is easy to modify the above construction to prove that this group has a subgroup isomorphic to the free Abelian group with continuum many generators.

To prove the second part, instead of a family of pairwise almost disjoint subsets of $\mathbb{N}$ use an independent family of subsets of $\mathbb{N}$ (e.g., [54, Proposition 9.2.5]). The proof that the corresponding automorphisms generate a free group is analogous to the proof of the first part. □

For $f \colon \mathbb{N} \to \mathbb{R}_+$ and $0 \leq p < q \leq \infty$ we write

$$A_{f,\pi}[p,q] = \{n : p \leq f(\pi(n))/f(n) \leq q\}.$$

For a function $f \colon \mathbb{N} \to \mathbb{R}_+$ such that $\mu_f(\mathbb{N}) = \infty$, consider the following two subgroups of the infinite symmetric group $S_\infty$ ($\mathcal{I}_f^*$ is the filter dual to $\mathcal{I}_f$).

$$G_f = \{\pi \in S_\infty : A_{f,\pi}[p,q] \in \mathcal{I}_f \text{ for some } 0 < p \leq q < \infty\}.$$
$$H_f = \{\pi \in S_\infty : \{n : f(\pi(n)) = f(n)\} \in \mathcal{I}_f^*.$$

Obviously, $H_f$ is a normal subgroup of $G_f$.

**Theorem 2.9.5.** *If $\mathcal{I}_f$ is a summable ideal not RK-isomorphjic to* Fin*, then $\operatorname{Aut}_{\mathrm{RK}}(\mathcal{I})$ is isomorphic to $G_f/H_f$.*

Proof. By Lemma 2.6.8, some $\pi \in S_\infty$ is an RK-automorphism of $\mathcal{I}_f$ if and only if it is in $G_f$ and every RK-automorphism of $\mathcal{I}_f$ is of the form $\Lambda(\pi)$ for some $\pi$ in $G_f$.

It remains to prove that $\pi$ is equal to the identity modulo $\mathcal{I}_f$ if and only if it belongs to $H_f$. The reverse inclusion is obvious. To prove the direct inclusion, assume towards contradiction that $\pi \notin H_f$ is not in the kernel. Therefore, the set $A = \{n : \pi(n) \neq n\}$ is not in $\mathcal{I}_f$. By [54, Lemma 9.4.5] there is a partition of $A$ into sets $A_j$, for $j < 6$, such that $\pi[A_j]$ is disjoint from $A_j$ for all $j$. At least one of these sets is not in $\mathcal{I}_f$, and it is moved by $\pi$. This contradicts the assumption that $\Lambda(\pi)$ is equal to the identity modulo $\mathcal{I}_f$. □

**Theorem 2.9.6.** *There is a summable ideal $\mathcal{I}_f$ with the following properties.*

(1) $\mathrm{Aut}_{\mathrm{RK}}(\mathcal{I}_f)$ *is isomorphic to a quotient of the group* $\prod_n S_{n!}$.
(2) *The only ideal $\mathcal{J} \supseteq \mathcal{I}_f$ RK-isomorphic to $\mathcal{I}_f$ is $\mathcal{I}_f$.*

Proof. Let $f$ be the nonincreasing function uniquely determined by the following conditions:

(i) $\mathrm{range}(f) = \{1/n! : n \in \mathbb{N}\}$,
(ii) the set $I(n) = \left\{ i : f(i) = \frac{1}{n!} \right\}$ is an interval in $\mathbb{N}$, and
(iii) $\mu_f(I(n)) = 1$ for all $n$.

For (1) it suffices to prove that for every RK-automorphism $h$ of $\mathcal{I}_f$ there is an RK-automorphism $h'$ of $\mathcal{I}$ such that $h' \in \prod_n S_{I(n)}$ and $\{j : h(j) \neq h'(j)\}$ belongs to $\mathcal{I}$. Lemma 2.6.8 implies that there are $p, q$ be such that the set

$$A[p,q] = \{n : p \leq \mu_g(h^{-1}(\{n\}))/\mu_f(\{n\}) \leq q\} \tag{2.16}$$

belongs to $\mathcal{I}_f^*$. Fix $\bar{k} > \max(q, 1/p)$ and $\bar{n}$ such that

$$f(\bar{n}-1) = \frac{1}{\bar{k}!} \;\;\&\;\; f(\bar{n}) = \frac{1}{(\bar{k}+1)!}.$$

Then all distinct $ij$ greater than $\bar{k}$ satisfy $\pi[I_i] \cap I_j = \emptyset$. Therefore, the restriction of $\tau$ to $A_{f,\tau} \setminus (\bar{n}+1)$ can be extended to $h' \in \prod_{n=1}^{\infty} S_{I_n}$. This $h'$ is as required and this completes the proof that $\mathrm{Aut}_{\mathrm{RK}}(\mathcal{I})$ is isomorphic to a quotient of $\prod_n S_{n!}$.

(2) Suppose that $h \colon \mathbb{N} \to \mathbb{N}$ is an RK-reduction of some $\mathcal{J} \supseteq \mathcal{I}_f$ to $\mathcal{I}_f$ that is also a RK-isomorphism. Since $\mathcal{I}_f$ contains an infinite subset of $\mathbb{N}$, we may assume that $h$ is a permutation of $\mathbb{N}$. Since $\mathcal{J} \supseteq \mathcal{I}_f$, Lemma 2.6.8 implies that there is $q < \infty$ such that the set

$$A(\cdot, q] = \left\{ i : \frac{f(\pi(i))}{f(i)} \leq q \right\}$$

belongs to $\mathcal{I}_f^*$. By the same lemma, in order to prove that $\mathcal{J} = \mathcal{I}_f$, it suffices to prove that the set

$$A[1/q, \cdot) = \left\{ i : \frac{f(\pi(i))}{f(i)} \geq \frac{1}{q} \right\}$$

belongs to $\mathcal{I}_f^*$. Otherwise, the set $A[1/q,\cdot) \notin \mathcal{I}_f$, and $\mu_f(A[1/q,\cdot)) = \infty$. Fix $\bar{n} \geq q/(q-2)$ and $k$ such that

$$f(\bar{n}-1) = \frac{1}{k!}, \qquad f(\bar{n}) = \frac{1}{(k+1)!}$$

and

$$\mu_f(\pi[A(q,\cdot)] \setminus \bar{n})) < 1.$$

Find $t \subseteq A(\cdot, q] \setminus (\bar{n}+1)$ such that $\mu_f(t) > \bar{n}$, and fix $\bar{m} > \max(t)$ and $k'$ such that

$$\frac{1}{k'!} = f(\bar{m}) > f(\bar{m}+1) = \frac{1}{(k'+1)!}.$$

Let

$$s = \{i \in [\bar{n}, \bar{m}] : f(i) \neq f(\pi^{-1}(i))\},$$

and note that all $l \in [k, k']$ satisfy $|I_l \cap s| \geq |I_l \cap t|$ because $h$ is a permutation. In particular, $\mu_f(s) \geq \mu_f(t)$. Also, since all $i \in t$ satisfy $f(\pi(i)) \leq f(i)/k$, we have

$$\mu_f(h[t]) \leq \frac{1}{k}\mu_f(t) < \frac{1}{q}\mu_f(t).$$

Similarly, since $f(h(i)) < f(i)/k$ for all $i \in (\bar{n}+1)$ such that $f(i) > \bar{n}$, we have

$$\mu_f((h[\bar{n}+1]) \cap [\bar{n}, \infty)) \leq \frac{\bar{n}}{k}.$$

Therefore,

$$\mu_f(s \setminus (h[\bar{n}+1] \cup t)) > \mu_f(t) - \frac{1}{q}\mu_f(t) - \frac{\bar{n}}{q} > \bar{n}\frac{q-2}{q} > 1.$$

As every $i \in s \setminus (\pi''(\bar{n}+1) \cup t)$ satisfies $f(h^{-1}(i)) > i$, since $f(i) \leq 1/(k+1)!$ for all $i \in s$, we have $s \subseteq h[A(q, \cdot)]$ and therefore $\mu_f(h[A(q, \cdot)]) \setminus (\bar{n}+1)) > 1$, contradicting the choice of $\bar{n}$. $\square$

CHAPTER 3

# Large hereditary sets

This Chapter is not an end in itself. It covers material, much of it well-known, needed in proofs of our main lifting theorems in Chapters 4, 6, and 9, and the readers not interesting in results beyond ZFC can go straight to §4. The latter Chapter contains results used in the former that require basics of nonmeager hereditary sets.

The characterisation of (co-)meagre subsets of $\mathcal{P}(\mathbb{N})$ given in Theorem 3.1.3 is a folklore[1] fact. Its use in analysis of nonmeager ideals on $\mathbb{N}$ goes back to [**87**] and [**149**], and it was adapted to nonmeager hereditary sets in [**40**]. §3.2 is devoted to nonmeagre hereditary sets. Another largeness property of hereditary subsets of $\mathcal{P}(\mathbb{N})$ used in our proofs discussed in and dating back to [**40**] is that of being *ccc over* Fin, i.e., having nontrivial intersection with every uncountable almost disjoint family. An apparently more precise version requires nontrivial intersections with uncountable tree-like families or perfect tree-like families (§3.3.1). In §3.4 we study the following question of coherence of (partial) liftings. Given is a homomorphism and two continuous maps, each one of which is a lifting to this homomorphism on a large hereditary set. How different can these maps be? The Chapter ends with §3.5 where we prove a well-known fact that a homomorphism has a Baire measurable lifting if and only if it has a continuous one and some specific variations of it.

## 3.1. Property of Baire

The main result of this section, Theorem 3.1.3, is the standard characterisation of subsets of $\mathcal{P}(\mathbb{N})$ that are (co)meagre in the Cantor set topology.

A topological space is *Polish* if it is separable and completely metrisable.

**Definition 3.1.1.** Assume $\mathcal{X}$ is a Polish space and $A \subseteq \mathcal{X}$.

(1) $A$ is said to be *meagre* if it can be covered by a countable union of nowhere dense sets.
(2) $A$ is said to be *comeagre* if its complement is meagre.
(3) $A$ has the *Property of Baire* if there is an open $U \subseteq \mathcal{X}$ such that $A \Delta U$ is meagre.

We will need a well-known characterisation of comeagre subsets of $\mathcal{P}(\mathbb{N})$ (Theorem 3.1.3) and its lesser known little brother, Lemma 3.1.2. If $I \Subset \mathbb{N}$ and $s \subseteq I$, then we write

$$[I, s] = \{a \subseteq \mathbb{N} : a \cap I = s\}.$$

The sets of this form comprise a basis for the compact metric topology on $\mathcal{P}(\mathbb{N})$.

**Lemma 3.1.2.** *For $A \subseteq \mathcal{P}(\mathbb{N})$ the following are equivalent.*

*(1) $A$ is nowhere dense in $\mathcal{P}(\mathbb{N})$.*

[1] With apologies to whoever proved it first.

(2) *For every $n$ there are $I \Subset [n, \infty)$ and $s \subseteq I$ such that $[I, s] \cap A = \emptyset$.*

(3) *For every $n$ there are $s_j \subseteq I_j \Subset \mathbb{N}$ for $j < n$, such that $n < \min(I_0)$ and $\max(I_j) < \min(I_{j+1})$ for all $j < n-1$ and $A \cap \bigcup_{j<n}[I_j, s_j] = \emptyset$.*

*If $D \subseteq \mathcal{P}(\mathbb{N})$ is dense, then (1)–(3) are also equivalent for every $A \subseteq D$.*

Proof. Fix a dense $D \subseteq \mathcal{P}(\mathbb{N})$ (possibly $D = \mathcal{P}(\mathbb{N})$) and $A \subseteq \mathcal{P}(\mathbb{N})$.

(1) implies (2): Assume $A$ is nowhere dense. Fix $n$ and let $u_j$, for $j < 2^n$, enumerate $\mathcal{P}(n)$. We will recursively choose $k_j$ and $v_j \subseteq [k_j, k_{j+1})$ so that $k_0 = n$ and the following holds. With

$$U_j = [n, u_j] \cap \bigcap_{i<j}[[k_i, k_{i+1}), v_i]$$

we will choose $k_{j+1}$ and $v_j$ so that $U_j \cap [[k_j, k_{j+1}), v_j] \cap A = \emptyset$. This is possible because $U_j$ is an open set and $A$ is nowhere dense, thus there is a nonempty open $W \subseteq U_j$ disjoint from $A$. Every nonempty open subset of $U_j$ has one of this form. This describes the construction, and the sets $I = [k_0, k_n)$ and $s = \bigcup_{j<n} v_j$ satisfy $[n, u_j] \cap [I, s] \cap A = \emptyset$ for all $j < 2^n$. Since $\bigcup_{j<2^n}[n, u_j] = \mathcal{P}(\mathbb{N})$, the set $[I, s]$ is as required.

(2) implies (3): Fix $A$ and assume that it satisfies (2). We will find $I_j$ and $s_j$ recursively. (1) asserts that a pair $I_0, s_0$ as required exists. If $I_j, s_j$ for $j < m < n-1$ had been found, applying (1) with $n = \max(I_{m-1})$ gives $I_m, s_m$ as required.

(3) trivially implies (2).

(2) implies (1): Fix $A \subseteq \mathcal{P}(\mathbb{N})$ that satisfies (3). In order to show that $A$ is nowhere dense, it suffices to prove that for every nonempty basic open set $V \subseteq \mathcal{P}(\mathbb{N})$ has a nonempty basic open subset $W$ that avoids $A$. Thus $V = [J, v]$ for some $J$ and $v$. Applying (2) with $n = \max(J)$ we obtain a basic open set $W = [J, v] \cap [I, s]$ disjoint from $A$. □

**Theorem 3.1.3.** *A subset $\mathcal{X}$ of $\mathcal{P}(\mathbb{N})$ is relatively comeagre in $\mathcal{P}(\mathbb{N})$ if and only if there are disjoint finite intervals $I_j \Subset \mathbb{N}$ and $s_j \subseteq I_j$, for $j \in \mathbb{N}$, such that*

$$\bigcap_m \bigcup_{j \geq m}[I_j, s_j] \subseteq \mathcal{X}.$$

*If such intervals exist, they can be chosen so that $\bigcup_j I_j = \mathbb{N}$.*

Proof. For the converse implication, suppose $I_j$ and $s_j$ are as stated. Since every basic open set has the form $[I, s]$, the open set $U_m := \bigcup_{j \geq m}[I_j, s_j]$ is dense. Therefore $\bigcap_m U_m$ is, by the Baire Category Theorem, comeagre.

For the direct implication, assume $\mathcal{X}$ is comeagre and fix dense open $U_n \subseteq \mathcal{P}(\mathbb{N})$ such that $\bigcap_n U_n \subseteq \mathcal{A}$. By replacing $U_n$ with $\bigcap_{j \leq n} U_j$, we may assume that $U_n \supseteq U_{n+1}$ for all $n$. By Lemma 3.1.2 we can recursively choose $I_n \Subset \mathbb{N}$ and $s_n \subseteq I_n$ so that $\min(I_{n+1}) = \max(I_n) + 1$ and $[I_n, s_n] \subseteq U_n$ for all $n$.

This describes the construction of the sequence $I_n$, $s_n$. If $a \restriction I_n = s_n$ for infinitely many $n$, then $a \in U_n$ for infinitely many $n$. Since the sets $U_n$ are decreasing, $a \in \bigcap_n U_n \subseteq \mathcal{A}$, as required. By the construction we have $\bigcup_n I_n = \mathbb{N}$. □

**Corollary 3.1.4.** *If $\mathcal{X} \subseteq \mathcal{P}(\mathbb{N})$ is comeagre, then there are a partition $\mathbb{N} = A_0 \sqcup A_1$ and sets $C_0 \subseteq A_0$ and $C_1 \subseteq A_1$ such that for every $X \subseteq \mathbb{N}$ both $(X \cap A_0) \cup C_1$ and $(X \cap A_1) \cup C_0$ belong to $\mathcal{X}$.*

Proof. Let $I(n)$ and $s(n)$ be as in Theorem 3.1.3. Then the sets

$$A_0 = \bigcup_{n \text{ even}} I(n), \quad A_1 = \bigcup_{n \text{ odd}} I(n), \quad C_0 = \bigcup_{n \text{ even}} s(n), \quad C_1 = \bigcup_{n \text{ odd}} s(n),$$

are as required. □

## 3.2. Nonmeagre hereditary sets

In this section we prove a variation of the Jalali–Naini and Talagrand characterisation of nonmeager hereditary subsets of $\mathcal{P}(\mathbb{N})$ dating back to [40] (Theorem 3.2.2) and some of its consequence whose value will become apparent only once we start stabilising and comparing liftings.

**Definition 3.2.1.** If $\mathcal{X}$ is a family of subsets of $\mathbb{N}$ by $\hat{\mathcal{X}}$ we denote the *hereditary closure* of $\mathcal{X}$,

$$\hat{\mathcal{X}} = \bigcup_{a\in X} \mathcal{P}(a).$$

Some $\mathcal{X} \subseteq \mathcal{P}(\mathbb{N})$ is called *hereditary* if $\mathcal{X} = \hat{\mathcal{X}}$.

**Theorem 3.2.2.** *A hereditary subset $\mathcal{X}$ of $\mathcal{P}(\mathbb{N})$ is nonmeagre if and only if for every sequence $s_i$, for $i \in \mathbb{N}$, of disjoint finite subsets of $\mathbb{N}$ there is an infinite $a \subseteq \mathbb{N}$ such that $\bigcup_{i\in a} s_i \in \mathcal{X}$.*

Proof. Theorem 3.1.3 implies that $\mathcal{X}$ is meagre if and only if there are disjoint intervals $I(n)$ and $s(n) \subseteq I(n)$, for $n \in \mathbb{N}$, such that the set $\bigcap_m \bigcup_{n\geq m}[I(n), s(n)]$ is disjoint from $\mathcal{X}$. □

The part of the following concerned with ideals is taken from [121]).

**Corollary 3.2.3.** *If $\mathcal{H}$ is a hereditary subset of $\mathcal{P}(\mathbb{N})$, then $\mathcal{H}$ is meagre if and only if there is a finite-to-one function $h\colon \mathbb{N} \to \mathbb{N}$ such that $A \mapsto h^{-1}(A)$ sends infinite sets to $\mathcal{H}$-positive sets.*

*If $\mathcal{I} \supseteq \mathbb{N}$ is an ideal on $\mathbb{N}$, then $\mathcal{I}$ is meagre if and only if* $\mathrm{Fin} \leq_{\mathrm{RB}} \mathcal{I}$.

*In particular, if $\mathcal{I}$ is an analytic ideal such that $\mathcal{I} \supseteq \mathrm{Fin}$ then $\mathrm{Fin} \leq_{\mathrm{RB}} \mathcal{I}$ and $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ embeds into $\mathcal{P}(\mathbb{N})/\mathcal{I}$.*

Proof. The first part is a reformulation of Theorem 3.2.2 and the second part is a special case. For the last part, assume $\mathcal{I}$ is analytic and includes Fin. It suffices to show that $\mathcal{I}$ is meager. Assume otherwise. Since analytic sets have the Property of Baire, there is a basic open subset $U$ of $\mathcal{P}(\mathbb{N})$ such that $\mathcal{I} \cap U$ is relatively comeager. As $\mathcal{I} \supseteq \mathrm{Fin}$, $\mathcal{I}$ is comeager in $\mathcal{P}(\mathbb{N})$. Since $A \mapsto \mathbb{N} \setminus A$ is an homeomorphism of $\mathcal{P}(\mathbb{N})$ onto itself, there is $A \in \mathcal{I}$ such that $\mathbb{N} \setminus A \in \mathcal{I}$, contradicting $\mathcal{I}$ being a proper ideal. □

Hereditary nonmeager sets behave differently from the ideals. For example, if $\mathcal{H}$ is hereditary and includes Fin, it not necessarily closed under finite modifications of its elements.

**Corollary 3.2.4.**
1. *The family of nonmeagre hereditary subsets of $\mathcal{P}(\mathbb{N})$ is closed under taking finite intersections of its elements.*
2. *The family of nonmeagre hereditary subsets of $\mathcal{P}(\mathbb{N})$ which are closed under finite modifications of their elements is closed under taking countable intersections.*

Proof. (1) Fix $k \geq 1$ and assume $\mathcal{H}_i$, for $i < k$, are nonmeagre and hereditary. Since $\mathcal{H} = \bigcap_{i<k} \mathcal{H}_i$ is hereditary, it suffices to prove that it is nonmeagre. Otherwise, by Corollary 3.2.3 there is a finite-to-one $h\colon \mathbb{N} \to \mathbb{N}$ such that $h^{-1}(A) \notin \mathcal{H}$ for all infinite $A$. By repeatedly applying Theorem 3.2.2 one finds infinite sets $A_0 \supseteq A_1 \supseteq \cdots \supseteq A_k$ such that $\bigcup_{j\in A_i} s_j \in \mathcal{H}_i$ for each $i < k$. $i = 1, \dots, k$. Then $A_{k-1} \in \mathcal{H}$; contradiction.

(2) Assume $\mathcal{H}_i$, for $i \in \mathbb{N}$, are nonmeagre and hereditary but $\mathcal{H} = \bigcap_{i\in\mathbb{N}} \mathcal{H}_i$ is meagre. By Corollary 3.2.3 there is a finite-to-one $h\colon \mathbb{N} \to \mathbb{N}$ such that $h^{-1}(A) \notin \mathcal{H}$ for all infinite $A$. Choose a decreasing sequence $A_i \subseteq \mathbb{N}$ such that $h^{-1}(A_i) \in \mathcal{H}_i$ for all $i$. Let $A \subseteq \mathbb{N}$ be infinite and such that $A \setminus A_i$ is finite for all $i$. Then $h^{-1}(A) \setminus h^{-1}(A_i)$ is finite for all $i$, hence $h^{-1} \in \mathcal{H}$; contradiction. □

**Corollary 3.2.5.** *Suppose that $\mathcal{H}$ is a hereditary nonmeagre subset of $\mathcal{P}(\mathbb{N})$.*

*(1) if $A \subseteq \mathbb{N}$ is infinite, then $\mathcal{H} \cap \mathcal{P}(A)$ is a hereditary relatively nonmeagre subset of $\mathcal{P}(A)$*

*(2) If $A \subseteq \mathbb{N}$ is infinite and co-infinite and $s_n$, for $n \in \mathbb{N}$, are disjoint finite subsets of $\mathbb{N} \setminus A$, then the set*

$$\{X \cap A : X \in \mathcal{H}, (\exists^\infty n) s_n \subseteq X\} \tag{3.1}$$

*is hereditary and relatively nonmeagre subset of $\mathcal{P}(A)$.*

*(3) If $\mathbb{N} = A_0 \sqcup A_1$ is a partition into infinite sets and $\mathcal{H}_j$ is a hereditary relatively nonmeagre subset of $\mathcal{P}(A_j)$ for $j < 2$, then $\mathcal{H}_0 \sqcup \mathcal{H}_1$ is a hereditary nonmeagre subset of $\mathcal{P}(\mathbb{N})$.*

Proof. (1) Clearly $\mathcal{H} \cap \mathcal{P}(A)$ is hereditary. Assume $\mathcal{H} \cap \mathcal{P}(A)$ is meagre. By Theorem 3.2.2 there are finite disjoint subsets $s(n)$ of $A$ such that $\bigcup_{n\in c} s(n) \notin \mathcal{H} \cap \mathcal{P}(A)$ for every infinite $c$. By the same theorem, $\mathcal{H}$ is meagre; contradiction.

(2) The set in (3.1) is hereditary because $\mathcal{H}$ is, and we can use Theorem 3.2.2 to prove that it is nonmeagre. Let $t_n$, for $n \in \mathbb{N}$, be disjoint finite subsets of $A$. Then $t_n \cup s_n$, for $n \in \mathbb{N}$, are disjoint finite subsets of $\mathbb{N}$. Theorem 3.2.2 implies that there is an infinite $Z \subseteq \mathbb{N}$ such that $X = \bigcup_{n\in Z} t_n \cup s_n$ belongs to $\mathcal{H}$. Then $X \cap A$ belongs to the set in (3.1). Since the sequence $t_n$ was arbitrary, Theorem 3.2.2 applied in $\mathcal{P}(A)$ implies that this set is a relatively nonmeagre subset of $\mathcal{P}(A)$.

(3) Clearly $\mathcal{H}_0 \sqcup \mathcal{H}_1 = (\mathcal{H}_0 \sqcup \mathcal{P}(A_1)) \cap (\mathcal{P}(A_0) \sqcup \mathcal{H}_1)$ and the two sets on the right-hand side are hereditary and nonmeager by the Kuratowski–Ulam theorem (Theorem A.1.5). The conclusion follows by Corollary 3.2.4. □

**Corollary 3.2.6.** *Suppose that and $\mathcal{X} \subseteq \mathcal{P}(\mathbb{N})$ is comeagre.*

*(1) If $\mathcal{H} \subseteq \mathcal{P}(\mathbb{N})$ is hereditary and nonmeagre, then there is a partition $\mathbb{N} = A_0 \sqcup A_1$ such that the set*

$$\{(X \cap A_0) \cup (Y \cap A_1) : X \in \mathcal{H} \cap \mathcal{X}, Y \in \mathcal{H} \cap \mathcal{X}\} \tag{3.2}$$

*is a hereditary and nonmeagre subset of $\mathcal{H}^2$.*

*(2) If $\mathcal{J}$ is a nonmeagre ideal on $\mathbb{N}$, then there are a partition $\mathbb{N} = A_0 \sqcup A_1$ and sets $C_0 \subseteq A_0$ and $C_1 \subseteq A_1$ such that for every $X \in \mathcal{J}$ both $(X \cap A_0) \cup C_1$ and $(X \cap A_1) \cup C_0$ belong to $\mathcal{J} \cap \mathcal{X}$.*

Proof. By Theorem 3.1.3 there are finite disjoint $I(n) \subseteq \mathbb{N}$ and $s(n) \subseteq I(n)$, for $n \in \mathbb{N}$, such that if $X \cap I(n) = s(n)$ for infinitely many $n$, then $X \in \mathcal{X}$. We may assume that $\bigcup_n I(n) = \mathbb{N}$ by enlarging these sets if necessary (the resulting set $\bigcap_n \bigcup_{m\geq n} [I(m), s(m)]$ is still dense $G_\delta$).

(1) Let $A_0 = \bigcup_n I(2n)$ and $A_1 = \bigcup_n I(2n+1)$. By Corollary 3.2.5 (2), the set $\{X \cap A_0 : X \in \mathcal{H}, (\exists^\infty n) I(2n+1) \subseteq X\}$ is hereditary and relatively nonmeagre subset of $\mathcal{P}(A_0)$. Since $\mathcal{H}$ is hereditary,

$$\{X \cap A_0 : X \in \mathcal{H}, (\exists^\infty n) X \cap I(n) = s(n)\}$$

is a hereditary and relatively nonmeagre in $\mathcal{P}(A_0)$. By an analogous argument, $\{X \cap A_1 : X \in \mathcal{H}, (\exists^\infty n) X \cap I(2n) = s(2n)\}$ is hereditary and nonmeagre in $\mathcal{P}(A_1)$, and the conclusion follows by Corollary 3.2.5 (3).

(2) By Theorem 3.2.2 there is an infinite $D \subseteq \mathbb{N}$ such that $\bigcup_{n\in d} s(n) \in \mathcal{J}$. Let $D = D_0 \cup D_1$ be a partition of $D$ into two infinite sets. Let

$$A_0 = \bigcup_{n\in D_0} I(n), \quad A_1 = \mathbb{N} \setminus A_0, \quad C_0 = \bigcup_{n\in D_0} s(n), \quad C_1 = \bigcup_{n\in D_1} s(n).$$

If $X \in \mathcal{J}$ then, since $\mathcal{J}$ is an ideal, both $Y_0 = (X \cap A_0) \cup C_1$ and $Y_1 = (X \cap A_1) \cup C_0$ belong to $\mathcal{J}$. Also, $Y_j \in \bigcap_{n\in d_j} [I(n), s(n)]$ for $j = 0, 1$, hence both $Y_0$ and $Y_1$ belong to $\mathcal{X}$, as required. □

The following simple consequence of Theorem 3.2.2 is used in the proof of our main lifting theorem which replaces thirty pages of [40, §3.8–§3.13].

**Lemma 3.2.7.** *If $\mathcal{H} \subseteq \mathcal{P}(\mathbb{N})$ is hereditary and nonmeagre, then there exists $k$ such that for every $s \Subset [k,\infty)$ the set*

$$\mathcal{H}[s] = \{a \subseteq \mathbb{N} : s \cup a \in \mathcal{H}\}$$

*is hereditary and nonmeagre. In particular, the intersection of $\mathcal{H}$ with every non-empty open subset of $\mathcal{P}([k,\infty))$ is relatively nonmeagre, and the set $\mathcal{H} \uplus \mathcal{P}(k)$ is hereditary, nonmeagre, and closed under finite changes of its elements.*

Proof. This set is clearly hereditary for every $k$. Assume it is meagre for all $k$. By Theorem 3.2.2, we can recursively find an increasing sequence $k(j)$ and $s(j) \subseteq [k(j), k(j+1))$ such that $\mathcal{H}[s(j)]$ is meagre for all $j$.

Thus $\bigcup_j \mathcal{H}[s(j)]$ is a meagre hereditary set, and by Theorem 3.2.2 there is a sequence of disjoint finite subsets of $\mathbb{N}$, $t(j)$, for $j \in \mathbb{N}$, such that the set

$$\{a \subseteq \mathbb{N} : (\exists^\infty j) t(j) \subseteq a\}$$

is disjoint from $\bigcup_j \mathcal{H}[s(j)]$. Recursively find increasing sequences $m(j), n(j)$ such that $u(j) = s(m(j)) \cup t(n(j))$ are pairwise disjoint.

By Theorem 3.2.2 there is an infinite $X \subseteq \mathbb{N}$ such that $a = \bigcup_{j\in X} v(j) \in \mathcal{H}$. Fix $j \in X$. Then $a \in \mathcal{H}[s(j)]$ and also $a \notin \bigcup_i \mathcal{H}[s(i)]$; contradiction.

For the second claim, it suffices to prove that the intersection of $\mathcal{H}$ with every basic open subset of $\mathcal{P}([k,\infty))$ is relatively nonmeagre. Since $\mathcal{H}$ is hereditary, this is equivalent to $\mathcal{H}[s]$ being nonmeagre for every $s \Subset [k,\infty)$. □

### 3.3. Ccc over Fin and tree-like almost disjoint families

A largeness property of ideals and hereditary sets, being ccc over Fin, is stronger than nonmeagerness. In §3.3.1 we discuss tree-like almost disjoint families and perfect tree-like almost disjoint families.

If $\mathcal{H}$ is hereditary in $\mathcal{P}(\mathbb{N})$, then a subset of $\mathcal{P}(\mathbb{N})$ is *$\mathcal{H}$-positive* if it does not belong to $\mathcal{H}$. Two sets of integers are *almost disjoint* if their intersection is finite. The following was defined for ideals in [40].

**Definition 3.3.1.** A hereditary set $\mathcal{H}$ is *ccc over* Fin if there is no uncountable family of $\mathcal{H}$-positive, pairwise disjoint modulo Fin, sets.

Since there is an uncountable family of almost disjoint subsets of $\mathbb{N}$, Fin is not ccc over Fin.

**Lemma 3.3.2.** *If a hereditary family $\mathcal{H}$ is ccc over* Fin *then it is nonmeagre.*
*If $\mathcal{I}$ and $\mathcal{J}$ are ideals, $\mathcal{I} \leq_{\mathrm{RB}} \mathcal{J}$, and $\mathcal{J}$ is ccc over* Fin*, then so is $\mathcal{I}$.*

Proof. For the first part, suppose that $\mathcal{H}$ is hereditary and meagre. By Corollary 3.2.3 there is a finite-to-one $h\colon \mathbb{N} \to \mathbb{N}$ such that $h^{-1}(A)$ is $\mathcal{H}$-positive for every infinite $A \subseteq \mathbb{N}$. Therefore the $h$-preimage of any uncountable family of almost disjoint subsets of $\mathbb{N}$ shows that $\mathcal{H}$ is not ccc over Fin. This argument also proves the second part. □

Suppose that $\mathcal{I}$ is an ideal on $\mathcal{P}(\mathbb{N})$. While '$\mathcal{I}$ is ccc over Fin" is stronger than "$\mathcal{I}$ is nonmeagre', it is, in general, weaker than "$\mathcal{P}(\mathbb{N})/\mathcal{I}$ is ccc" (Example 5.4.3). Moreover, David Fremlin observed that while a ccc over Fin ideal has to be nonmeagre, there exist a ccc over Fin ideals of Lebesgue measure zero (see [69, page 17]). Also, there are nonmeagre ideals that are not ccc over Fin.

**Remark 3.3.3.** Nonmeagre, hereditary sets closed under finite changes of their elements are also called *groupwise dense*, and the minimal cardinality of the family of groupwise dense sets whose intersection is not groupwise dense is called *groupwise density*. These notions were introduced in [11] in order to provide a succinct combinatorial formulation of a statement with applications to diverse mathematical structures (see [9]). For more on cardinal characteristics of the continuum see [10].

**3.3.1. Tree-like almost disjoint families.** Infinite sets $A$ and $B$ of integers are *almost disjoint* if their intersection is finite, and $A$ *almost includes* $B$, $A \supseteq^* B$, if $B \setminus A$ is finite.

For $s$ in $\{0,1\}^{<\mathbb{N}}$ or in $\mathbb{N}^{<\mathbb{N}}$ let $|s|$ denote its length. Since we use the convention that $n = \{0, \dots, n-1\}$, $|s|$ also denotes the domain of $s$.

The following notion, introduced in [162] and [94] (but see [140, II.3.8 and II.4]), will be very useful in setting up an open partition (see Definition 6.3.3).

**Definition 3.3.4.** A family $\mathcal{A}$ of almost disjoint sets of integers is *tree-like* if there is an ordering $\prec$ on its domain $\mathcal{D} = \bigcup \mathcal{A}$ such that $\langle \mathcal{D}, \prec \rangle$ is a tree of height $\omega$ and each element of $\mathcal{A}$ is included in a unique maximal branch of this tree.

Tree-like almost disjoint families appeared implicitly in Shelah's proof that all automorphisms of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ are trivial ([139]) and explictily under the name of neat almost disjoint families in [162]. They were studied further in [94].

The following lemma will not be used explicitly, and proving it may help furnish the intuition.

**Lemma 3.3.5.** *For an almost disjoint family $\mathcal{A}$ the following are equivalent.*

*(1) $\mathcal{A}$ is tree-like.*
*(2) There is an injective $f\colon \mathbb{N} \to \{0,1\}^{<\mathbb{N}}$ such that the image of every $A \in \mathcal{A}$ is included in a branch of the tree $\mathbb{N}^{<\mathbb{N}}$, and for different $A$ and $B$ in $\mathcal{A}$ the corresponding branches are different.*
*(3) There is a finite-to-one $f\colon \mathbb{N} \to \{0,1\}^{<\mathbb{N}}$ such that the image of every $A \in \mathcal{A}$ is included in a branch of the tree $\mathbb{N}^{<\mathbb{N}}$, and for different $A$ and $B$ in $\mathcal{A}$ the corresponding branches are different.*

*Moreover, (2) and (3) are equivalent to their variants in which the range of $f$ is required to be a downwards closed subset $\mathbb{N}^{<\mathbb{N}}$.* □

**Definition 3.3.6** (Perfect tree-like almost disjoint family)**.** Suppose that $J(s)$, for $s \in \{0,1\}^{<\mathbb{N}}$, are nonempty pairwise disjoint nonempty finite subsets of $\mathbb{N}$. For $f \in \{0,1\}^{\mathbb{N}}$ let

$$J(f) = \bigcup \{J_{f \restriction n} : f \in \{0,1\}^{\mathbb{N}}\}.$$

Then $J(f) \cap J(g) = \bigcup_{s \sqsubseteq f \wedge g} J(s)$ for $f \neq g$. Therefore, the family

$$\mathcal{A}\{J_s\} = \{J(f) : f \in \{0,1\}^{\mathbb{N}}\}$$

is tree-like, and even a perfect subset of $\mathcal{P}(\mathbb{N})$.

Any family of the form $\mathcal{A}\{J(s)\}$ is called a *perfect tree-like almost disjoint family.*

Every perfect tree-like family is a tree-like almost disjoint family that is a perfect subset of $\mathcal{P}(\mathbb{N})$, and therefore of cardinality $\mathfrak{c}$. Not every subset of $\mathcal{P}(\mathbb{N})$ that is perfect and tree-like almost disjoint family satisfies the requirements of Definition 3.3.6, but this should not cause inconvenience.

The analogue of the following lemma for uncountable almost disjoint families, used in the proof of the OCA lifting theorem, uses Martin's Axiom (Lemma A.5.1).

**Lemma 3.3.7.** *If $\mathcal{A}$ is a perfect tree-like almost disjoint family, then there is a perfect tree-like almost disjoint family $\mathcal{B}$ such that every element of $\mathcal{B}$ includes $2^{\aleph_0}$ elements of $\mathcal{A}$.*

Proof. We have $\mathcal{A} = \mathcal{A}\{J_s\}$ for pairwise disjoint finite sets $J_s$, for $s$ in $\{0,1\}^{<\mathbb{N}}$. For $s \in \{0,1\}^n$ let

$$I_s = \bigcup \{J_t : 2n \leq |t| < 2(n+1), (\forall j < n) t(2j+1) = s(j)\}.$$

This is a family of finite, pairwise disjoint, sets. Let $\mathcal{B} = \mathcal{A}\{I_s\}$. Then every $f$ satisfies

$$I(f) = \bigcup_n I_{f \restriction n} = \bigcup \{J(g) : g(2j+1) = f(j) \text{ for all } j\}$$

and $\mathcal{B}$ is as required. □

## 3.4. Almost liftings

We move on to study coherence between two almost liftings of a given homomorphism.

**Lemma 3.4.1.** *Suppose that $\mathcal{I}$ is an analytic ideal, that $\Phi \colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})/\mathcal{I}$ is a homomorphism, and that $\mathcal{H}_j$ for $j = 0, 1$ is a nonmeagre hereditary subset of $\mathcal{P}(\mathbb{N})$ closed under finite changes of its elements.*

*(1) If $\Theta_i \colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ is a continuous lifting of $\Phi$ on $\mathcal{H}_i$ for $i = 0, 1$, then each $\Theta_i$ is a lifting of $\Phi$ on a relatively comeagre subset of $\mathcal{H}_0 \cup \mathcal{H}_1$.*

*(2) If $\mathcal{K}$ is an analytic approximation to $\mathcal{I}$ and $\Theta_i \colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ is a continuous $\mathcal{K}$-approximation to $\Phi$ on $\mathcal{H}_i$ for $i = 0, 1$, then each $\Theta_i$ is a $\mathcal{K}^2$-approximation to $\Phi$ on a relatively comeagre subset of $\mathcal{H}_0 \cup \mathcal{H}_1$.*

Proof. By taking $\mathcal{K} = \mathcal{I}$ and and noting that in this case $\mathcal{K}^2 = \mathcal{K}$, one sees that it suffices to prove the second part. Towards this, note that the set

$$\mathcal{Y} = \{A : \Psi_0(A) \Delta \Psi_1(A) \in \mathcal{K} \sqcup \text{Fin}\}$$

is Borel. By Theorem 3.2.2, $\mathcal{H} = \mathcal{H}_0 \cap \mathcal{H}_1$ is hereditary and nonmeagre. Since $\mathcal{Y}$ is analytic and includes $\mathcal{H}$, it is comeagre. Hence for a relatively comeagre set of $A$

in $\mathcal{H}_0$ we have that $\Theta_1(A)\Delta\Phi_*(A) \subseteq (\Theta_1(A)\Delta\Theta_0(A)) \cup (\Theta_0(A)\Delta\Phi_*(A))$ belongs to $\mathcal{K}^2 \mathbin{\underline{\cup}} \mathrm{Fin}$. Therefore $\Theta_1$ is a $\mathcal{K}^2$-approximation to $\Phi$ on a relatively comeagre subset of $\mathcal{H}_0$. Analogous proof shows that $\Theta_0$ is a $\mathcal{K}^2$-approximation to $\Phi$ on a relatively comeagre subset of $\mathcal{H}_1$, and the coclusion follows. □

The following coherence property is a more precise version of Lemma 3.4.1 (see also Claim 6.5.1 for the simpler case when $\mathcal{I} = \mathrm{Fin}$).

**Lemma 3.4.2.** *Suppose that $\mathcal{I}$ is an ideal, $\Phi\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})/\mathcal{I}$ is a homomorphism, $\mathcal{K}$ is a closed approximation to $\mathcal{I}$, and $\Psi_i$ is a continuous $\mathcal{K}$-approximation to $\Phi$ on a nonmeagre hereditary set $\mathcal{H}_i$, for $i = 0, 1$. Then there are $k$ and $m'$ in $\mathbb{N}$ such that all $A \subseteq [k, \infty)$ satisfy $(\Psi_0(A)\Delta\Psi_1(A)) \setminus m' \in \mathcal{K}^{10}$.*

Proof. By Theorem 3.2.2, $\mathcal{H} = \mathcal{H}_0 \cap \mathcal{H}_1$ is hereditary and nonmeagre. We claim that there is $m'$ such that all $A \in \mathcal{H}$ satisfy $(\Psi_0(A)\Delta\Psi_1(A)) \setminus m' \in \mathcal{K}^6$. Since both $\Psi_0$ and $\Psi_1$ are $\mathcal{K}$-approximations to $\Phi$, every $a \in \mathcal{H}$ satisfies $\Psi_0(A)\Delta\Psi_1(A) \in \mathcal{K}^2 \mathbin{\underline{\cup}} \mathrm{Fin}$. Let

$$g(A) = \min\{m : (\Psi_0(A)\Delta\Psi_1(A)) \setminus m \in \mathcal{K}^2\}.$$

By the continuity of $\Psi_0$ and $\Psi_1$, the set

$$\mathcal{X}_m = \{A \in \mathcal{H} : g(A) \leq m\}$$

is relatively closed in $\mathcal{H}$ for every $m$. Since $\mathcal{H}$ is nonmeagre, we can find $m$ large enough for $\mathcal{X}_m$ to have nonempty interior (relative to $\mathcal{H}$). Let $k$ and $u \subseteq k$ be such that the clopen set $U = \{A : A \cap k = u\}$ satisfies $U \cap \mathcal{H} \subseteq \mathcal{X}_m$. Then all $A \subseteq [k, \infty)$ satisfy

$$(\Psi_0(u \cup A)\Delta\Psi_1(u \cup A)) \setminus m \in \mathcal{K}^2. \tag{3.3}$$

By Lemma 3.2.7 and increasing $k$ if needed we may assume that $\mathcal{H} \cap \mathcal{P}([k, \infty))$ is dense in $\mathcal{P}([k, \infty))$.

Since $\Psi_0$ is a $\mathcal{K}$-approximation to the homomorphism $\Phi$ on $\mathcal{H}$, for every $A \in \mathcal{H}$ and every $s \in \mathrm{Fin}$ we have $\Psi_0(A)\Delta\Psi_0(A\Delta s)\Delta\Psi_0(s) \in \mathcal{K}^3 \mathbin{\underline{\cup}} \mathrm{Fin}$. Thus we can set

$$f_0(A) = \min\{l : (\Psi_0(A)\Delta\Psi_0(A\Delta s)\Delta\Psi_0(s)) \setminus l \in \mathcal{K}^3 \text{ for all } s \subseteq k\}.$$

Let $f_1\colon \mathcal{H} \to \mathbb{N}$ be the analogously defined function with $\Psi_0$ replaced with $\Psi_1$. The sets $\{A \in \mathcal{H} : f_j(A) > n\}$ for $j = 0, 1$ are, by the continuity of $\Psi_j$, relatively open for every $n$. Since $\mathcal{H}$ is nonmeagre,

$$\mathcal{Z} = \{a \in \mathcal{H} : \max(f_0(a), f_1(a)) \leq m'\}$$

has nonempty interior for some $m' \geq m$. We claim that $m'$ is as required. Fix $A \in \mathcal{H}$ such that $\min(A) \geq k$. Then (writing $x =^{l,\mathcal{L}} y$ if $(x\Delta y) \setminus l \in \mathcal{L}$ and applying (3.3) to $A$ and to $u \cup A = u\Delta A$ in the second equality)

$$\Psi_0(A) =^{m',\mathcal{K}^3} \Psi_0(u\Delta A)\Delta\Psi_0(s) =^{m',\mathcal{K}^4} \Psi_1(u\Delta A)\Delta\Psi_1(s) =^{m',\mathcal{K}^3} \Psi_1(A)$$

thus $(\Psi_0(A)\Delta\Psi_1(A)) \setminus m' \in \mathcal{K}^{10}$ as promised. Since $\mathcal{H} \cap \mathcal{P}([k, \infty))$ is dense in $\mathcal{P}([k, \infty))$ and both $\Psi_0$ and $\Psi_1$ are continuous, this holds for all $A \subseteq [k, \infty)$, as required. □

## 3.5. From Baire measurable to continuous liftings

Lemma 3.5.3 (1) below is well known (see [161, p. 132]) and it shows that another natural order coincides with $\leq_{\mathrm{BE}}$. Proposition 3.5.4 was tacitly assumed in [40] as obvious, we finally provide a (not completely trivial) proof.

Stabilisers were used to analyse and produce liftings implicitly in [139] and [162] and finally isolated by Just ([93], [94]). In [40] stabilisers were used indiscriminately. We will follow the elegant approach of [101] and [102].

**Definition 3.5.1.** Suppose that $F\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$, $\mathbb{N} = A_0 \sqcup A_1$, $C_0 \subseteq A_0$, and $C_1 \subseteq A_1$. Let

$$\Theta_0(B) = F((B \cap A_0) \cup C_1)\Delta\Phi_*(C_1),$$
$$\Theta_1(B) = F((B \cap A_1) \cup C_0)\Delta\Phi_*(C_0),$$
$$\Theta(B) = \Theta_0(B)\Delta\Theta_1(B).$$

Then $\Theta$ is said to be the *stabilisation of* $F$ associated to $A_0, A_1, C_0$, and $C_1$.

Stabilisations will be used in proofs of our lifting theorems (Theorem 4.1.2, Theorem 7.1.1). V. Kanovei and M. Reeken ([101]) and D.H. Fremlin ([69, Proposition 1C]) have proved an analogue of Lemma 3.5.3 (1) for Lebesgue-measurable liftings: Every homomorphism between two quotients $\mathcal{P}(\mathbb{N})/\mathcal{I}$ that has a Lebesgue-measurable lifting has a continuous lifting.

Approximations to an ideal were defined in Definition 1.2.1: A hereditary subset $\mathcal{K}$ of $\mathcal{P}(\mathbb{N})$ is an approximation to $\mathcal{I}$ if $\mathcal{I} \subseteq \mathcal{K} \mathbin{\underline{\cup}} \mathrm{Fin}$.

**Definition 3.5.2.** Assume that $\mathcal{I}$ is an ideal on $\mathbb{N}$, $\mathcal{K}$ is an approximation to $\mathcal{I}$, $\mathcal{X} \subseteq \mathcal{P}(\mathbb{N})$, and $F\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$. A map $\Gamma\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ is a *$\mathcal{K}$-approximation to $F$ on $\mathcal{X}$* if

$$\Gamma(A)\Delta F(A) \in \mathcal{K} \mathbin{\underline{\cup}} \mathcal{I}$$

for all $A \in \mathcal{X}$. If $\Phi$ is a homomorphism, then $F$ is a *$\mathcal{K}$-approximation to* $\Phi$ if it is a $\mathcal{K}$-approximation to a lifting of $\Phi$.

Lemma 3.5.3 (1) shows that a (group, or Boolean algebra) homomorphism between quotients $\mathcal{P}(\mathbb{N}/\mathcal{J}$ and $\mathcal{P}(\mathbb{N})/\mathcal{I}$ has a Baire measurable lifting if and only if it has a continuous lifting, as well as three other similar-sounding statements. The approximation $\mathcal{K}$ is not required to be closed or even Borel. Parts (3) and (4) of this lemma will be used to study almost liftings in the context of $\mathrm{OCA}_{\mathrm{T}}$ and MA($\sigma$-linked) in §6.

**Lemma 3.5.3.** *Suppose that $\mathcal{I}$ is an ideal on $\mathbb{N}$ with an approximation $\mathcal{K}$, and and $\Phi\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})/\mathcal{I}$ is a Boolean algebra homomorphism or a group homomorphism.*

1. *If $\Phi$ has a Baire-measurable lifting on a comeagre set, then it has a continuous lifting.*
2. *If $\Phi$ has a Baire measurable $\mathcal{K}$-approximation on a comeagre set for some approximation $\mathcal{K}$ to $\mathcal{I}$, then it has a continuous $\mathcal{K}^2$-approximation.*
3. *If $\Phi$ has a Baire-measurable lifting on a nonmeagre ideal $\mathcal{J}$, then it has a continuous lifting on $\mathcal{J}$.*
4. *If $\Phi$ has a Baire-measurable $\mathcal{K}$-approximation on a nonmeagre ideal $\mathcal{J}$, for some approximation $\mathcal{K}$ to $\mathcal{I}$, then it has a continuous $\mathcal{K}^2$-approximation on $\mathcal{J}$.*

Proof. (1) Let $\Phi_*\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ be a Baire measurable lifting of $\Phi$ on a comeagre set $\mathcal{T}$. Then the restriction of $\Phi_*$ to some dense $G_\delta$ subset $\mathcal{X}$ of $\mathcal{T}$ is continuous.

By Corollary 3.1.4 there are a partition $\mathbb{N} = A_0 \sqcup A_1$ and sets $C_0 \subseteq A_0$ and $C_1 \subseteq A_1$ such that for every $X \subseteq \mathbb{N}$ both $(X \cap A_0) \cup C_1$ and $(X \cap A_1) \cup C_0$ belong to $\mathcal{X}$. Let $\Theta$ be the stabilisation of $\Phi_*$ associated to $A_0, A_1, C_0$, and $C_1$.

It is continuous because $(B \cap A_\varepsilon) \cup C_{1-\varepsilon}$ is in $G$ for $\varepsilon = 0, 1$. It suffices to check that $\Theta$ is a lifting of $\Phi$ in case when the latter is a group homomorphism. For $B \subseteq A_\varepsilon$ we have $\Phi_*(B) =^{\mathcal{I}} \Phi_*(B \cup C_{1-\varepsilon}) \Delta \Phi_*(C_{1-\varepsilon})$, and for every $B \subseteq \mathbb{N}$ we have $B = (B \cap A_0) \Delta (B \cap A_1)$, and therefore $\Theta(B)^{\mathcal{I}} = \Theta_0(B \cap A_0) \Delta \Theta_1(B \cap A_1) =^{\mathcal{I}} \Phi_*(B)$. $\Psi$ is a lifting for $\Phi$.

Analogous argument proves (2).

(3) Suppose that $\Phi$ has a Baire-measurable lifting on a nonmeagre ideal $\mathcal{J}$. Corollary 3.2.6 asserts that there are a partition $\mathbb{N} = A_0 \sqcup A_1$ and sets $C_0 \subseteq A_0$ and $C_1 \subseteq A_1$ such that for every $X \in \mathcal{J}$ both $(X \cap A_0) \cup C_1$ and $(X \cap A_1) \cup C_0$ belong to $\mathcal{J} \cap X$. Therefore the construction analogous to that used in (1) and (2) with Corollary 3.2.6 replacing Corollary 3.1.4 proves (3).

The proof of (4) is analogous. □

**Proposition 3.5.4.** *Suppose that $\mathcal{I}$ is an analytic ideal and $\Phi\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})/\mathcal{I}$ is a homomorphism. The following are equivalent.*

*(1) $\Phi$ has a Baire-measurable almost lifting.*
*(2) $\Phi$ has a continuous almost lifting.*
*(3) $\Phi$ is decomposable.*

Proof. The implication from (1) to (3) is Lemma 3.5.3 (3) and the implications from (3) to (2) and from (2) to (1) are vacuous. □

CHAPTER 4

# Lifting theorems I: From Baire measurable to completely additive

In the present Chapter we are concerned with analysis of Baire measurable liftings of homomorphisms.

**Definition 4.0.1.** Suppose that $\Phi_*$ is a lifting of a homomorphism $\Phi\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})/\mathcal{I}$. Topological properties of $\Phi_*$, such as continuous, Borel, or Baire measurable, are defined with respect to the Cantor set topology on $\mathcal{P}(\mathbb{N})$.[1]

A lifting is called *completely additive* if there is a function $h\colon \mathbb{N} \to \mathbb{N}$ such that

$$\Phi_*(A) = h^{-1}(A)$$

for all $A \in \mathcal{P}(\mathbb{N})$.

Lemma 3.5.3 implies that a homomorphism from $\mathcal{P}(\mathbb{N})$ into $\mathcal{P}(\mathbb{N})/\mathcal{I}$ has a Baire measurable lifting if and only if it has a continuous one. Clearly a completely additive lifting is automatically continuous. This Chapter is devoted to analysis of the question when the existence of a continuous lifting implies the existence of one that is completely additive, and its results are summarised in Theorem 4.1.2. Ideals $\mathcal{I}$ that have the property that every homomorphism of $\mathcal{P}(\mathbb{N})$ into $\mathcal{P}(\mathbb{N})/\mathcal{I}$ with a Baire measurabe lifting has a completely additive one are studied in §4.1. In §4.2 we introduce the Fubini property and prove Fubini property of some ideals (Theorem 4.2.3). Characterisation of the Fubini property using Katětov order is given in §4.2.1. In Theorem 4.3.1 we prove that the Fubini property implies the Radon–Nikodym property. An ideal that fails the Radon–Nikodym property is constructed in Theorem 4.4.1.

While in [40] stabilisers introduced by Just had played key role in finding liftings, they are almost completely absent from the present proofs. Instead, we use ideas from [101] and [102].

## 4.1. The Radon–Nikodym property

The main result of this section is Theorem 4.1.2, whose proof is given in §4.2 and §4.3. The Radon–Nikodym property introduced below should not be confused with the Nikodym property ([33]), which is an analog of the Uniform Boundedness Principle (i.e., Banach–Steinhaus Theorem) for the ring associated with the ideal at hand.

**Definition 4.1.1.** Suppose $\mathcal{I}$ is an ideal on $\mathbb{N}$.

(1) $\mathcal{I}$ has the *Radon–Nikodym property* if every Boolean algebra homomorphism $\Phi\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})/\mathcal{I}$ with a Baire measurable lifting has a completely additive lifting.

[1] These are often called *topologically simple* liftings, e.g., in [59].

(2) $\mathcal{I}$ has the *group Radon–Nikodym property* if every group algebra homomorphism $\Phi\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})/\mathcal{I}$ with a Baire measurable lifting has a continuous lifting which is a group homomorphism.

**Theorem 4.1.2.** *All of the following ideals have the Radon–Nikodym property.*

*(1) Non-pathological analytic P-ideals.*
*(2) Non-pathological $F_\sigma$-ideals*
*(3) Summable ideals (Example 1.3.5).*
*(4) Ideals $\emptyset \times \mathrm{Fin}$ and $\mathrm{Fin} \times \emptyset$ (Example 1.3.5).*
*(5) Density ideals (Definition 1.7.1).*
*(6) Erdös–Ulam ideals (Example 1.3.5).*
*(7) Matrix summability ideals (Definition 1.8.1).*
*(8) Ordinal ideals $\mathcal{I}_{\beta,\alpha}$, where $\alpha$ and $\beta$ are indecomposable ordinals.*
*(9) Weiss ideals $\mathcal{W}_\alpha$, where $\alpha$ is a multiplicatively indecomposable ordinal.*

Proof. The Fubini property introduced in §4.2 implies the Radon–Nikodym property (Theorem 4.3.1), hence in each case it suffices to prove the former. This was done in Lemma 4.2.5 for (1) and (2), and in Corollary 4.2.8 for a class of ideals larger than the one in (8).[2]

(3), (6), and (7) are consequences of (1): For summable ideals this is vacuous, all Erdös–Ulam ideals are non-pathological analytic P-ideals because Theorem 2.7.8 implies that they are density ideals, and all matrix summability ideals are nonpathological analytic P-ideals by [158, Corollary 3.12].

(4) Define a submeasure $\varphi$ on $\mathcal{P}(\mathbb{N}^2)$ by $\varphi(A) = \max\{2^{-n} : A \not\subseteq [n,\infty) \times \mathbb{N}\}$. It is non-pathological and lower semicontinuous, and clearly $\emptyset \times \mathrm{Fin} = \mathrm{Exh}(\varphi)$. The submeasure $\psi$ defined on $\mathcal{P}(\mathbb{N}^2)$ by $\psi(A) = \min\{n : A \subseteq n \times \mathbb{N}\}$ is lower semicontinuous, non-pathological, and $\mathrm{Fin} \times \emptyset = \mathrm{Fin}(\psi)$.

For (9), see [101, Theorem 30]. □

Before proceeding, we should describe continuous group homomorphisms and continuous Boolean algebra homomorphisms. from $\mathcal{P}(\mathbb{N})$ into itself.

**Lemma 4.1.3.** *Some $\Phi\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ is a continuous group homomorphism if and only if there are finite $s_n \subseteq \mathbb{N}$ such that $\Phi(a) = \{n : |a \cap s_n| \text{ is odd}\}$ for all $a$.*

*It is a continuous Boolean algebra homomorphism if and only if $s_n$ is a singleton for all $n$.*

Proof. We need to prove that every closed subgroup $\Gamma$ of $\mathcal{P}(\mathbb{N})$ of index 2 is of the form $\{a : |a \cap s| \text{ is even}\}$ for some nonempty, finite $s \subseteq \mathbb{N}$. Fix such $\Gamma$ and note that it is clopen.

Thus the set $s = \{n : \{n\} \notin \Gamma\}$ is finite and nonempty. We claim that every $b \in \mathcal{P}(s) \cap \Gamma$ has even cardinality. Otherwise, fix such $b$ of minimal possible odd cardinality. By the definition of $s$, $|b| \geq 3$. Since $\Gamma$ has index 2, for some $c \in \Gamma$ both $b \cap c$ and $b \setminus c$ are nonempty. One of these sets has an odd cardinality, contradicting the minimality of $b$. Thus $\Gamma \subseteq \{a \subseteq \mathbb{N} : |a \cap s| \text{ is even}\}$. But the set on the right-hand side is a closed subgroup of $\mathcal{P}(\mathbb{N})$ of index 2, and therefore it is equal to $\Gamma$.

We know that $\Phi$ is a continuous group homomorphism if and only if there is a function $h\colon \mathbb{N} \to \mathbb{N}$ such that $\Phi(a) = h^{-1}(a)$. With $s_n = \{h(n)\}$, this is equivalent to $\Phi(a) = \{n : |a \cap s_n| \text{ is odd}\}$. □

[2] See also Theorem 4.2.3 for closure properties of the class of ideals with the Fubini property.

In the original definition of the Radon–Nikodym property given in [40] it was required that $\ker(\Phi) \supseteq \mathrm{Fin}$. There was no compelling reason for this restriction, and the following is proven by composing $\Phi$ with the quotient map $\pi_{\mathcal{I}}\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})/\mathcal{I}$ (and noting that it has a continuous lifting, namely the identity map on $\mathcal{P}(\mathbb{N})$).

**Lemma 4.1.4.** *Suppose $\mathcal{J}$ is an ideal on $\mathbb{N}$.*

*(1) If $\mathcal{J}$ has the Radon–Nikodym property, then for every ideal $\mathcal{I}$ on $\mathbb{N}$, every Boolean algebra homomorphism $\Phi\colon \mathcal{P}(\mathbb{N})/\mathcal{I} \to \mathcal{P}(\mathbb{N})/\mathcal{J}$ with a Baire measurable lifting has a completely additive lifting.*

*(2) If $\mathcal{J}$ has the group Radon–Nikodym property, then for every ideal $\mathcal{I}$ on $\mathbb{N}$, every group algebra homomorphism $\Phi\colon \mathcal{P}(\mathbb{N})/\mathcal{I} \to \mathcal{P}(\mathbb{N})/\mathcal{J}$ with a Baire measurable lifting has a completely additive lifting.* □

For $\leq_{\mathrm{BE}}$ (Baire-embeddability) and $\leq_{\mathrm{RK}}$ (the Rudin–Keisler order) see Definition 2.3.2.

**Corollary 4.1.5.** *If $\mathcal{I}$ and $\mathcal{J}$ are ideas on $\mathbb{N}$ and $\mathcal{J}$ has the Radon–Nikodym property, then $\mathcal{I} \leq_{\mathrm{BE}} \mathcal{J}$ if and only if $\mathcal{I} \leq_{\mathrm{RK}} \mathcal{J}$.* □

By $\mathcal{I}^*$ we denote the *dual filter* of $\mathcal{I}$, namely the filter of all sets whose complement is in $\mathcal{I}$, and by $\mathcal{I}_+$ we denote the coideal of all *sets*, namely all sets of integers which are not in $\mathcal{I}$. An isomorphism of quotients $\Phi\colon \mathcal{P}(\mathbb{N})/\mathcal{I} \to \mathcal{P}(\mathbb{N})/\mathcal{J}$ is *trivial* if there are sets $A \in \mathcal{I}^*$, $B \in \mathcal{J}^*$ and a bijection $h\colon B \to A$ such that $\Phi_h$ is a lifting of $\Phi$. Note that (a) below gives a satisfactory answer to the basic question (see §1.4). Part (b) of Proposition 4.1.6 below for $\mathcal{I} = \mathrm{Fin}$ was first proven by Velickovic ([161]).

**Proposition 4.1.6.** *If ideals $\mathcal{I}$ and $\mathcal{J}$ have the Radon–Nikodym property then every Baire isomorphism of quotients $\mathcal{P}(\mathbb{N})/\mathcal{I}$ and $\mathcal{P}(\mathbb{N})/\mathcal{J}$ is trivial. In particular,*

*(a) The quotients over $\mathcal{I}$ and $\mathcal{J}$ are Baire-isomorphic if and only if the corresponding ideals are Rudin–Keisler isomorphic (Definition 2.1.3).*

*(b) All Baire automorphisms of $\mathcal{P}(\mathbb{N})/\mathcal{I}$ are trivial.*

Proof. Let $\Phi\colon \mathcal{P}(\mathbb{N})/\mathcal{I} \to \mathcal{P}(\mathbb{N})/\mathcal{J}$ be a Baire isomorphism and let $h_1, h_2$ be such that $\Phi_{h_1}$ is a lifting of $\Phi$ and $\Phi_{h_2}$ is a lifting of $\Phi^{-1}$. If $g = h_1 \circ h_2$ is the identity on a set in $\mathcal{I}^*$, then $\Phi_{h_1}$ is a lifting of $\Phi$ and $h_1$ witnesses that $\mathcal{I}$ and $\mathcal{J}$ are isomorphic. Let $B$ be the set of all $x$ such that $g(x) \neq x$. Then we can split $B$ into disjoint sets $B_1, B_2, B_3$ so that $g''B_i$ is disjoint from $B_i$ for $i = 1, 2, 3$ (see e.g., [18, Lemma 9.1]). This easily implies each $B_i$ is in $\mathcal{I}$, and therefore the set of fixed points for $g$ is in $\mathcal{I}^*$, as required. Both (a) and (b) follow immediately. □

## 4.2. The Fubini property

Identify $\mathcal{P}(\mathbb{N})$ with $\{0,1\}^{\mathbb{N}}$ and consider the product measure $\lambda$ on this space (on $\{0,1\}$ take the uniform probability measure). Alternatively, identify $\mathcal{P}(\mathbb{N})$ with the compact group $\mathbb{Z}/2\mathbb{Z}$ and equip it with the Haar measure. One easily sees that this is the same measure. Definition 4.2.2, Theorem 4.2.3, and Theorem 4.3.1 are taken from [101] and [102], where $\lambda$ was referred to as the Lebesgue measure. We consider $\mathbb{N}$ with the discrete topology, and $\mathbb{N} \times \mathcal{P}(\mathbb{N})$ as a Polish space. For $\mathcal{X} \subseteq \mathbb{N} \times \mathcal{P}(\mathbb{N})$, its horizontal and vertical sections are

$$\mathcal{X}_m = \{A : (m, A) \in \mathcal{X}\}$$
$$\mathcal{X}^A = \{m : (m, A) \in \mathcal{X}\}.$$

A proof of the following lemma is included for the sake of those readers who tend to object to the phrase 'everything is Borel' and require proofs.

**Lemma 4.2.1.** *Fix $\mathcal{X} \subseteq \mathbb{N} \times \mathcal{P}(\mathbb{N})$. If $\mathcal{X}$ is Borel, then the function $A \mapsto \mathcal{X}^A$ is Borel-measurable. If in addition $\mathcal{Y} \subseteq \mathcal{P}(\mathbb{N})$ is Borel, $\{A : \mathcal{X}^A \in \mathcal{Y}\}$ is Borel.*

Proof. Note that $\mathcal{X}$ is Borel if and only if each $\mathcal{X}_n$ is Borel. Thus if $\mathcal{X}$ is Borel, then for every $n \in \mathbb{N}$ we have $\{A : n \in \mathcal{X}^A\} = \mathcal{X}_n$, thus the preimage of every clopen subset of $\mathcal{P}(\mathbb{N})$ is Borel, as required. Since the preimage of a Borel set under a Borel-measurable map is Borel, the last claim follows. □

**Definition 4.2.2.** An ideal $\mathcal{J}$ on $\mathbb{N}$ has the *Fubini property* if for every $\varepsilon > 0$ and every Borel $\mathcal{X} \subseteq \mathbb{N} \times \mathcal{P}(\mathbb{N})$, if $\{m \in \mathbb{N} : \lambda(\mathcal{X}_m) \geq \varepsilon\}$ is $\mathcal{I}$-positive then $\lambda\{A : \mathcal{X}^A \notin \mathcal{I}\} \geq \varepsilon$.

An equivalent reformulation of the Fubini property is given in Lemma 4.2.4 below. The following theorem whose proof occupies the remainder of this section summarises results on the Fubini property that we will need, but see also Theorem 4.2.9 below.

**Theorem 4.2.3.** *The class of ideals with the Fubini property on $\mathbb{N}$ has the following closure properties.*

1. *If $\mathcal{I}$ is countably generated, then it has the Fubini property.*
2. *If $\mathcal{I}$ is the pointwise limit of a sequence of Borel ideals $\mathcal{I}_n$, for $n \in \mathbb{N}$ with the Fubini property (i.e., $A \in \mathcal{I}_\infty$ if and only if $(\forall^\infty n) A \in \mathcal{I}_n$) then $\mathcal{I}_\infty$ has the Fubini property.*
3. *If $\mathcal{I}$ is an intersection of an arbitrary family of ideals with the Fubini property, then $\mathcal{I}$ has the Fubini property.*
4. *If $\mathcal{I}_n$, for $n \in \mathbb{N}$, are Borel ideals with the Fubini property and $\mathcal{I}_n \subseteq \mathcal{I}_{n+1}$ for all $n$ then $\mathcal{I} = \bigcup_n \mathcal{I}_n$ is a Borel ideal with the Fubini property.*
5. *If $\mathcal{J}$ and $\mathcal{J}_n$, for $n \in \mathbb{N}$, have the Fubini property, then so does $\mathcal{I} = \bigoplus_{\mathcal{J}} \mathcal{J}_n$.*
6. *If $\varphi$ is a non-pathological lower semicontinuous submeasure on $\mathbb{N}$, then both $\mathrm{Exh}(\varphi)$ and $\mathrm{Fin}(\varphi)$ have the Fubini property.*
7. *Suppose that $\varphi_n$, for $n \in \mathbb{N}$, are non-pathological lower semicontinuous submeasures on $\mathbb{N}$ such that $\limsup_n \|\varphi_n\| > 0$. Then the ideal $\mathcal{D}_\varphi$ (see Lemma 1.8.3) has the Fubini property.*
8. *For every indecomposable countable ordinal $\alpha$, the ideal $\mathcal{O}_\alpha$ has the Fubini property.*

Proof. Proofs of (6), (7) are in Lemma 4.2.5 and Lemma 4.2.6, respectively and strengthenings of (8) can be found in Theorem 4.2.7 and Corollary 4.2.8.

Each of the proofs of (1)–(4) starts with the words 'fix $\varepsilon > 0$ and a Borel set $\mathcal{X} \subseteq \mathbb{N} \times \mathcal{P}(\mathbb{N})$ such that the set

$$B = \{n : \lambda(\mathcal{X}_n) \geq \varepsilon\}$$

is $\mathcal{I}$-positive' and ends by concluding that $\lambda(\{A : \mathcal{X}^A \notin \mathcal{I}\}) \geq \varepsilon$. The middle parts of the proofs follow.

(1) Let $A_n \subseteq \mathbb{N}$ be sets that generate $\mathcal{I}$. By replacing $A_n$ with $\bigcup_{j \leq n} A_j$, we may assume that $A_n \subseteq A_{n+1}$ and therefore $\mathcal{I} = \bigcup_n \mathcal{P}(A_n)$.

Since $B \notin \mathcal{I}$, we can choose $j(n) \in B \setminus A_n$ for all $n$. Since the measure $\lambda$ is finite, we have $\lambda(\bigcap_m \bigcup_{n\geq m} \mathcal{X}_{j(n)}) \geq \lim_m \lambda(\bigcup_{n\geq m} \mathcal{X}_{j(n)} \geq \varepsilon$. But $A \in \bigcup_m \bigcap_{n\geq m} \mathcal{X}_{j(n)}$ implies $\mathcal{X}^A \not\subseteq \bigcup_n \mathcal{X}_n$, thus $\lambda(\{A : \mathcal{X}^A \notin \mathcal{I}\} \geq \varepsilon$ as required.

(2) Suppose that $\mathcal{I}_\infty = \mathcal{I}$ is the pointwise limit of ideals $\mathcal{I}_n$. For $n \leq \infty$ let

$$\mathcal{Y}_n = \{A : \mathcal{X}^A \notin \mathcal{I}_n\}.$$

By Lemma 4.2.1, this is a Borel set, and the sets $\mathcal{Y}_n$ converge to $\mathcal{Y}_\infty$ pointwise. Fix $\delta > 0$ and $n$ large enough to have $B \notin \mathcal{I}_n$ and $\lambda(\mathcal{Y}_\infty \Delta \mathcal{Y}_n) < \delta$. Since $\mathcal{I}_n$ has the Fubini property, we have $\lambda(\mathcal{Y}_n) \geq \varepsilon$, and therefore $\lambda(\mathcal{Y}_\infty) \geq \varepsilon - \delta$. Since $\delta > 0$ was arbitrary, $\lambda(\mathcal{Y}_\infty) \geq \varepsilon$, the Fubini property of $\mathcal{I}_\infty$ follows.

(3) Assume $\mathcal{I} = \bigcap_{\alpha\in\mathbb{I}} \mathcal{J}_\alpha$. Then $B \notin \mathcal{J}_\alpha$ for some $\alpha$.

Clearly $\{A : \mathcal{X}^A \notin \mathcal{I}\} \supseteq \{A : \mathcal{X}^A \notin \mathcal{J}_\alpha\}$. By the Fubini property of $\mathcal{J}_\alpha$ we have $\lambda(\{A : \mathcal{X}^A \notin \mathcal{J}_\alpha\} \geq \varepsilon$, and the conclusion follows.

(4) This is a special case of (2).

(5) Fix $\mathcal{J}$ and $\mathcal{J}_n$, for $n \in \mathbb{N}$, with the Fubini property such that $\mathcal{I} = \bigoplus_{\mathcal{J}} \mathcal{J}_n$. In this proof $\mathcal{X} \subseteq \mathbb{N}^2 \times \mathcal{P}(\mathbb{N})$, since $\mathcal{I}$ is an ideal on $\mathbb{N}^2$. Thus $B \subseteq \mathbb{N}^2$, and

$$B' = \{n \in \mathbb{N} : B_n \notin \mathcal{J}_n\}$$

is $\mathcal{J}$-positive. Fix $m \in B'$ and for $A \subseteq \mathbb{N}$ consider the 'middle section' of $\mathcal{X}$,

$$\mathcal{X}^A_{(m,\cdot)} = \{n : (m, n, A) \in \mathcal{X}\}$$

By the Fubini property of $\mathcal{J}_m$, $\lambda(\{A : \mathcal{X}^A_{(m,\cdot)} \notin \mathcal{J}_m\} \geq \varepsilon$.

By Lemma 4.2.1, $\mathcal{Y} = \{(m, A) : \mathcal{X}^A_{(m,\cdot)} \notin \mathcal{J}_m\}$ is a Borel set, and by the previous line $\lambda(\mathcal{Y}_m) \geq \varepsilon$. By the Fubini property of $\mathcal{J}$, $\lambda(\{A : \mathcal{Y}^A \notin \mathcal{J}\}) \geq \varepsilon$. However, $\mathcal{Y}^A \notin \mathcal{J}$ implies $\{m : \mathcal{X}^A_{(m,\cdot)} \notin \mathcal{J}_m\} \notin \mathcal{J}$, which is equivalent to $\mathcal{X}^A \notin \bigoplus_{\mathcal{J}} \mathcal{J}_m$. Therefore $\lambda(\{A : \mathcal{X}^A\}) \geq \varepsilon$, as required. □

**Lemma 4.2.4.** *For every ideal $\mathcal{I}$ on $\mathbb{N}$ the following are equivalent.*

*(1) For every $\varepsilon > 0$ and every Borel $\mathcal{X} \subseteq \mathbb{N} \times \mathcal{P}(\mathbb{N})$, if $\{n : \lambda(\mathcal{X}_n) \geq \varepsilon\}$ is $\mathcal{I}$-positive then $\lambda(\{A : \mathcal{X}^A \in \mathcal{I}_+\}) \geq \varepsilon$.*

*(2) For every $\varepsilon > 0$ and every Borel $\mathcal{X} \subseteq \mathbb{N} \times \mathcal{P}(\mathbb{N})$, if $\{n : \lambda(\mathcal{X}_n) \geq \varepsilon\}$ is $\mathcal{I}$-positive then $\mathcal{X}^A \in \mathcal{I}_+$ for some $A \in \mathcal{P}(\mathbb{N})$.*

Proof. Only the converse implication requires a proof. Suppose that (2) holds. Towards contradiction, assume that for some $\delta < \varepsilon$ there is a Borel set $\mathcal{X} \subseteq \mathbb{N}\times\mathcal{P}(\mathbb{N})$ such that the set $\mathcal{B} = \{A \mid \mathcal{X}^A \in \mathcal{I}_+\}$ satisfies $\lambda(\mathcal{B}) = \delta$.

The set $\mathcal{Y} = \mathcal{X} \setminus \mathbb{N}\times\mathcal{B}$ is Borel and the set $\{n : \lambda(\mathcal{Y}_n) \geq \varepsilon - \delta\}$ is $\mathcal{I}$-positive. By (2) there is $A \in \mathcal{P}(\mathbb{N})$ such that $\mathcal{Y}^A$ is $\mathcal{I}$-positive. However, $A \notin \mathcal{B}$ by construction and therefore $\mathcal{Y}^A \in \mathcal{I}$; contradiction. □

The converse of the following lemma is also true for analytic P-ideals, see Theorem 4.2.9 below (for $\mathrm{Exh}(\varphi)$ see Definition 1.4.3).

**Lemma 4.2.5.** *If $\varphi$ is a non-pathological lower semicontinuous submeasure on $\mathbb{N}$, then both $\mathrm{Exh}(\varphi)$ and $\mathrm{Fin}(\varphi)$ have the Fubini property.*

Proof. Fix a non-pathological lower semicontinuous submeasure $\varphi$. We will first prove that $\mathcal{I} = \mathrm{Fin}(\varphi)$ has the Fubini property.

It suffices to show that (2) of Lemma 4.2.4 holds. Fix $\varepsilon > 0$ and a Borel $\mathcal{X} \subseteq \mathbb{N} \times \mathcal{P}(\mathbb{N})$ such that $\{n : \lambda(\mathcal{X}_n) \geq \varepsilon\}$ is $\mathcal{I}$-positive. Since $\lambda$ is $\sigma$-additive, for a large enough $K < \infty$ the set $\mathcal{C} = \{A \mid \varphi(\mathcal{Y}^A) > K\}$ satisfies $\lambda(\mathcal{C}) < (\varepsilon - \delta)/2$. Let

$\mathcal{Z} = \mathcal{Y} \setminus (\mathbb{N} \times \mathcal{C})$. Then the set $D = \{n : \lambda(\mathcal{Z}_n) \geq (\varepsilon - \delta)/2\}$ satisfies $\varphi(D) = \infty$ but $\varphi(\mathcal{Z}^A) \leq K$ for all $A \in \mathcal{P}(\mathbb{N})$. Since $\varphi$ is non-pathological, some measure $\mu \leq \varphi$ satisfies $\mu(D) > K\varepsilon^{-1}$.

Since both $\lambda$ and $\mu$ are $\sigma$-additive Borel measures, we can apply Fubini's Theorem to the product measure $\mu \times \lambda$ on the $\sigma$-algebra of Borel subsets of $\mathbb{N} \times \mathcal{P}(\mathbb{N})$. However, $\int \mu(\mathcal{Z}^A)\, d\lambda(A) \leq K$ and $\int \lambda(\mathcal{Z}_n)\, d\mu(n) > K$; contradiction.

It remains to prove that $\mathcal{J} = \mathrm{Exh}(\varphi)$ has the Fubini property. It suffices to show that (2) of Lemma 4.2.4 holds. Fix $\varepsilon > 0$ and a Borel $\mathcal{X} \subseteq \mathbb{N} \times \mathcal{P}(\mathbb{N})$ such that the set $B = \{n : \lambda(\mathcal{X}_n) \geq \varepsilon\}$ is $\mathcal{J}$-positive. Let $\delta > 0$ be such that $\varphi(B \setminus k) > \delta$ for all $k \in \mathbb{N}$. Towards contradiction assume that $\mathcal{X}^A \in \mathcal{J}$ for all $A$. Therefore for every $A$ there is $k$ such that $\varphi(\mathcal{X}^A \setminus k) < \varepsilon\delta$. Let $k(A)$ be the minimal such $k$. Since $\varphi$ is lower semicontinuous, the set $\{A : k(A) = k\}$ is Borel for all $k$ and by $\sigma$-additivity of $\lambda$ there is $k$ such that the set $\mathcal{C} = \{A : \varphi(\mathcal{X}^A \setminus k\}$ satisfies $\lambda(\mathcal{C}) < \varepsilon\delta\}) > 1 - \frac{\varepsilon}{2}$. Let $\mathcal{Y} = \mathcal{X} \cap (\mathbb{N} \times \mathcal{C})$. Then $\lambda(\mathcal{Y}_n) > \varepsilon/2$ for all $n \in B$, and $\varphi(\mathcal{Y}^A \setminus k) < \varepsilon\delta$ for all $A \in \mathcal{P}(\mathbb{N})$. Since $\varphi$ is non-pathological, there is a measure $\mu \leq \varphi$ such that $\mu(B \setminus k) > \delta$. Computing $(\mu \times \lambda)(\mathcal{Y})$ in two ways, we obtain $\int \mu(\mathcal{Y}^A)\, d\lambda(A) < \varepsilon\delta$ and $\int \lambda(\mathcal{Y}_n)\, d\mu(n) > \varepsilon\delta$, contradicting Fubini's theorem. □

The following is a special case of [158, Corollary 3.12] (see §1.8.1 and Lemma 1.8.3 for ideals of the form $\mathcal{D}_\varphi$; also see Lemma 4.2.11 for a slightly different proof).

**Lemma 4.2.6.** *Suppose that $\varphi_n$, for $n \in \mathbb{N}$, are non-pathological lower semicontinuous submeasures on $\mathbb{N}$ such that $\limsup_n \|\varphi_n\| > 0$. Then the ideal*

$$\mathcal{D}_\varphi = \{A \subseteq \mathbb{N} : \limsup_n \varphi_n(A) = 0\}$$

*as defined in Lemma 1.8.3 has the Fubini property.*

Proof. We will use Lemma 4.2.4. Fix $\varepsilon > 0$ and let $\mathcal{X} \subseteq \mathbb{N} \subseteq \mathcal{P}(\mathbb{N})$ be a Borel set such that the set $B = \{n : \lambda(\mathcal{X}_n) \geq \varepsilon\}$ is $\mathcal{D}_\varphi$-positive. Towards contradiction, assume that $\mathcal{X}^A \in \mathcal{D}_\varphi$ for all $A \in \mathcal{P}(\mathbb{N})$. Let $\delta > 0$ be such that $\limsup_n \mu_n(B) > \delta$. Since all $\varphi_n$ are lower semicontinuous, by $\sigma$-additivity of $\lambda$ there is $k$ such that $\mathcal{C} = \{A : (\forall n \geq k)\varphi_n(\mathcal{X}^A) < \varepsilon\delta\}$ satisfies $\lambda(\mathcal{C}) > 1 - \frac{\varepsilon}{2}$. Let $\mathcal{Y} = \mathcal{X} \cap (\mathbb{N} \times \mathcal{X})$. Find $n > k$ such that $\varphi_n(B) > \delta$. Since $\varphi_n$ is non-pathological, there is a measure $\mu \leq \varphi_n$ such that $\mu(B) > \delta$. As in the proof of Lemma 4.2.5, by computing $(\mu \times \lambda)(\mathcal{Y})$ two ways we obtain $\varepsilon\delta < \varepsilon\delta$; contradiction. □

We now turn to ordinal ideals (§1.9). The ideals occurring in the following are a special case of the ideals introduced in Definition 1.9.1.

**Theorem 4.2.7.** *For every pair of countable ordinals $\alpha \leq \beta$ the ideal*

$$\mathcal{I}_{\beta,\alpha} = \{A \subseteq \omega^\beta : \mathrm{otp}(A) < \omega^\alpha\}$$

*has the Fubini property.*

Proof. The proof proceeds by induction on $\alpha$. If $\alpha = 0$, then $\mathcal{I}_{\beta,\alpha} = \mathrm{Fin}(\omega^\beta)$ is countably generated and the assertion follows from Theorem 4.2.3 (1).

Fix a limit ordinal $\alpha$ and assume that $\mathcal{I}_{\beta,\gamma}$ holds for all $\gamma < \alpha$ and all $\beta$. If $\alpha_i$ is an increasing sequence with $\alpha = \sup_i \alpha_i$, then for every $\beta$ we have $\mathcal{I}_{\beta,\alpha} = \bigcap_i \mathcal{I}_{\beta,\alpha_i}$, and the Fubini property of $\mathcal{I}_{\beta,\alpha}$ follows by Theorem 4.2.3 (4)

Now assume that $\mathcal{I}_{\beta,\alpha}$ has the Fubini property for all $\beta$. To prove that $\mathcal{I}_{\beta,\alpha+1}$ has the Fubini property for all $\beta$, proceed by induction on $\beta \geq \alpha + 1$. Fix ordinals $\beta_i \leq \beta_{i+1} < \beta$ such that $\beta = \sum_i \beta_i$ (if $\beta = \gamma + 1$ let $\beta_i = \gamma$ for all $i$, and if $\beta$ is limit choose $\beta_i$ strictly increasing). Then $\omega^\beta = \sum_i L_i$, where $L_i \cong \omega^{\beta_i}$ for each $i$, and the following is a nice exercise.

For every $A \subseteq \omega^\beta$, $\operatorname{otp}(A) < \omega^{\alpha+1}$ if and only if $(\forall i) \operatorname{otp}(A \cap L_i) < \omega^{\alpha+1}$ and $(\forall^\infty i) \operatorname{otp}(A \cap L_i) < \omega^\alpha$.

This translates to $\mathcal{I}_{\beta,\alpha+1} = (\bigoplus_\emptyset \mathcal{I}_{\beta_i,\alpha+1}) \cap (\bigoplus_{\mathrm{Fin}} \mathcal{I}_{\beta_i,\alpha})$ which follows from Theorem 4.2.3, parts (2) and (5). □

Since every well-order $L$ can be written as sum of finitely many indecomposable ordinals, Theorem 4.2.7 implies the following.

**Corollary 4.2.8.** *If $L$ is a countable well-ordered set and $\alpha$ is a countable ordinal, then the ideal $\mathcal{O}_{L,\alpha} = \{A \subseteq L : \operatorname{otp}(A) < \omega^\alpha\}$ has the Fubini property.* □

**4.2.1. Characterisation of the Fubini property and its permanence properties.** Motivated by a question of Louveau, in [147, Theorem 2.1] Solecki characterised filters that satisfy Fatou's Lemma in terms of Katětov order (§2.2). This is naturally recast as a characterisation of ideals with the Fubuni property. The following is [83, Theorem 3.1] (also see [86, Corollary 5.6] and [82, Corollary 5.26]).

**Theorem 4.2.9.** *For every ideal $\mathcal{I}$ on $\mathbb{N}$ the following are equivalent.*

*(1) $\mathcal{I}$ has the Fubini property.*
*(2) $\mathcal{S} \not\leq_K \mathcal{I} \restriction A$ for every $\mathcal{I}$-positive $A$.*

*If in addition $\mathcal{I}$ is an analytic P-ideal, then the above are equivalent to each of the following.*

*(3) $\mathcal{I}$ is non-pathological.*
*(4) $\mathcal{I} \restriction A \leq_K \mathcal{Z}_0$ for every $\mathcal{I}$-positive $A$.* □

Another equivalence, announced in [101] and proved using [147, §1] in [83, Corollary 3.2], is that $\mathcal{I}$ fails the analog of Fatou's lemma.

Theorem 4.2.9 has an immediate corollary (see also [158, Corollary 3.12]).

**Proposition 4.2.10.** *An intersection of any family of ideals on $\mathbb{N}$ with the Fubini property has the Fubini property.*

*An intersection of a countable family of nonpathological analytic P-ideals is a nonpathological analytic P-ideal.*

Proof. It is straightforward to check that the family of ideals with the Fubini property is closed under finite intersections. Suppose that $\mathcal{F}$ is a family of ideals on $\mathbb{N}$ such that $\mathcal{I} = \bigcap \mathcal{F}$ does not have the Fubini property. By Theorem 4.2.9, there is an $\mathcal{I}$-positive $A$ such that $\mathcal{S} \leq_{\mathrm{K}} \mathcal{I} \restriction A$. Let $h\colon A \to \Omega$ be such that $h^{-1}(X) \in \mathcal{I}$ for all $X \subseteq \Omega$. If $\mathcal{J} \in \mathcal{F}$ is such that $A$ is $\mathcal{J}$-positive, then $h$ witnesses that $\mathcal{S} \leq_{\mathrm{K}} \mathcal{J} \restriction A$, hence $\mathcal{J}$ does not have the Fubini property by Theorem 4.2.9.

If $\mathcal{I}_n$, for $n \in \mathbb{N}$, are nonpathological analytic P-ideals, then by Theorem 4.2.9 and the first part $\mathcal{I} = \bigcap_n \mathcal{I}_n$ has the Fubini property. It is also analytic, and it is a P-ideal by Lemma 1.3.3. Again by Theorem 4.2.9, it is nonpathological. □

The proof of the second part of Proposition 4.2.10 is rather silly, it would be nice to have a direct construction of a nonpathological submeasure instead.

We have an alternative proof of Lemma 4.2.5, partly based on the proof of [158, Proposition 3.1] (for $\mathcal{D}_\varphi$ see Lemma 1.8.3).

**Lemma 4.2.11.** *Every matrix summability ideal has the Fubini property.*

*If $\varphi_n$, for $n \in \mathbb{N}$, are nonpathological lower semicontinuous submeasures and the ideal $\mathcal{D}_\varphi$ is dense, then $\mathcal{D}_\varphi$ has the Fubini property.*

PROOF. It suffices to prove the second part, and to prove it it suffices to prove that $\mathcal{D}_\varphi$ is the intersection of ideals with the Fubini property. Fix a $\mathcal{D}_\varphi$-positive $A$ and $\varepsilon > 0$ such that the set $Z = \{n : \varphi_n(A) > \varepsilon\}$ is infinite.

Since $\mathcal{D}_\varphi$ is dense, every $s \Subset \mathbb{N}$ satisfies $\lim_n \varphi_n(m) = 0$. Hence there are an infinite $Z' \subseteq Z$ and disjoint $I_n \Subset \mathbb{N}$, for $n \in Z'$, such that $\varphi_n(A \cap I_n) > \varepsilon$ for all $n \in Z'$. Let $\psi_n$ denote the restriction of $\varphi_n$ to $\mathcal{P}(I_n)$; this is a nonpathological submeasure because $\varphi_n$ is nonpathological. Then the generalised density ideal $\mathcal{Z} = \{B \subseteq \bigcup_n I_n : \limsup_{n \in Z'} \psi_n(B \cap I_n) = 0\}$ is nonpathological, included in $\mathcal{D}_\varphi$, and $A$ is $\mathcal{Z}$-positive. By Theorem 4.2.9, it has the Fubini property. $\square$

### 4.3. The Fubini property implies the Radon–Nikodym property

The following is [102, Theorem 10, Theorem 13], see also [101]. The special case when $\mathcal{J}$ is a non-pathological P-ideal is the main result of [40, §1].

**Theorem 4.3.1.** *If a Borel ideal $\mathcal{J}$ has the Fubini property then it has both the group Radon–Nikodym property and the Radon–Nikodym property.*

In a few lines from here we will prove the group case of Therorem 4.3.1, and the remainder of this section will be devoted to its Boolean algebra case.

For a set $A$ consider $\mathcal{P}(A)$ as a probability measure space, with respect to the product measure $\lambda$. We use the same symbol $\lambda$ for the measure on $\mathcal{P}(A)$ for every $A$, and for the measure on $\mathcal{P}(A)^2$. The set will always be clear from the context.

**Lemma 4.3.2.** *Suppose that ideal $\mathcal{I}$ has the Fubini property and that $\Theta\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ is a continuous lifting of a group homomorphism $\Phi\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})/\mathcal{I}$. If*

$$\lambda(\{(a,b) : m \in \Theta(b)\Delta\Theta(c)\Delta\Theta(b\Delta c)\} < 1/6$$

*for all $m$, then there is a continuous group homomorphism $\tilde{\Theta}$ such that $\Theta(a)\Delta\tilde{\Theta}(a) \in \mathcal{I}$ for all $a$.*

PROOF. The set $\mathcal{X} = \{(m,b,c) : m \in \Theta(b)\Delta\Theta(c)\Delta\Theta(b\Delta c)\}$ is Borel set because $\Theta$ is continuous. For $b \in \mathcal{P}(\mathbb{N})$ the function

$$\Theta_b(a) = \Theta(a\Delta b)\Delta\Theta(b)$$

is a continuous lifting of $\Phi$.

Fix $m$. Then $\lambda(\{(a,b) : m \in \Theta_b(a)\Delta\Theta(a)\}) < 1/6$. Moreover, for every $a$ and every pair $b,c$ we have that $m \in \Theta_b(a)\Delta\Theta_c(a)$ implies that one of $m \in \Theta(b)\Delta\Theta(c)\Delta\Theta(b\Delta c)$ or $m \in \Theta(a\Delta b)\Delta\Theta(a\Delta c)\Delta\Theta(b\Delta c)$ applies. Since the set of pairs $b,c$ such that either of these two possibilities applies has measure $< 1/6$, we conclude that $\lambda(\{(b,c) : m \in \Theta_b(a)\Delta\Theta_c(a)\}) < 1/3$. There is therefore $G_m(a) \in \{\emptyset, \{m\}\}$ such that the set

$$\mathcal{Z}_a = \{b : \Theta_b(a) \cap \{m\} = G_m(a)\}$$

has measure $> 2/3$.

We claim that $G_m\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\{m\})$ is a group homomorphism. Fix $a$ and $b$. Each of the sets $\mathcal{Z}_a$, $\mathcal{Z}_b$, and $\{y\Delta b : y \in \mathcal{Z}_{a\Delta b}\}$ has measure $> 2/3$, hence we can choose $x$ in their intersection. Then

$$\begin{aligned} G_m(a)\Delta G_m(b) &= (\Theta(a\Delta x)\Delta\Theta(x)\Delta\Theta(b\Delta x)\Delta\Theta(x)) \cap \{m\} \\ &= (\Theta((a\Delta b)\Delta(b\Delta x))\Delta\Theta(b\Delta x)) \cap \{m\} \\ &= G_m(a\Delta b) \end{aligned}$$

as required. Thus $G_m$ is a Lebesgue-measurable group homomorphism.

Then $\tilde{\Theta}(a) = \bigcup_m G_m(a)$ is a Lebesgue-measurable group homomorphism, and it remains to prove that it is Borel-measurable. Equivalently, we need to prove that the graph of $G$ is a Borel set. By Suslin's separation theorem, it will suffice to prove that the graph of $G$ is both analytic and coanalytic.

Towards this, by e.g., repeatedly applying Luzin's theorem, we can find an $F_\sigma$ set $\mathcal{Y}$ of full measure such that the restriction of $G$ to $\mathcal{Y}$ is Borel-measurable. For every $a$ there is $y \in \mathcal{Y}\Delta(a\Delta\mathcal{Y})$. Thus $G(a) = b$ is equivalent to each of the following statements:

There is $y \in \mathcal{Y}\Delta(a\Delta\mathcal{Y})$ such that $\tilde{\Theta}(y)\Delta\tilde{\Theta}(y\Delta a) = b$.

For every $y \in \mathcal{Y}\Delta(a\Delta\mathcal{Y})$ we have $\tilde{\Theta}(y)\Delta\tilde{\Theta}(y\Delta a) = b$.

Thus the graph of $\tilde{\Theta}$ is both analytic and coanalytic, and $\tilde{\Theta}$ is Borel-measurable. Finally, Pettis's theorem ([**105**]) implies that every Borel-measurable homomorphism between Polish groups is continuous, in particular that $\tilde{\Theta}$ is continuous. □

The following lemma will be used to show that for Borel Fubini ideals the group Radon–Nikodym property implies the Radon–Nikodym property.

**Lemma 4.3.3.** *If $a \subseteq \mathbb{N}$ is finite and $|a| \geq 2$, then the measure of the set*

$$\{(x,y) \in \mathcal{P}(\mathbb{N}) : |x \cap a| \textit{ and } |y \cap a| \textit{ are odd and } |(x \cup y) \cap a| \textit{ is even}\}$$

*is at least* $7/64$.

Proof. Clearly the measure of the set in question is equal to the measure of

$$\mathcal{X} = \{(x,y) \in \mathcal{P}(a) : |x| \text{ and } |y| \text{ are odd and } |x \cup y \text{ is even}\}).$$

First we prove that for nonempty $c$, $|\{x \subseteq c : |x| \text{ is odd}\}| = 2^{|c|-1}$. If $|c|$ is odd then this is obvious, since the transformation $x \mapsto c \setminus x$ is bijective.

If $|c|$ is even and $c$ is nonempty, fix $m \in c$. Then

$$\begin{aligned} \{x \subseteq c : |x| \text{ is odd}\} =& \{x \cup \{m\} : x \subseteq c \setminus \{m\} \text{ and } |x| \text{ is even}\} \\ & \cup \{x : x \subseteq c \setminus \{m\} \text{ and } |x| \text{ is odd}\}. \end{aligned}$$

The cardinality of each of the two sets on the right-hand side is half the cardinality of $\mathcal{P}(c \setminus \{m\})$, and the desired conclusion follows.

Fix $a$. If $(x,y) \in \mathcal{P}(a)^2$ then $(x,y) \in \mathcal{X}$ if and only if $|x|$ is odd, $|y \setminus x|$ is odd, and $|x \cap y|$ is even. For every nonempty $x \subseteq a$ with $|x| = k < |a|$ the set $\mathcal{Y}_{a,x} = \{d : |c \cap d| \text{ even}, |d \setminus c| \text{ odd}\}$ has cardinality $2^{k-1} \cdot 2^{|a|-k-1} = 2^{|a|-2}$.

If $|a|$ is even, then $x \subseteq a$ of odd cardinality is a proper nonempty subset, and therefore in this case $|\mathcal{X}| = |\{x \subseteq a : |x| \text{ is odd}\}| \cdot 2^{|a|-4} = 2^{2|a|-8}$, and the measure of $\mathcal{X}$ is $|\mathcal{X}|2^{-2|a|} = 1/8$.

If $|a|$ is odd, then for every $x \subseteq a$ such that $x \neq a$ and $|x|$ is odd we still have $|\mathcal{Y}_{a,x}| = 2^{|a|-4}$. Therefore $|\mathcal{X}| = (2^{|a|-1} - 1) \cdot 2^{|a|-4}$. The measure of $\mathcal{X}$ is the smallest when $|a| = 3$, in which case it is $7/64$. □

**Lemma 4.3.4.** *Suppose that $\mathcal{I}$ has the Fubini property and that $\Phi\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})/\mathcal{I}$ is a Boolean algebra homomorphism that has a lifting $\Theta\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ which is a continuous group homomorphism. Then $\Phi$ has a lifting which is a continuous Boolean algebra homomorphism.*

Proof. By Lemma 4.1.3 there are finite $s_n \subseteq \mathbb{N}$ such that

$$\Theta(a) = \{n : |a \cap s_n| \text{ is odd}\}.$$

Since $\Theta(\mathbb{N})\Delta\mathbb{N} \in \mathcal{I}$, the set $C = \{n : s_n = \emptyset\}$ belongs to $\mathcal{I}$.

We claim that $C = \{n : |s_n| \geq 2\} \in \mathcal{I}$.

Assume otherwise and let $\mathcal{X} = \{(n, b, c) : n \in (\Theta(b) \cap \Theta(c)) \setminus \Theta(b \cup c)\}$. Then

$$\mathcal{X} = \{(n, b, c) : |b \cap s_n| \text{ and } |c \cap s_n| \text{ are odd and } |(b \cup c) \cap s_n \text{ is even}\}.$$

By Lemma 4.3.3, $\lambda(\mathcal{X}_n) \geq 7/64$ for every $n \in C$, and therefore by the Fubini property of $\mathcal{I}$ there is $(b, c)$ such that $\mathcal{X}^{(b,c)} \notin \mathcal{I}$; contradiction.

Therefore the set $D$ of $n$ such that $s_n$ is not a singleton belongs to $\mathcal{I}$. For $n \in D$ redefine $s_n = \{p_n\}$ where $p_n$ is the $n$-th digit of $\pi$ (or whatever). Then $\tilde{\Theta}(a) = \{n : |a \cap s_n| \text{ is odd}\}$ still lifts $\Phi$ and it is a Boolean algebra homomorphism by Lemma 4.1.3. □

Proof of Theorem 4.3.1. To prove the group case, fix a group homomorphism $\Phi\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})/\mathcal{I}$ with a Baire measurable lifting. By Lemma 3.5.3 (1) it has a continuous lifting. It will suffice to assume that it has a Lebesgue-measurable lifting $\Theta$. By Lemma 4.3.2 it will suffice to prove that the set of $m \in \mathbb{N}$ such that

$$\lambda(\{(a, b) : m \in \Theta(b)\Delta\Theta(c)\Delta\Theta(b\Delta c)\} \geq 1/6$$

belongs to $\mathcal{I}$. Otherwise, by the Fubini property of $\mathcal{I}$ there is a pair $(b, c) \in \mathcal{P}(\mathbb{N})$ the set $\Theta(b)\Delta\Theta(c)\Delta\Theta(b\Delta c) \notin \mathcal{I}$; contradiction.

The Boolean algebra case follows from the group case and Lemma 4.3.4. □

The Radon–Nikodym property of ordinal ideals $\mathcal{O}_\alpha$ (Definition 1.9.1) may be related to a question of Galvin ([70]) concerning the partially ordered sets $P(\alpha) = \langle \mathcal{P}(\alpha)/\mathcal{O}_\alpha, \subseteq^{\mathcal{O}_\alpha}\rangle$. The Radon–Nikodym property of $\mathcal{O}_\alpha$ implies, for example, that there is no definable Boolean algebra monomorphism of $P(\omega^3)$ into $P(\omega^\omega)$. Galvin ([70]) asked whether there is (provably without using any additional set-theoretic axioms) a *strictly increasing* mapping from $P(\omega^3)$ into $P(\omega^\omega)$ (see [36, §3], [35] for more details).

## 4.4. Failure of the Radon–Nikodym property

Ulam-stability, together with the notion of an asymptotically additive liftings, has played a crucial role in discovering the lifting results of [40]. These had been replaced by Kanovei–Reeken's results on Fubini property, supplemented by Biba's trick in the proof of the new OCA Lifting Theorem (Theorem 6.1.2). This section is the only place where Ulam-stability makes an (implicit) appearance. For more, see [59, §4].

**Theorem 4.4.1.** *There are an $F_\sigma$ P-ideal $\mathcal{I}$ and a homomorphism $\Psi\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})/\mathcal{I}$ with a continuous lifting but no completely additive lifting.*

The proof is given after the following result, taken from [37] and Theorem [40, Theorem 1.8.2].

**Theorem 4.4.2.** *For every $m$ there are finite sets $I$ and $J$, a submeasure $\varphi$ on $J$, and $\Psi\colon \mathcal{P}(I) \to \mathcal{P}(J)$ with the following properties for all subsets $a$ and $b$ of $I$.*

*(1)* $\varphi((J \setminus \Psi(a))\Delta\Psi(I \setminus a)) < 1/m$.

*(2)* $\varphi(\Psi(a) \cup \Psi(b))\Delta\Psi(a \cup b)) < 1/m$.

*(3) For every $h\colon J \to I$ some $a \in \mathcal{P}(I)$ satisfies $\varphi(h^{-1}(a)\Delta\Psi(a)) > 1$.*

Proof. Fix a finite set $I$ such that $|I| > 2^{3m+1}$. For $j \in I$, consider the principal ultrafilter

$$\langle j\rangle = \{s \in \mathcal{P}(I) : k \in s\}$$

and let $J = \mathcal{P}(\mathcal{P}(I) \setminus \{\langle j\rangle : j \in I\}\}$. By capital letters $X, Y, Z$ we denote elements of $J$. For $X \in J$ let

$$\mathcal{C}_X = \{Z \in J : (Z\Delta\langle k\rangle) \cap X \neq \emptyset \text{ for all } k \in I\}.$$

Define $\varphi\colon \mathcal{P}(J) \to [0,\infty)$ by

$$\varphi(A) = \frac{1}{3m}\min\{|X| : X \in J \text{ and } A \subseteq \mathcal{C}_X\}.$$

Since $X \subseteq Y$ implies $\mathcal{C}_X \subseteq \mathcal{C}_Y$ and $\mathcal{C}_X \cup \mathcal{C}_Y \subseteq \mathcal{C}_{X\cup Y}$, the function $\varphi$ is monotonic and subadditive, and therefore a submeasure. Let

$$\Psi(s) = \{X \in J : s \in X\}.$$

Every $a \subseteq I$ satisfies $(\mathcal{P}(J) \setminus \Psi(a))\Delta\Psi(\mathcal{P}(I) \setminus a) \subseteq \mathcal{C}_{\{a,\mathcal{P}(I)\setminus a\}}$, and therefore $\varphi((\mathcal{P}(J)\setminus\Psi(a))\Delta\Psi(\mathcal{P}(I)\setminus a)) \leq \frac{2}{3m}$. Similarly, $(\Psi(a)\cup\Psi(b))\Delta\Psi(a\cup b) \subseteq \mathcal{C}_{\{a,b,a\cup b\}}$ for all $a$ and $b$ in $\mathcal{P}(I)$, thus $\varphi((\Psi(a) \cup \Psi(b))\Delta\Psi(a \cup b)) \leq \frac{1}{m}$.

To verify (3), fix $h\colon J \to I$ and assume towards contradiction that for all $a \in \mathcal{P}(I)$ we have $\varphi(h^{-1}(a)\Delta\Psi(a)) \leq 1$. For $k \in I$ fix $X(k) \in J$ of cardinality $\leq 3m$ such that $\Psi(\{k\})\Delta h^{-1}(\{k\}) \subseteq \mathcal{C}_{X(a)}$. Since $\{Y \in J : (Y\Delta\langle k\rangle) \cap X = \emptyset\} \supseteq \mathcal{C}_X$ for every $X$ and $\Psi(\{k\}) = \{X \in J : \{k\} \in X\}$, we have

$$\begin{aligned} h^{-1}(\{k\}) &\supseteq \{Y \in \Psi(\{k\}) : (Y\Delta\langle k\rangle) \cap X(k) = \emptyset\} \\ &= \{Y \in J : \{k\} \in Y \text{ and } (Y\Delta\langle k\rangle) \cap X(k) = \emptyset\}. \end{aligned}$$

Since $|I| > 2^{3m+1}$, the cardinality of this set is not smaller than

$$|\mathcal{P}(\mathcal{P}(I) \setminus (\{\langle j\rangle : j \in I\} \cup X(k))\}| = 2^{2^{|I|}-|I|-3m} > |I|^{-1} \cdot 2^{2^{|I|}-|I|}.$$

The sets $h^{-1}(\{k\})$, for $k \in I$, are pairwise disjoint, and therefore

$$|J| > \left|\bigcup_{k\in I} h^{-1}(\{k\})\right| \geq 2^{2^{|I|}-|I|} = |J|;$$

contradiction. □

A result for group homomorphisms analogous to Theorem 4.4.2 appears in [41], see also [100].

Proof of Theorem 4.4.1. Fix $\varepsilon_j > 0$, for $j \in \mathbb{N}$, such that $\sum_j \varepsilon_j < \infty$. By Theorem 4.4.2 there are finite sets $I_j$ and $J_j$, submeasure $\varphi_j$ on $J_j$, and $\Psi_j\colon \mathcal{P}(I_j) \to \mathcal{P}(J_j)$ such that $\Psi_j$ is an $\varepsilon_j$-approximate homomorphism with respect to $\varphi_j$ that cannot be 1-approximated (with respect to $\varphi_j$) by a homomorphism.

We may assume that $\mathbb{N} = \bigsqcup_j I_j = \bigsqcup_j J_j$. Let $\varphi = \sum_j \varphi_j$. This is a lower semicontinuous submeasure and $\mathcal{I} = \mathrm{Exh}(\varphi) = \mathrm{Fin}(\varphi)$ is our ideal. The function $\Psi(A) = \bigcup_j \Psi_j(A \cap I_j)$ is a continuous lifting of a homomorphism from $\mathcal{P}(\mathbb{N})$ into $\mathcal{P}(\mathbb{N})/\mathcal{I}$.

We claim that this homomorphism does not have a completely additive lifting. Assume otherwise and fix $h\colon \mathbb{N} \to \mathbb{N}$ such that $A \mapsto h^{-1}(A)$ is such a lifting. We claim that

$$\lim_j \varphi(\Psi(I_j))\Delta h^{-1}(I_j)) = 0.$$

Otherwise, assume $\varepsilon > 0$ is such that $X = \{j\varphi(\Psi(I_j))\Delta h^{-1}(I_j)) > \varepsilon\}$ is infinite. By using the lower semicontinuity of $\varphi$ and the fact that the sets $h^{-1}(I_j)$ are disjoint, we can recursively find an infinite $Y \subseteq X$ such that $\varphi(h^{-1}(I_j) \cap \bigcup_{i\in Y\setminus\{j\}} I_i) < 2^{-j}$ for all $j \in Y$. This implies that $\Psi(\bigcup_{j\in Y} I_j)\Delta h^{-1}(\bigcup_{j\in Y} I_j)$ does not belong to $\mathrm{Exh}(\varphi)$; contradiction.

Fix $n$ large enough to have $\varphi(\Psi(I_j))\Delta h^{-1}(I_j)) < 1/2$ for $j \geq n$. By the choice of $I_j, J_j$ and $\Psi_j$, there is $A_j \subseteq I_j$ such that $\varphi_j(h^{-1}(A_j)\Delta\Psi_j(A_j)) > 1/2$. Then $A = \bigcup_{j\geq n} A_j$ witnesses that $A \mapsto h^{-1}(A)$ is not a lifting. □

With $\mathcal{KL}_{\mathcal{P}(\mathbb{N})}$ denoting the ideal defined in §1.6.0.1, a proof similar to the proof of Theorem 4.4.2 shows that (using the notation from §1.6.0.1) the function

$$A \mapsto \textstyle\bigcup_{p\in\mathbb{I}} f(p, A)$$

where $f(p, A) = \emptyset$ if $A \notin U^p$ and $f(p, A) = A\Delta A_{s^p}$ if $A \in U^p_s$ lifts a homomorphism from $\mathcal{P}(\mathbb{N})$ into $\mathcal{P}(\mathbb{I} \times \mathbb{N})/\mathcal{LV}_{\mathcal{P}(\mathbb{N})} \times \emptyset$ that has no completely additive lifting (see [100] for details).

## 4.5. Applications of lifting theorems, I

The following variant of Proposition 4.1.6 gives a strong answer to the basic question for non-pathological analytic P-ideals.

**Proposition 4.5.1.** *Suppose that $\mathcal{I}$ and $\mathcal{J}$ are analytic ideals and $\mathcal{J}$ has the Fubini property. Then the following holds.*

*(1) $\mathcal{P}(\mathbb{N})/\mathcal{I}$ and $\mathcal{P}(\mathbb{N})/\mathcal{J}$ are Baire-isomorphic if and only if $\mathcal{I}$ and $\mathcal{J}$ are RK-isomorphic.*

*(2) $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is Baire-embeddable into $\mathcal{P}(\mathbb{N})/\mathcal{J}$ if and only if $\mathcal{I}$ is RK-reducible to $\mathcal{J}$.*

Proof. By Theorem 4.3.1, the ideal $\mathcal{J}$ has the Fubini property and therefore both $\mathcal{I}$ and $\mathcal{J}$ have the Radon–Nikodym property, hence the conclusion follows by Proposition 4.1.6. □

**Theorem 4.5.2.** *If $\mathcal{I}$ and $\mathcal{J}$ are analytic ideals, $\mathcal{P}(\mathbb{N})/\mathcal{J}$ is Baire-embeddable into $\mathcal{P}(\mathbb{N})/\mathcal{I}$, and $\mathcal{I}$ has the Fubini property, then $\mathcal{J}$ has the Fubini property.*

Proof. Since $\mathcal{I}$ has the Fubini property, it has the Radon–Nikodym property and $\mathcal{J} \leq_{\mathrm{BE}} \mathcal{I}$ implies $\mathcal{J} \leq_{\mathrm{RK}} \mathcal{I}$. If $\mathcal{J}$ fails the Fubini property then Theorem 4.2.9 implies that $\mathcal{S} \leq_{\mathrm{K}} \mathcal{J}$. Together with $\mathcal{J} \leq_{\mathrm{RK}} \mathcal{I}$ this implies $\mathcal{S} \leq_{\mathrm{K}} \mathcal{I}$, and by Theorem 4.2.9, $\mathcal{I}$ fails the Fubini property; contradiction. □

The following is an immediate corollary of Theorem 2.6.10 and the Radon–Nikodym property of summable ideals.

**Theorem 4.5.3.** *The quasi-ordered set of all dense summable ideals with respect to $\leq_{\mathrm{RB}}$ has the following properties.*

*(1) It has no maximal elements.*

*(2) It has no minimal elements.*

(3) *It includes an isomorphic copy of* $\langle \mathcal{P}(\mathbb{N})/\operatorname{Fin}, \subseteq^* \rangle$.

(4) *If* $\mathcal{I}_f <_{\mathrm{RB}} \mathcal{I}_g$ *then some* $h$ *satisfies* $\mathcal{I}_f <_{\mathrm{RB}} \mathcal{I}_h <_{\mathrm{RB}} \mathcal{I}_g$. □

**Theorem 4.5.4.** *If density ideals* $\mathcal{Z}_\mu$ *and* $\mathcal{Z}_\nu$ *satisfy (for* $G_{\nu\delta}$ *and* $F_\mu$ *see Definition 2.7.15)*

$$\lim_i \operatorname{at}^+(\mu_i) = \lim_i \operatorname{at}^+(\nu_i) = 0 \;\;\&\;\; G_{\nu\delta} \circ F_\mu = o(n)$$

*for all* $\delta > 0$*, then the quotients* $\mathcal{P}(\mathbb{N})/\mathcal{Z}_\mu$ *and* $\mathcal{P}(\mathbb{N})/\mathcal{Z}_\nu$ *are not Baire-isomorphic.*

Proof. These ideals are not RK-isomorphic by Theorem 2.7.16. Assume that their quotients are Baire-isomorphic. Since the density ideals have the Radon–Nikodym property (Theorem 4.1.2), Proposition 4.5.1 implies that the ideals $\mathcal{Z}_\mu$ and $\mathcal{Z}_\nu$ are isomorphic; contradiction. □

CHAPTER 5

# ZFC results about quotients

In this chapter we isolate some absolute properties of quotients $\mathcal{P}(\mathbb{N})/\mathcal{I}$ that can be used to prove non-isomorphism of several analytic quotients in ZFC (see Proposition 5.1.7). Sequential topology on Boolean algebras is defined and used to prove that $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is a complete metric space for every analytic P-ideal (Proposition 5.2.2) and that the quotients over $\mathsf{nwd}$ and $\mathsf{null}$ are not isomorphic (Proposition 5.2.5, taken from [62]). By a result of Fremlin ([66]) included as Theorem 5.3.1, the measure algebra of Maharam character $\mathfrak{c}$ embeds into the quotient over $\mathcal{Z}_0$. By §5.4, embeddability between analytic quotients differs from Baire-embeddablity, provably in ZFC.[1]

## 5.1. Small sets and deep sets

The results of this section had been adapted from [49]. Proposition 5.1.7 shows that no quotient over a density ideal is isomorphic to one over an LV-ideal, and that there are at least two isomorphism types of dense density ideals.

The following is a standard model-theoretic definition. The readers of [54] should take note that $\aleph_1$-saturated structures were called 'countably saturated' there.

**Definition 5.1.1.** A quotient $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is $\aleph_1$-*saturated* if every consistent countable consistent type over it is realised in it.

**Definition 5.1.2.** Let $\mathcal{I}$ be an ideal. A set $A \subseteq \mathbb{N}$ is $\mathcal{I}$-*small* if there are $A_s \subseteq \mathbb{N}$, for $s \in \{0,1\}^{<\mathbb{N}}$, such that all $s$ the following holds.

1. $A_{\langle\rangle} = A$,
2. $A_s = A_{s\frown 0} \cup A_{s\frown 1}$,
3. $A_{s\frown 0} \cap A_{s\frown 1} = \emptyset$, and
4. For every $b \in \{0,1\}^{\mathbb{N}}$, if $X \setminus A_{b\restriction n} \in \mathcal{I}$ for all $n$, then $X \in \mathcal{I}$.

A set $A \subseteq \mathbb{N}$ is $\mathcal{I}$-*deep* if the quotient $\mathcal{P}(A)/(\mathcal{I} \restriction A)$ is $\aleph_1$-saturated (see Definition A.2.1). Let

$$\mathsf{S}_{\mathcal{I}} = \{A \subseteq \mathbb{N} : A \text{ is } \mathcal{I}\text{-small}\},$$
$$\mathsf{D}_{\mathcal{I}} = \{A \subseteq \mathbb{N} : A \text{ is } \mathcal{I}\text{-deep}\},$$

**Lemma 5.1.3.** *Suppose that $\mathcal{I}$ is an ideal on $\mathbb{N}$.*

*(1) $\mathsf{S}_{\mathcal{I}}$ is an ideal that includes $\mathcal{I}$.*
*(2) $\mathsf{D}_{\mathcal{I}}$ is an ideal that includes $\mathcal{I}$.*
*(3) An isomorphism between $\mathcal{P}(\mathbb{N})/\mathcal{I}$ and $\mathcal{P}(\mathbb{N})/\mathcal{J}$ sends equivalence classes of $\mathcal{I}$-small sets into equivalence classes of $\mathcal{J}$-small sets and the equivalence classes of $\mathcal{I}$-deep sets into the equivalence classes of $\mathcal{J}$-deep sets.*

[1] In the realm of C*-algebras this does not happen, as the wonderful [160] shows.

*(4)* $\mathcal{I} \subseteq \mathsf{D}_\mathcal{I}$, $\mathcal{I} \subseteq \mathsf{S}_\mathcal{I}$, *and* $\mathsf{D}_\mathcal{I} \cap \mathsf{S}_\mathcal{I} \subseteq \mathcal{I}$. □

**Proposition 5.1.4.** *(1) If* $\mathcal{Z}_\varphi$ *is an EU-ideal, then* $\mathsf{S}_{\mathcal{Z}_\mu} = \mathcal{P}(\mathbb{N})$.

*(2) If* $\mathcal{Z}_\varphi$ *is a generalised density ideal such that* $\limsup_n \mu_n(I_n) = \infty$, *then* $\mathsf{S}_{\mathcal{Z}_\mu}$ *is a proper* $F_\sigma$ *ideal properly including* $\mathcal{Z}_\mu$.

*(3) If* $\mathcal{Z}_\mu$ *is a density ideal then every* $\mathcal{Z}_\mu$*-positive set* $A$ *contains a positive subset that belongs to* $\mathsf{S}_{\mathcal{Z}_\mu}$.

*(4) If* $\mathcal{I}$ *is a dense LV-ideal, then* $\mathsf{S}_\mathcal{I} = \mathcal{I}$.

*(5) If* $\mathcal{I}$ *is a dense generalised density ideal, then* $\mathsf{D}_\mathcal{I} = \mathcal{I}$.

Proof. (1) It suffices to prove that $\mathbb{N} \in \mathsf{S}_{\mathcal{Z}_\varphi}$. Recursively define $A_s$ as in Definition 5.1.2 and so that for every $s$ we have $\limsup_n |\varphi_n(A \cap I_n) - 2^{-|s|}| = 0$. Then all $b \in 2^\mathbb{N}$ and $X$ such that $X \setminus A_{b\restriction n} \in \mathcal{Z}_\varphi$ for all $n$ satisfy $X \in \mathcal{Z}_\varphi$.

(2) We will prove that

$$\mathsf{S}_{\mathcal{Z}_\varphi} = \{A : \limsup_n \varphi_n(A) < \infty\}.$$

This is clearly an $F_\sigma$ ideal. Fix a $\mathcal{Z}_\varphi$-positive $A \subseteq \mathbb{N}$.

Suppose that $\limsup_n \varphi_n(A) = \infty$. Let $A_s$, for $s \in \{0,1\}^{<\mathbb{N}}$, be as in Definition 5.1.2. Recursively choose $s_j$, for $j \in \mathbb{N}$, so that $s_0 = \langle\rangle$, $s_j \sqsupset s_{j+1}$, and $\limsup_n \varphi_n(A_{s_j}) = \infty$ for all $j$. Choose an increasing sequence $n_j$, for $j \in \mathbb{N}$, such that $\mu_{n_j}(A_{s_j}) \geq j$. The set $X = \bigcup_j (A_{s_j} \cap I_{n_j})$ is included in each $A_{s_j}$ modulo finite, and it is $\mathcal{Z}_\mu$-positive. Therefore, $A \notin \mathsf{S}_{\mathcal{Z}_\varphi}$.

Now suppose that $\limsup_n \mu_n(A) < \infty$. We may assume $A$ is $\mathcal{Z}_\varphi$-positive, in which case $\mathcal{Z}_\varphi \restriction A$ is isomorphic to an EU-ideal by Theorem 2.7.8. This implies $A \in \mathsf{S}_{\mathcal{Z}_\varphi}$ by (1).

Clause (3) follows immediately from the characterization of $\mathsf{S}_{\mathcal{Z}_\varphi}$ given in (2).

(4) Let $\varphi_n$ and $\varphi = \sup_n \varphi_n$ be the submeasures defining $\mathcal{I}$. By (LV2), with $\varphi_\infty(A) = \lim_n \varphi(A \setminus n))$, we have

$$\varphi_\infty(A \cup B) = \max(\varphi_\infty(A), \varphi_\infty(B)).$$

Note that $\mathcal{I} = \mathrm{Exh}(\varphi) = \{A : \varphi^\infty(A) = 0\}$. Hence, if $A_{\langle\rangle}$ is positive and $A_s$ are as in Definition 5.1.2, one can then recursively choose a branch $b$ in $\{0,1\}^\mathbb{N}$ such that $\varphi_\infty(A_{b\restriction n}) = \varphi_\infty(A_{\langle\rangle}) = \delta > 0$ for all $n$. Then there are finite pairwise disjoint sets $s_n \subseteq A_{b\restriction n} \setminus A_{b\restriction(n-1)}$ such that $\varphi(s_n) \geq \delta/2$ for all $n \geq 1$. Then the set $X = \bigcup_n s_n$ is not in $\mathrm{Exh}(\varphi)$, but $X \setminus A_{b\restriction m}$ is finite for all $m$.

(5) Assume $\mathcal{I}$ is a dense density ideal or a dense LV-ideal. If $\varphi$ is the natural lower semicontinuous submeasure such that $\mathcal{I} = \mathrm{Exh}(\varphi)$ and $A$ is a positive set, recursively construct $\mathcal{I}$-positive sets $A = A_1 \supset A_2 \supset A_3 \ldots$ such that $\varphi(A_n) < 1/n$ for all $n$. Then the only lower bound for $A_n$ is $[\emptyset]_\mathcal{I}$. □

We return to the ideal introduced in Lemma 2.8.1. For $A \subseteq \mathbb{N}^2$ and $m \in \mathbb{N}$ let $\mu_m(A) = \sum_{(m,n)\in A} 1/mn$ and $\mathcal{I}_\infty = \mathrm{Exh}(\sup_m \mu_m)$.

**Lemma 5.1.5.** *If* $A \subseteq \mathbb{N}$ *is* $\mathcal{I}_\infty$*-positive then the following are equivalent.*

*(1) The quotient* $\mathcal{P}(A)/()\mathcal{I}_\infty \restriction A)$ *is* $\aleph_1$*-saturated.*

*(2)* $\mathcal{I}_\infty \restriction A$ *is summable.*

*(3)* $(\exists B \in \mathrm{Fin} \times \emptyset) A \setminus B \in \mathcal{I}_\infty$.

Proof. All summable ideals are $F_\sigma$, hence by Corollary 11.1.8 (2) implies (1). If (3) holds, then $\mathcal{I}_\infty \restriction A$ is obviously summable.

To prove that (1) implies (3), assume that the latter fails. Then there is $\varepsilon > 0$ such that the set

$$C = \{n : \mu_n(A) \geq \varepsilon\}$$

is infinite. We may assume $\mathrm{at}^+(\mu_n) < \varepsilon/2$ for all $n \in C$. For $n \in C$ find $B_n \subseteq A \cap I_n$ such that $\mu_n(B_n) \geq \varepsilon/2$, and let $B = \bigcup_{n \in C} B_n$. Then $\mathcal{I}_\infty \restriction B$ is a proper dense density ideal. Proposition 5.1.4 (5) asserts that $\mathsf{D}_{\mathcal{I}_\infty \restriction B} = \mathcal{I}$, i.e., (1) fails. □

**Lemma 5.1.6.** *Every $C \subseteq \mathbb{N}^2$ set is either in $\mathsf{D}_{\mathcal{I}_\infty}$ or it includes an $\mathcal{I}_\infty$-positive set in $\mathsf{S}_{\mathcal{I}_\infty}$.*

Proof. If $C \notin \mathsf{D}_{\mathcal{I}_\infty}$, then by Lemma 5.1.5 the set $A = \{n : C \restriction \{n\} \times \mathbb{N} \notin \mathcal{I}_\infty\}$ is infinite. For each $n \in A$ pick $J_n \subseteq C \cap (\{n\} \times \mathbb{N})$ such that $\mu_n(J_n) \geq 1$ and let $B = \bigcup_{n \in A} J_n$. Then $\mathcal{I}_\infty \restriction B$ is a dense density ideal, so by (3) of Proposition 5.1.4 it contains a positive set in $\mathsf{S}_{\mathcal{I}_\infty}$. □

If $\mathcal{J}_1$ and $\mathcal{J}_2$ are ideals such that $\mathcal{J}_1 \cap \mathcal{J}_2 \supseteq \mathcal{I}$, we say that $\mathcal{J}_1$ and $\mathcal{J}_2$ form a *pregap* over $\mathcal{I}$. A pregap is *split* by $C \subseteq \mathbb{N}$ if $\mathcal{J}_1 \restriction C \subseteq \mathcal{I}$ and $\mathcal{J}_2 \restriction (\mathbb{N} \setminus C) \subseteq \mathcal{I}$. If no $C$ splits a pregap, we say that it is a *gap* over $\mathcal{I}$. By (4) of Lemma 5.1.3, $\mathsf{S}_{\mathcal{I}}$ and $\mathsf{D}_{\mathcal{I}}$ form a pregap over $\mathcal{I}$.

Recall that by Lemma 1.7.6 and Lemma 2.8.2 we have $\mathcal{Z}_0 \oplus \mathcal{Z}_\infty \sim_{\mathrm{RK}} \mathcal{Z}_\infty$ and $\mathcal{I}_\infty \sim_{\mathrm{RK}} \mathcal{I}_\infty \oplus \mathcal{I}_{1/n} \sim_{\mathrm{RK}} \mathcal{I}_\infty \oplus \mathcal{Z}_0 \sim_{\mathrm{RK}} \mathcal{I}_\infty \oplus \mathcal{Z}_\infty$. It turns out that two sum of the ideals $\mathcal{Z}_0, \mathcal{Z}_\infty, \mathcal{LV}, \mathcal{I}_{1/n}, \emptyset \times \mathrm{Fin}$ and $\mathcal{I}_\infty$ are either isomorphic by these two lemmas, or they have nonisomorphic quotients.

**Proposition 5.1.7.** *Quotients over analytic P-ideals listed below are pairwise nonisomorphic.*[2]

(1) $\mathcal{Z}_\infty$, $\mathcal{Z}_0$, $\mathcal{LV}$, $\mathcal{Z}_\infty \oplus \mathcal{LV}$, $\mathcal{Z}_0 \oplus \mathcal{LV}$,
(2) $\mathcal{Z}_\infty \oplus \mathcal{I}_{1/n}$, $\mathcal{Z}_0 \oplus \mathcal{I}_{1/n}$, $\mathcal{LV} \oplus \mathcal{I}_{1/n}$, $\mathcal{Z}_\infty \oplus \mathcal{LV} \oplus \mathcal{I}_{1/n}$, $\mathcal{Z}_0 \oplus \mathcal{LV} \oplus \mathcal{I}_{1/n}$,
(3) $\mathcal{Z}_\infty \oplus (\emptyset \times \mathrm{Fin})$, $\mathcal{Z}_0 \oplus (\emptyset \times \mathrm{Fin})$, $\mathcal{LV} \oplus (\emptyset \times \mathrm{Fin})$, $\mathcal{Z}_\infty \oplus \mathcal{LV} \oplus (\emptyset \times \mathrm{Fin})$, $\mathcal{Z}_0 \oplus \mathcal{LV} \oplus (\emptyset \times \mathrm{Fin})$,
(4) $\mathcal{I}_{1/n}$, $\emptyset \times \mathrm{Fin}$,
(5) $\mathcal{I}_\infty$, $\mathcal{I}_\infty \oplus \mathcal{LV}$, $\mathcal{I}_\infty \oplus \emptyset \times \mathrm{Fin}$, $\mathcal{I}_\infty \oplus \mathcal{LV} \oplus \emptyset \times \mathrm{Fin}$.

Proof. By Lemma 5.1.3, we only need to prove that the pairs of ideals $\mathsf{S}_{\mathcal{I}}$ and $\mathsf{D}_{\mathcal{I}}$ associated to these the fifteen ideals listed above are sufficiently different. By (1)–(4) of Proposition 5.1.4, the five ideals in (1) all have different $\mathsf{S}_{\mathcal{I}}$ and $\mathsf{D}_{\mathcal{I}} = \mathcal{I}$, by (5) of Proposition 5.1.4. Since $\mathsf{D}_{\mathrm{Fin}} = \mathcal{P}(\mathbb{N})$, for the ideals $\mathcal{I}$ in (2) the ideal $\mathsf{D}_{\mathcal{I}}$ is generated by a single set over $\mathcal{I}$. Since $\mathsf{D}_{\emptyset \times \mathrm{Fin}} = \mathrm{Fin} \times \emptyset$, an ideal generated by a countable family of infinite pairwise disjoint sets, for the ideals $\mathcal{I}$ in (3) the ideal $\mathsf{D}_{\mathcal{I}}$ is generated by a countable family of infinite pairwise disjoint sets.

The only two ideals $\mathcal{I}$ on the list such that $\mathsf{S}_{\mathcal{I}} = \mathcal{I}$ are Fin and $\emptyset \times \mathrm{Fin}$, hence the quotients over the ideals in (4) are not isomorphic to any of the others. Since one of them is $\aleph_1$-saturated and the other is not, they are not isomorphic to each other.

By Lemma 5.1.6, ideals $\mathsf{S}_{\mathcal{I}_\infty}$ and $\mathsf{D}_{\mathcal{I}_\infty}$ form a gap over $\mathcal{I}_\infty$. Since every $\mathcal{J}$ among $\mathcal{Z}_\infty, \mathcal{Z}_0, \mathcal{LV}, \mathcal{I}_{1/n}$, and $\emptyset \times \mathrm{Fin}$ satisfies either $\mathsf{D}_{\mathcal{J}} = \mathcal{J}$ or $\mathsf{S}_{\mathcal{J}} = \mathcal{J}$, all ideals $\mathcal{I}$ listed in (1)–(4) have the property that $\mathsf{S}_{\mathcal{I}}$ and $\mathsf{D}_{\mathcal{I}}$ are separated. Therefore no quotients over an ideals in (5) is isomorphic to the quotient over any of the ideals in (1)–(4).

[2]These ideals are separated in five lists only for readability.

It remains to distinguish the quotients over the ideals in (5). Clause (4) of Proposition 5.1.4 implies that any ideal of the form $\mathcal{J} = \mathcal{I} \oplus \mathcal{LV}$ has a positive set $A$ such that $\mathsf{S}_{\mathcal{J}} \restriction A = \mathsf{D}_{\mathcal{J}} \restriction A = \mathcal{J} \restriction A$. On the other hand, if $A \notin \mathsf{D}_{\mathcal{I}_\infty}$, then $A$ has an $\mathcal{I}_\infty$-positive subset $B$ such that $\mathcal{I}_\infty \restriction B$ is a density ideal, hence $A \in \mathsf{S}_{\mathcal{I}_\infty}$. The ideal $\mathcal{I}_\infty \oplus \emptyset \times \mathrm{Fin}$ has this property as well. Therefore neither of the quotients over $\mathcal{I}_\infty$ or $\mathcal{I}_\infty \oplus \mathcal{LV}$ is isomorphic to any quotient over an ideal of the form $\mathcal{I} \oplus \mathcal{LV}$.

Finally, if $\mathcal{J} \in \{\mathcal{I}_\infty, \mathcal{I}_\infty \oplus \mathcal{LV}\}$ and $A$ is $\mathsf{D}_{\mathcal{J}}$-positive, then by Lemma 5.1.6 it has a $\mathcal{J}$-positive subset $B$ such that $\mathcal{J} \restriction B$ is a dense density ideal, and therefore has a positive subset in $\mathsf{S}_{\mathcal{J}}$. But any ideal of the form $\mathcal{J} = \mathcal{I} \oplus \emptyset \times \mathrm{Fin}$ clearly has a positive set $A$ such that $\mathcal{J} \restriction A = \emptyset \times \mathrm{Fin}$, hence $A$ has no $\mathcal{J}$-positive subsets in $\mathsf{S}_{\mathcal{J}}$. Therefore, neither of the quotients over $\mathcal{I}_\infty$ or $\mathcal{I}_\infty \oplus \mathcal{LV}$ is isomorphic to the quotients over $\mathcal{I}_\infty \oplus \emptyset \times \mathrm{Fin}$ or $\mathcal{I}_\infty \oplus \mathcal{LV} \oplus \emptyset \times \mathrm{Fin}$. □

## 5.2. Sequential topology on quotients

In this section we study submeasures on Boolean algebras $\mathcal{P}(\mathbb{N})/\mathcal{I}$ (see §1.4.1, §1.4.3). Given an ideal $\mathcal{I}$ on a set $X$, we write $\pi_{\mathcal{I}}$ for the quotient map,

$$\pi_{\mathcal{I}} \colon \mathcal{P}(X) \to \mathcal{P}(X)/\mathcal{I}.$$

It will sometimes be convenient to write $[A]_{\mathcal{I}}$ for $\pi_{\mathcal{I}}(A)$, and even to drop the subscript and write $[A]$ when $\mathcal{I}$ is clear from the context.

**5.2.1. Submeasure $\varphi_\infty$ and completeness of $\mathcal{P}(\mathbb{N})/\operatorname{Exh}(\varphi)$ for lower semicontinuous $\varphi$.** Fix a lower semicontinuous submeasure $\varphi$ on a (countable) set $X$ and let $\mathcal{I} = \operatorname{Exh}(\varphi)$. For all $A$ and $B$ in $\mathcal{P}(X)$ we have $\pi_{\mathcal{I}}(A) = \pi_{\mathcal{I}}(B)$ if and only if $\lim_n \varphi((A \Delta B) \setminus n) = 0$ (limit exists the because $\varphi$ is nonincreasing). Therefore the following is straightforward to verify.

**Lemma 5.2.1.** *Suppose that $\varphi$ is a lower semicontinuous submeasure on $\mathbb{N}$ and let $\mathcal{I} = \operatorname{Exh}(\varphi)$. Then*

$$\varphi_\infty([A]_{\mathcal{I}}) = \lim_n \varphi(A \setminus n) \tag{5.1}$$

*defines a strictly positive submeasure on $\mathcal{P}(\mathbb{N})/\mathcal{I}$.* □

I am indebted to Jon Keith and Paolo Leonetti for noticing that the proof of [40, Lemma 1.3.3 (c)] was incomplete and for completing (no pun intended!) the proof in [109, Theorem 2.4] (see also [108]). The proof of this fact given in Proposition 5.2.2 below is the proof from [62] (with details added), which combines the ideas from [40], [109], and [2]. Its case when $\varphi$ is a submeasure corresponding to an EU-ideal is [95, Lemma 3].

**Proposition 5.2.2.** *If $\varphi$ is a lower semicontinuous submeasure on $\mathbb{N}$ then with $\mathcal{I} = \operatorname{Exh}(\varphi)$, $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is a complete metric space with respect to the metric defined by (for $A, B$ in $\mathcal{P}(\mathbb{N})/\mathcal{I}$)*

$$d_\varphi(A, B) = \min(1, \varphi_\infty(A \Delta B)).$$

Proof. It will be convenient to have $A, B, C$ range over $\mathcal{P}(\mathbb{N})$ and write $[A]$ for $[A]_{\mathcal{I}}$. Fix a Cauchy sequence $[A_n]$, for $n \in \mathbb{N}$, in $\mathcal{P}(\mathbb{N})/\mathcal{I}$. By passing to a subsequence we may assume that $d_\infty([A_m], [A_n]) < 2^{-m}$ for all $m < n$.

We now find sets $\tilde{A}_m$, for $m \in \mathbb{N}$, such that $[A_m] = [\tilde{A}_m]$ for all $m$ and $\varphi(\tilde{A}_{m+1} \Delta \tilde{A}_m) < 2^{-m}$ recursively as follows. Set $\tilde{A}_0 = A_0$. Suppose that $\tilde{A}_m$ has been chosen. Let $k(m)$ be large enough so that $X_m = (A_m \Delta A_{m+1}) \setminus k(m)$ satisfies

$\varphi(X_m) < 2^{-m}$ and let $\tilde{A}_{m+1} = A_m \Delta X_m$. Since $\mathcal{I} \supseteq \mathrm{Fin}$, we have $[\tilde{A}_{m+1}] = [A_{m+1}]$ and $\tilde{A}_m \Delta \tilde{A}_{m+1} = X_m$ is as required. Let

$$B_n = \bigcup_{m \geq n} \tilde{A}_m.$$

Then $B_n \supseteq B_{n+1}$, $B_n \Delta B_{n+1} = \tilde{A}_n \setminus \bigcup_{j=n+1} \tilde{A}_j$, and

$$\varphi(B_n \Delta B_{n+1}) \leq \varphi(\tilde{A}_n \setminus \tilde{A}_{n+1}) < 2^{-n}.$$

Also, $\tilde{A}_n \Delta B_n = B_n \setminus \tilde{A}_n \subseteq \bigcup_{j=1}^{\infty} (\tilde{A}_{n+j} \setminus \tilde{A}_{n+j-1})$. Therefore

$$\varphi(\tilde{A}_n \Delta B_n) \leq \sum_{j \geq 1} 2^{-n-j+2} = 2^{-n+2}$$

and it will suffice to find limit of the Cauchy sequence $[B_n]$, for $n \in \mathbb{N}$.

Let $C = \bigcap_n B_n$. Using the lower semicontinuity of $\varphi$ we have

$$\varphi(B_n \Delta C) = \varphi(B_n \setminus C) = \lim_{k \to \infty} \varphi((B_n \setminus C) \cap k).$$

For a fixed $k \in \mathbb{N}$ we have $C \cap k = B_{n'} \cap k$ for all large enough $n'$. Such $k$ and $n'$ satisfy $(B_n \setminus C) \cap k = \bigcup_{j=0}^{n'-n-1} (B_{n+j} \setminus B_{n+j+1}) \cap k$ and therefore

$$\varphi((B_n \setminus C) \cap k) < \sum_{j \geq 0} 2^{-n-j} = 2^{-n+1}.$$

By lower semicontinuity, this implies $\varphi(B_n \setminus C) \leq 2^{-n+1}$ and $\lim_n [B_n] = [C]$, and this gives $\lim_n [A_n] = [C]$. Since we started with an arbitrary Cauchy sequence, this completes the proof. □

Let us take a closer look at the sequential topology used in the proof of Proposition 5.2.2. In a Boolean algebra $\mathbb{B}$, by $\bigvee_n A_n$ we denote the supremum of elements $A_n \in \mathbb{B}$ (if it exists) and by $\bigwedge_n A_n$ we denote their infimum (if it exists). A Boolen algebra is called *σ-complete* if these objects exist for every sequence $(A_n)$. Quotients $\mathcal{P}(\mathbb{N})/\mathcal{I}$ are rarely $\sigma$-complete (otherwise Proposition 11.1.3 would not be of much use).

**Definition 5.2.3.** Suppose that $A_n$, for $n \in \mathbb{N}$, is a sequence in a Boolean algebra $\mathbb{B}$. If for all $m$ in $\mathbb{N}$ the elements $B_m = \bigvee_{n \geq m} A_n$ and $C_m = \bigwedge_{n \geq m} A_n$ exist, and $\bigwedge_m B_m = \bigvee_m C_m = A$, then we say that the sequence $A_n$ *converges to* $A$ in the sequential topology on $\mathbb{B}$.

The proof of Proposition 5.2.2 relies on the fact that on the quotient over an analytic P-ideal the metric topology and the sequential topology agree and shows the following.

**Proposition 5.2.4.** *If $\mathcal{I}$ is an analytic P-ideal, then the sequential topology on the quotient $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is completely metrisable, and a complete metric is given by $\varphi_\infty$ for a submeasure $\varphi$ such that $\mathcal{I} = \mathrm{Exh}(\varphi)$.* □

The ideals $\mathsf{nwd}$, $\mathsf{null}$, and $\mathcal{Z}_s$ were introduced in Definition 1.8.7 and Definition 1.8.6. The first part of the following is taken from [62].

**Proposition 5.2.5.** *The quotients over* $\mathsf{nwd}$ *and* $\mathsf{null}$ *are nonisomorphic.*

*The quotients over* $\mathsf{nwd}$ *and $\mathcal{Z}_s$ are nonisomorphic.*

*None of these quotients is isomorphic to the quotient over any analytic P-ideal*

PROOF. Write $\mathcal{I} = \mathsf{nwd}$ and $\mathcal{J} = \mathsf{null}$. For the first part, it suffices to prove that the sequential topology on $\mathcal{P}(\mathbb{Q})/\mathcal{I}$‘ is not Hausdorff and that on $\mathcal{P}(\mathbb{Q})/\mathcal{J}$ is.

Enumerate $\mathbb{Q}$ as $q_n$, for $n \in \mathbb{N}$ and let $U_{n,j}$ be the open $1/j$-ball in $\mathbb{Q}$ around $q_n$ and note that $\lim_j [U_{n,j}]_\mathcal{I} = [\emptyset]_\mathcal{I}$. Suppose that $\mathcal{U}$ is an open neighbourhood of some $[A]_\mathcal{I}$ in $\mathcal{P}(\mathbb{Q})/\mathcal{I}$. Then for every $n$ some $j$ satisfies $[U_{n,j}]_\mathcal{I} \in \mathcal{U}$. Therefore, if $\mathcal{V}$ is an open neighbourhood of $[\emptyset]_\mathcal{I}$, then we can recursively find $f \colon \mathbb{N} \to \mathbb{N}$ such that $A_k = \bigcup_{n \leq k} U_{n,f(n)}$ satisfies $[A_k]_\mathcal{I} \in \mathcal{V}$ for all $k$. However, clearly $\lim_k [A_k]_\mathcal{I} = [\mathbb{Q}]_\mathcal{I}$. Wince $\mathcal{V}$ was arbitrary, this shows that the closure of every open neighbourhood of $[\emptyset]_\mathcal{I}$ contains $[\mathbb{Q}]_\mathcal{I}$.

In order to prove that the sequential topology on $\mathcal{P}(\mathbb{Q})/\mathcal{J}$ is Hausdorff it suffices to prove (writing $\lim_n$ for limits in the sequential topology and $\lambda$ for the Lebesgue measure) that if $\lim_n [A_n]_\mathcal{J} = [A]_\mathcal{J}$ then $\lim_n \lambda(\overline{A_n \Delta A}) = 0$. Suppose the contrary, that $A_n$, for $n \in \mathbb{N}$, and $A$ are subsets of $\mathbb{Q}$ such that $\lim_n [A_n]_\mathcal{J} = [A]_\mathcal{J}$, but there is $\varepsilon > 0$ such that $\lim_n \lambda(\overline{A_n \Delta A}) \geq \varepsilon$ for infinitely many $n$. Since $\lim_n [A_n]_\mathcal{J}$ exists, there are $B_n$ and $C_n$, for $n \in \mathbb{N}$, such that

$$[B_m]_\mathcal{J} = \bigvee\nolimits_{n \geq m} [A_n]_\mathcal{J}, \quad [C_m]_\mathcal{J} = \bigwedge\nolimits_{n \geq m} [A_n]_\mathcal{J}, \text{ and } \bigwedge\nolimits_m [B_m]_\mathcal{J} = \bigvee\nolimits_m [C_m]_\mathcal{J} = [A]_\mathcal{I}.$$

Assume for a moment that $\lambda(\overline{A \setminus C_m}) \geq \varepsilon/2$ for infinitely many $m$, and therefore for all $m$. We may modify sets $C_m$ without changing their $\mathcal{J}$-equivalence classes so that $C_m \subseteq C_{m+1} \subseteq A$ for all $m$. The set $F = \bigcap_m \overline{A \setminus C_m}$ then has measure at least $\varepsilon/2$. For each $m$ fix $x_m \in A \setminus C_m$ such that $d(x_m, F) \leq F$; since $F$ is separable, this can be done so that $X = \{x_m m \in \mathbb{N}\}$ has $F$ in its closure. Then $[X]_\mathcal{J} \neq [\emptyset]_\mathcal{J}$ and $[X]_\mathcal{J} \cap [C_m]_\mathcal{J} = [\emptyset]_\mathcal{J}$ for all $m$, contradicting $[A]_\mathcal{J} = \bigvee_m [C_m]_\mathcal{J}$.

Therefore $\lambda(\overline{B_m \setminus A}) \geq \varepsilon/2$ for all $m$. This assumption leads to contradiction by an argument analogous to the above. By straightforward induction on the sequential rank, the closure of a subset of $\mathcal{P}(\mathbb{Q})/\mathcal{J}$ is closed in the metric topology associated with $\lambda$, and the topology is therefore Hausdorff.

Since the sequential topology has purely algebraic definition, an isomorphism between Boolean algebras is automatically a homeomorphism and we conclude that $\mathsf{nwd}$ and $\mathsf{null}$ give rise to nonisomorphic quotients.

By [68, Proposition 2E], the quotient over $\mathcal{Z}_s$ is weakly $\sigma$-distributive. To see that the quotient over $\mathsf{nwd}$ does not have this property, choose partitions of $\mathbb{Q}$ into clopen sets $\mathbb{Q} = \bigcup_n U_{m,n}$ such that $\operatorname{diam}(U_{m,n}) \leq 1/m$ for all $m, n$ and for all $m, n$ the set $\{j : U_{m+1,j} \subseteq U_{m,n}\}$ is infinite. Let $A_{m,n} = \bigcup_{j \geq n} U_{m,j}$ for $m, n$. Then for every $f \colon \mathbb{N} \to \mathbb{N}$ the set $\bigwedge A_{m,f(m)}$ the set is nowhere dense. $\square$

In [50] I claimed that the quotients over the ideals $\mathsf{null}$ and $\mathcal{Z}_s$ are not isomorphic and that this follows from [32], but I hadn't been able to reconstruct the proof. In [131] it was shown that (in ZFC) there are continuum many Borel ideals on $\mathbb{N}$ with nonisomorphic quotients.

### 5.3. The measure algebra embeds

Consider the measure algebra $(\mathbb{B}_\mathfrak{c}, \theta_\mathfrak{c})$ associated with the Haar measure on $2^\mathfrak{c}$. Up to a measure-preserving isomorphism, it is the unique probability measure algebra of Maharam character $\mathfrak{c}$; this is Maharam's theorem (see e.g., [66, §331]). This algebra is generated by a family $A_\xi$, for $\xi < \mathfrak{c}$, of sets which, for all disjoint finite subsets $F$ and $G$ of $\mathfrak{c}$, satisfies $\theta_\mathfrak{c}(\bigwedge_{\xi \in F} A_\xi \wedge \bigwedge_{\eta \in G} A_\eta^\complement) = 2^{-|F|-|G|}$. We consider this algebra as a metric space with respect to the metric $d(A, B) = \theta_\mathfrak{c}(A \Delta B)$

and $\mathcal{P}(\mathbb{N})/\mathcal{Z}_0$ with respect to the metric $\mu_\infty$ as in Lemma 5.2.1. An embedding $\Phi\colon \mathbb{B} \to \mathbb{B}'$ between Boolean algebras is called *regular* if the image of every maximal antichain in $\mathbb{B}$ is a maximal antichain in $\mathbb{B}'$. It is not difficult to prove that the standard embedding of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ into $\mathcal{P}(\mathbb{N})/\mathcal{I}$ for a dense analytic P-ideal (this is the last sentence of Corollary 3.2.3) is not regular. The following is [**67**, Proposition 491P].

**Theorem 5.3.1.** *There is an isomorphic, isometric, and regular embedding of* $(\mathbb{B}_\mathfrak{c}, \theta_\mathfrak{c})$ *into* $(\mathcal{P}(\mathbb{N})/\mathcal{Z}_0, \mu_\infty)$.

The conclusion holds for any other EU-ideal in place of $\mathcal{Z}_0$, with a little extra work or by using RK-bi-embeddability of EU-ideals, Proposition 2.7.10.

Proof. By Theorem 2.7.8, $\mathcal{Z}_0$ is the density ideal associated with a partition of $\mathbb{N}$ into finite sets $I_n$, with $|I_n| = 2^n$, and $\mu_n$ the normalised counting measure $\mu_n$ on $I_n$. Identify $I_n$ with $\{0,1\}^n$, and for $j < n$ let $a_{n,j} = \{s \in \{0,1\}^n : s(j) = 0\}$. Then for any two disjoint subsets $F$ and $G$ of $n$ we have

$$\mu_n(\bigwedge_{j\in F} a_{n,j} \wedge \bigwedge_{j \in G} a_{n,j}^{\complement}) = 2^{-|F|-|G|}.$$

For each $x \in [0,1]$ choose $k(x,n)$, for $n \in \mathbb{N}$, so that $0 \leq k(x,n) < 2^n$ and $\lim_n k(x,n)2^{-n} = x$. Re-enumerate $A_\xi$, for $\xi < \mathfrak{c}$ as $A_x$, for $x \in [0,1]$ and define

$$\Phi(A_x) = [\bigcup_n a_{n,k(x,n)}]_{\mathcal{Z}_0}.$$

Since all $x \neq y$ satisfy $k(x,n) \neq k(y,n)$ for all but finitely many $n$, any two finite disjoint subsets $F$ and $G$ of $[0,1]$ satisfy $\bigwedge_{x\in F} \Phi(A_x) \wedge \bigwedge_{x\in G} \Phi(A_x)^{\complement}) = 2^{-|F|-|G|}$. Therefore, $\Phi$ can be extended to an isometric embedding from the subalgebra $\mathbb{B}_0$ of $\mathbb{B}_\mathfrak{c}$ generated by $A_x$, for $x \in [0,1]$, into $\mathcal{P}(\mathbb{N})/\mathcal{Z}_0$. Since $\mathbb{B}_0$ is dense in $\mathbb{B}_\mathfrak{c}$, this embedding extends to an (isometric) embedding as required.

It remains to prove that $\Phi$ is a regular embedding. Since $\mathbb{B}_\mathfrak{c}$ carries a strictly positive measure, every antichain $\mathcal{A}$ in it is countable and $\sum_{A\in\mathcal{A}} \theta_\mathfrak{c}(A) = 1$. Since $\Phi$ is isometric, $\mathcal{B} = \Phi[\mathcal{A}]$ is an antichain in $\mathcal{P}(\mathbb{N})/\mathcal{Z}_0$ and $\sum_{B\in\mathcal{B}} \mu_\infty(B) = 1$. If $C \in \mathbb{N}/\mathcal{Z}_0$ and $C \cap B = [0]_{\mathcal{Z}_0}$ for all $B \in \mathcal{B}$, then $\mu_\infty(C \bigcup F) = 0$ for every $F \Subset \mathcal{B}$, and therefore $\mu_\infty(C) = 0$, implying that $\mathcal{B}$ is a maximal antichain. □

Closely related is the main result of [**46**], where it was proved that the regular open algebra of $\mathcal{P}(\mathbb{N})/\mathcal{Z}_0$ is isomorphic to that of the product of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ and the homogeneus probability measure algebra of Maharam character $\mathfrak{c}$. Given Theorem 5.3.1, the proof of this fact is rather simple: One embeds $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ into $\mathcal{P}(\mathbb{N})/\mathcal{Z}_0$ by collapsing the interval $I_n$ as in the proof of the former to $n$. Once it is verified that this embedding is regular, it remains to prove (using the embedding $\Phi$) that the quotient is forced to be isomorphic to the ultraproduct $\prod_{\dot{\mathcal{U}}}(\mathcal{P}(I_n), \mu_n)$, where $\mathcal{U}$ is the $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$-generic ultrafilter on $\mathbb{N}$. See the proofs in §11.2 for related arguments.

## 5.4. Homomorphisms without Baire measurable liftings

As we have already remarked, all automorphisms of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ can be trivial. This is not true for homomorphisms. If $\mathcal{I}$ is a maximal nonprincipal ideal on $\mathcal{P}(\mathbb{N})$, then $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is the two-element Boolean algebra and is therefore embeddable into every other Boolean algebra. This embedding does not have Baire measurable lifting, but it has a lifting whose graph is covered by graphs of two continuous (even constant) functions. It is neither injective nor surjective, but we will see that there are nontrivial homomorphisms with either one of these properties and therefore demonstrate the sharpness of Shelah's result by showing that only automorphisms can be expected to be trivial. Recall the order $\leq^+_{\mathrm{BE}}$ on analytic ideals, defined in Definition 2.3.2 by $\mathcal{I} \leq^+_{\mathrm{BE}} \mathcal{J}$ if and only if $\mathcal{I} \leq_{\mathrm{BE}} \mathcal{J} \restriction A$ for some $\mathcal{J}$-positive set $A$.

**Example 5.4.1.** A monomorphism between two analytic quotients with no Baire measurable lifting. Let $\Phi_1 \colon \mathcal{P}(\mathbb{N})/\mathcal{I} \to \mathcal{P}(\mathbb{N})/\mathcal{J}_1$ be a monomorphism between analytic quotients which has a Baire lifting and let $\mathcal{J}_2$ be an analytic ideal. Let $\mathcal{K}$ be a maximal ideal extending $\mathcal{I}$ and let $\Phi_2 \colon \mathcal{P}(\mathbb{N})/\mathcal{K} \to \mathcal{P}(\mathbb{N})/\mathcal{J}_2$ be a monomorphism. Let $\mathcal{J} = \mathcal{J}_1 \oplus \mathcal{J}_2$ and consider the amalgamation (Definition 2.3.3)

$$\Phi = \Phi_1 \oplus \Phi_2 \colon \mathcal{P}(\mathbb{N})/\mathcal{I} \to \mathcal{P}(\mathbb{N} \oplus \mathbb{N})/\mathcal{J}.$$

Then $\Phi$ is a monomorphism of $\mathcal{P}(\mathbb{N})/\mathcal{I}$ into $\mathcal{P}(\mathbb{N})/\mathcal{J}$ which has no Baire lifting. If we assume otherwise and let $\Phi_*$ be a continuous (see Lemma 3.5.3) lifting of $\Phi$, then we would have

$$\mathcal{K} = \{C : \Phi_*(C)\Delta F(C) \in \mathcal{J}\}$$

hence $\mathcal{K}$ would be a nonprincipal maximal analytic ideal, which is impossible by a classical result of Sierpinski (or by, say, Theorem 3.2.2).

Recall the following quasi-order $\leq^+_{\mathrm{BE}}$ introduced in Definition 2.3.2.

> $\mathcal{I} \leq^+_{\mathrm{BE}} \mathcal{J}$ if and only if $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is Baire-embeddable into $\mathcal{P}(A)/\mathcal{J}$ for some $\mathcal{J}$-positive set $A$.

**Proposition 5.4.2.** *If $\mathcal{I}$ and $\mathcal{J}$ are analytic ideals and $\mathcal{I} \leq^+_{\mathrm{BE}} \mathcal{J}$ then $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is isomorphically embeddable into $\mathcal{P}(\mathbb{N})/\mathcal{J}$.*

Proof. Define the embedding as in Example 5.4.1: Fix $A \in \mathcal{J}^+$ and an embedding $\Phi \colon \mathcal{P}(\mathbb{N})/\mathcal{I} \to \mathcal{P}(\mathbb{N})/\mathcal{J}$ with Baire measurable lifting $\Phi_*$. Choose an ultraflilter $\mathcal{U}$ disjoint from $\mathcal{I}$ and let $F(X) = \Phi_*(X)$ if $X \notin \mathcal{U}$ and $F(X) = \Phi_*(X) \cup (\mathbb{N} \setminus A)$ if $X \in \mathcal{U}$. □

By Corollary 2.6.9, if $\mathcal{I}$ and $\mathcal{J}$ are dense summable ideals then $\mathcal{I}$ is RB-reducible to the restriction of $\mathcal{J}$ to a positive set and vice versa. Therefore, each one of $\mathcal{P}(\mathbb{N})/\mathcal{I}$ and $\mathcal{P}(\mathbb{N})/\mathcal{J}$ is embeddable into the other, but Theorem 4.5.3 implies that the Baire-embeddability relation between quotients of dense summable ideals is rather complicated.

**Example 5.4.3** (An epimorphism between two analytic quotients without a Baire-measurable lifting.)**.** Let $\{\mathcal{U}_n\}$ be a sequence of ultrafilters on $\mathbb{N}$ and define $F_{\mathcal{U}} \colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ by

$$F_{\mathcal{U}}(A) = \{n : A \in \mathcal{U}_n\}.$$

This is clearly an endomorphism of $\mathcal{P}(\mathbb{N})$. If the sequence $\{\mathcal{U}_n\}$ is *discrete*, i.e., for some sequence $\{A_n\}$ of pairwise disjoint subsets of $\mathbb{N}$ we have $A_n \in \mathcal{U}_n$ for every $n$, then $F_{\mathcal{U}}$ is an epimorphism. To see this, pick an arbitrary $B \subseteq \mathbb{N}$. Then there is $C \subseteq \mathbb{N}$ such that $A_n \subseteq^* C$ if $n \in B$ and $A_n \perp C$ if $n \notin B$. This proves that $F_{\mathcal{U}}$ is an epimorphism of $\mathcal{P}(\mathbb{N})$, and therefore a lifting of an epimorphism $\Phi \colon \mathcal{P}(\mathbb{N})/\operatorname{Fin} \to \mathcal{P}(\mathbb{N})/\operatorname{Fin}$. Now assume that moreover each $\mathcal{U}_n$ is a nonprincipal ultrafilter. Then $\Phi$ does not have a Baire lifting: Assuming otherwise implies it has a continuous lifting as well (Lemma 3.5.3 (1)), therefore $\ker(\Phi)$ is an $F_\sigma$ ideal. But $\ker(\Phi)$ is nonmeagre because it includes an intersection of countably many maximal nonprincipal ideals (see e.g., Theorem 3.2.2 (b)), so it does not have a property of Baire—a contradiction.

When writing [40], I did not know better but to conjecture that a sufficiently strong forcing axiom implies every homomorphism between analytic quotients is of the form $F_{\mathcal{U}}$ for a sequence of ultrafilters $\{\mathcal{U}_n\}$. A ZFC counterexample has been constructed by Alan Dow in [27].

CHAPTER 6

# Lifting theorems II: Using forcing axioms

This Chapter, together with Chapter 9, contains our central lifting results. It is largely based on [57]. The main result of his chapter is the OCA Lifting Theorem, Theorem 6.1.2, implying that under forcing axioms for a countably 80-determined ideal (i.e., every known $F_{\sigma\delta}$ ideal) $\mathcal{I}$, every homomorphism $\Phi$ of $\mathcal{P}(\mathbb{N})$ into $\mathcal{P}(\mathbb{N})/\mathcal{I}$ has a continuous lifting. If the ideal in addition has the Fubini property, then Theorem 4.1.2 implies that $\Phi$ has a completely additive lifting. This shows that for ideals to which these theorems apply (and this class is contractually considerably larger) the only isomorphisms between their quotients are the ones obtained from RK-isomorphisms. The situation for embeddings is almost as simple.

The definition of countably determined ideals is in §1.10. For decomposable homomorphisms and almost liftings see Definition 6.1.1. The proof of Theorem 6.1.2 is considerably simpler than the proof of its precursor in [40], but a guide to this proof may come handy. It is provided in §6.1.1. Briefly, it consists of proving that various ideals of the form $\mathcal{J}^{\mathcal{K}}_{\text{something or another}}(\Phi)$ (Definition 6.2.2) have certain largeness properties. Proposition 6.3.1 is a local version of the main result, expressed as 'the ideal $\mathcal{J}_\sigma$ intersects every perfect tree-like almost disjoint family nontrivially'. This largeness property, introduced in [21], enabled us to weaken the axioms $\mathrm{OCA_T}+\mathrm{MA}$ used in [40] to $\mathrm{OCA_T}+\mathrm{MA}(\sigma\text{-linked})$ and even $\mathrm{OCA_T}$ alone in simpler cases. Proposition 6.4.1 implies that a set in $\mathcal{J}^{\mathcal{K}}_\sigma$ cannot be partitioned into infinitely many $\mathcal{J}_{\text{cont}}$-positive sets, and it is the only part of the proof that remained unchanged over the intervening decades. The punchline, involving uniformisation via the legendary Biba's trick, is in §6.5 (the case of countably generated ideals, extracted as a warm-up) and §6.6 (the general case). Parts of the proof are sewn together in §6.7.

Applications of the OCA lifting theorem are given in Chapter 7.

## 6.1. Preliminaries

Most efforts of this Chapter go into the proof of the $\mathrm{OCA_T}$ lifting theorem, Theorem 6.1.2, which is an improvement of eponymous theorems in [40] and [57] and its variations.

Amalgamation of homomorphisms was introduced in Definition 2.3.3, and the following is the flip side of this notion.

**Definition 6.1.1.** Suppose that $\mathcal{I}$ is an ideal on $\mathbb{N}$ and $\Phi\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})/\mathcal{I}$ is a Boolean algebra homomorphism. Then $\Phi$ is called *decomposable* if there are homomorphisms $\Phi_1$ and $\Phi_2$ such that $\Phi_2$ has a continuous lifting, $\ker(\Phi_1)$ is nonmeagre, and $\Phi = \Phi_1 \oplus \Phi_2$.

A function $\Theta\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ is an *almost lifting* of $\Phi$ if the set

$$\{A \in \mathcal{P}(\mathbb{N}) : [\Theta(A)]_{\mathcal{I}} = \Phi(A)\}$$

includes a nonmeagre ideal.

An ideal is *ccc over* Fin if it intersects every uncountable almost disjoint (with respect to Fin) family (Definition 3.3.1). See Definition 1.10.1 for 'countably $d$-determined' and Definition 4.2.2 for the Fubini property.

**Theorem 6.1.2** (OCA lifting theorem)**.** *Assume* $\mathrm{OCA_T}$ *and* $\mathrm{MA}(\sigma\text{-linked})$ *and that* $\mathcal{I}$ *is a countably 80-determined ideal. Then every homomorphism from* $\mathcal{P}(\mathbb{N})$ *into* $\mathcal{P}(\mathbb{N})/\mathcal{I}$ *has a continuous lifting on an ideal which is ccc over* Fin *and is decomposable. If in addition* $\mathcal{I}$ *has the Fubini property, then* $F$ *can be chosen to be completely additive.*

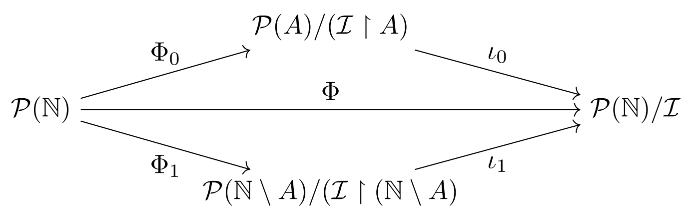



FIGURE 1. A decomposable homomorphism $\Phi$; $\ker(\Phi_0)$ is nonmeager and $\Phi_1$ has a continuous lifting.

A guide to the proof of Theorem 6.1.2 is given in §6.1.1 and the complete proof is assembled in §6.7.

For countably generated ideals we do not even need $\mathrm{MA}(\sigma\text{-linked})$.

**Theorem 6.1.3.** *Assume* $\mathrm{OCA_T}$. *If* $\mathcal{I}$ *is a countably generated ideal on* $\mathbb{N}$ *and* $\Phi\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})/\mathcal{I}$ *is a homomorphism, then* $\Phi$ *has a completely additive lifting on a nonmeagre ideal.*

The proof of Theorem 6.1.3 shows that the relevant nonmeagre ideal has an apparently stronger property, that it intersects every perfect tree-like almost disjoint family (Definition 3.3.6). If $\mathrm{MA}(\sigma\text{-linked})$ holds in addition to $\mathrm{OCA_T}$ then Theorem 6.1.2 implies that $\Phi$ has a completely additive lifting on an ideal which is ccc over Fin, but this proof is an instance of shooting fly with a cannon; by Corollary A.5.5, this already follows from Theorem 6.1.3.

The case when $\mathcal{I} = \mathrm{Fin}$ of Theorem 6.1.3 was proved from $\mathrm{OCA_T}$ and MA in [**162**] and from $\mathrm{OCA_T}$ in [**21**, Theorem 3.3], and Theorem 6.1.2 for analytic P-ideals was proved from $\mathrm{OCA_T}$ and MA in [**40**, Theorem 1.9.2]. The proof for Fin given here (included in the proof of Theorem 6.1.3) is slightly shorter than the one in [**21**]. It uses 'Biba's trick', first used in the proof of [**21**, Proposition 5.6] and developed further in [**57**]. A more sophisticated use of Biba's trick is given in the proof of Proposition 6.6.2, which is a partial result towards Theorem 6.1.2.

The proof of the original (weaker) variant of Theorem 6.1.2 from $\mathrm{OCA_T}$ and MA given in [**40**] used a very different strategy. A revised version of this proof in the case when $\mathcal{I} = \mathrm{Fin}$ is given in §A.7.

**6.1.1. Guide to the proof of the OCA lifting theorem.** Fix a countably 80-determined ideal $\mathcal{I}$ and a homomorphism $\Phi\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})/\mathcal{I}$. The proof proceeds by showing that the hereditary sets $\mathcal{J}^{\mathcal{K}}_{\mathrm{cont}}, \mathcal{J}^{\mathcal{K}}_{\sigma}$ (for $\mathcal{K}$ a closed approximation to $\mathcal{I}$) defined in Definition 6.2.2 are 'large'. Parts of the proof impose various requirements on $\mathcal{I}$ using different assumptions.

In Proposition 6.3.1 we prove that $\mathrm{OCA_T}$ implies that if $\mathcal{I}$ is countably 16-determined and $\Phi\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})/\mathcal{I}$ is a homomorphism, then the ideal $\mathcal{J}_{\mathrm{cont}}$ intersects every perfect tree-like almost disjoint family nontrivially and that if $\mathcal{I}$ is countably 32-determined and $\mathrm{MA}(\sigma\text{-linked})$ holds then $\mathcal{J}_{\mathrm{cont}}$ is ccc over Fin.

This is achieved in two stages. The first stage (Lemma 6.3.2 (3)) uses $\mathrm{OCA_T}$ and $\mathrm{MA}(\sigma\text{-linked})$ to show that for every closed approximation $\mathcal{K}$ to $\mathcal{I}$ the hereditary set $\mathcal{J}^{\mathcal{K}^{32}}_{\mathrm{cont}}(\Phi)$ has nonempty intersection with every uncountable almost disjoint family. (The first two parts of this lemma are $\mathrm{OCA_T}$-only results with slightly weaker conclusions.)

The second stage is a (known) ZFC result, Proposition 6.4.1, asserting that if a set in $\mathcal{J}_\sigma$ is partitioned into countably many sets, then all but finitely many of them belong to $\mathcal{J}_{\mathrm{cont}}$.

Finally, in §6.5 $\mathrm{OCA_T}$ is used to uniformise local liftings attached to sets in $\mathcal{J}_{\mathrm{cont}}$ and produce a lifting on the ideal $\mathcal{J}_{\mathrm{cont}}$. This part of the proof uses Biba's trick, implicit in [21] and explicit in [57].

If in addition $\mathcal{I}$ is a Fubini ideal, then $\Phi$ is decomposable by Proposition 3.5.4.

## 6.2. Approximations to a homomorphism

This section contains no nontrivial results, only its definitions are of any value. Even some of them (e.g., $\mathcal{K}$-approximations) are taken from earlier chapters. Nevertheless, since the proof of Theorem 6.1.2 largely consists of analysing the ideals $\mathcal{J}^{\mathcal{K}}_{\text{😎}}(\Phi)$ for various choices of 😎 introduced in Definition 6.2.2, skipping this section is not an option.

**6.2.1. Coherence of $\mathcal{K}$-approximations.** Approximations to an ideal $\mathcal{I}$ were defined in Definition 1.2.1: A hereditary subset $\mathcal{K}$ of $\mathcal{P}(\mathbb{N})$ is an approximation to $\mathcal{I}$ if $\mathcal{I} \subseteq \mathcal{K} \sqcup \mathrm{Fin}$. By Definition 3.5.2, a function $\Theta\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ is a $\mathcal{K}$-*approximation to* $\Phi$ if for every $A \subseteq \mathbb{N}$ we have

$$\Phi_*(A) \Delta \Theta(A) \in \mathcal{K} \sqcup \mathrm{Fin}\,.$$

If this holds for all $A \in \mathcal{X}$ for some $\mathcal{X} \subseteq \mathcal{P}(\mathbb{N})$, then we say that $\Theta$ is a $\mathcal{K}$-*approximation to* $\Phi$ *on* $\mathcal{X}$.

As in Definition 1.2.1, we write $\mathcal{X}^2$ for $\mathcal{X} \sqcup \mathcal{X}$. The following variation on standard stabilisation trick is a second-hand corollary, as a immediate consequence of Corollary 3.2.6.

**Corollary 6.2.1.** *Suppose that $\mathcal{I}$ is an ideal on $\mathbb{N}$ with a closed approximation $\mathcal{K}$, $\Phi\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})/\mathcal{I}$ is a homomorphism, and $\mathcal{H}$ is a hereditary nonmeagre subset of $\mathcal{P}(\mathbb{N})$,*

(1) *If $\Phi$ has a Baire measurable $\mathcal{K}$-approximation on $\mathcal{H}^2$, then it has a continuous $\mathcal{K}^2$-approximation on $\mathcal{H}$.*

*(2) If $\Phi$ has a $\mathcal{K}$-approximation on $\mathcal{H}^2$ whose graph can be covered by graphs of countably many Baire measurable functions, then it has a $\mathcal{K}^2$-approximation on $\mathcal{H}$ whose graph can be covered by graphs of countably many continuous functions.*

Proof. For the first part, suppose $\Theta$ is a Baire measurable $\mathcal{K}$-approximation to $\Phi$ on $\mathcal{H}^2$. Every Baire measurable function between Polish spaces is continuous on a comeagre set $\mathcal{X}$ (Lemma A.1.1). By Corollary 3.2.6, there are a partition $\mathbb{N} = A_0 \sqcup A_1$ and sets $C_0 \subseteq A_0$ and $C_1 \subseteq A_1$ such that for every $X \in \mathcal{H}$ both $(X \cap A_0) \cup C_1$ and $(X \cap A_1) \cup C_0$ belong to $\mathcal{H}^2 \cap \mathcal{X}$. Then

$$X \mapsto (\Theta((X \cap A_0) \cup C_1) \cap \Phi_*(A_0)) \cup (\Theta((X \cap A_1) \cup C_0) \cap \Phi_*(A_1))$$

is a continuous $\mathcal{K}^2$-approximation to $\Phi$ on $\mathcal{H}$.

For the second part, fix Baire measurable functions $\{\Theta_m\}$ whose graphs cover the graph of a $\mathcal{K}$-approximation to $\Phi$ on $\mathcal{H}^2$. Apply Corollary 3.2.6 as before. For all $m$ and $n$, the function

$$X \mapsto (\Theta_m((X \cap A_0) \cup C_1) \cap \Phi_*(A_0)) \cup (\Theta_n((X \cap A_1) \cup C_0) \cap \Phi_*(A_1))$$

is continuous, and graphs of these functions cover the graph of a $\mathcal{K}^2$-approximation to $\Phi$ on $\mathcal{H}$. $\square$

**6.2.2. Ideals associated with approximations to a homomorphism $\Phi$.** The proof of Theorem 6.1.2 is a struggle to prove that the ideals as in Definition 6.2.2 below are appropriately large. This method goes back to the original rigidity proof, that in an oracle-cc forcing extension all automorphisms of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ are trivial ([139], also see [59, §5]). In this proof, one shows that if $\mathcal{J}_{\mathrm{cont}}$ (as defined below) is dense (in the sense that every infinite $A \subseteq \mathbb{N}$ has an infinite subset that belongs to $\mathcal{J}_{\mathrm{cont}}$), then that it is a P-ideal, and finally that it is equal to $\mathcal{P}(\mathbb{N})$. Being a P-ideal is used to assure that a certain poset is ccc. Courtesy of Biba, we can skip this part. Instead we prove that $\mathcal{J}_{\mathrm{cont}}$ intersects every perfect tree-like almost disjoint family (Proposition 6.3.1) and then, if $\Phi$ is an isomorphism, that it is equal to $\mathcal{P}(\mathbb{N})$ (or, in case when $\Phi$ is a homomorphism, that a single continuous function provides a lifting of $\Phi$ on $\mathcal{J}_{\mathrm{cont}}$).

**Definition 6.2.2.** If $\mathcal{I}$ is an ideal on $\mathbb{N}$ and is $\mathcal{K}$ is a closed approximation to $\mathcal{I}$, then for a homomorphism $\Phi\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})/\mathcal{I}$ let[1]

$$\begin{aligned}
\mathcal{J}^{\mathcal{K}}_{\mathrm{cont}}(\Phi) &= \{A : \Phi \text{ has a continuous } \mathcal{K}\text{-approximation on } \mathcal{P}(A)\},\\
\mathcal{J}^{\mathcal{K}}_{\mathrm{cont}*}(\Phi) &= \{A : \Phi \text{ has a continuous } \mathcal{K}\text{-approximation}\\
&\qquad \text{on a relatively nonmeagre hereditary subset of } \mathcal{P}(A)\},\\
\mathcal{J}^{\mathcal{K}}_{\sigma}(\Phi) &= \{A : \Phi \text{ has a } \mathcal{K}\text{-approximation on } \mathcal{P}(A) \text{ whose graph}\\
&\qquad \text{can be covered by graphs of countably many Borel functions}\}.
\end{aligned}$$

We will omit the parameter $\Phi$ and write $\mathcal{J}^{\mathcal{K}}_{\mathrm{cont}}$, $\mathcal{J}^{\mathcal{K}}_{\mathrm{cont}*}$, or $\mathcal{J}^{\mathcal{K}}_{\sigma}$ whenever $\Phi$ is clear from the context (that would be always, since we will be dealing with at most one homomorphism at a time and all homomorphisms will be denoted $\Phi$). We also write $\mathcal{J}_{\mathrm{cont}} = \mathcal{J}^{\mathcal{I}}_{\mathrm{cont}}$, $\mathcal{J}_{\mathrm{cont}*} = \mathcal{J}^{\mathcal{I}}_{\mathrm{cont}*}$, and $\mathcal{J}_{\sigma} = \mathcal{J}^{\mathcal{I}}_{\sigma}$.

[1] A related ideal $\mathcal{J}_{\mathrm{dec}}(\Phi)$ is defined and used only in the revision of the proof from [40] given in §A.7.

**Lemma 6.2.3.** *Suppose that $\mathcal{I}$ is an ideal with approximations $\mathcal{K}$ and $\mathcal{L}$ and that $\Phi\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})/\operatorname{Fin}$ is a homomorphism*

1. *We have $\mathcal{J}^{\mathcal{K}} \sqcup \mathcal{J}^{\mathcal{L}} \subseteq \mathcal{J}^{\mathcal{K}\sqcup\mathcal{L}}$, $\mathcal{J}^{\mathcal{K}}_{\mathrm{cont}*} \sqcup \mathcal{J}^{\mathcal{L}}_{\mathrm{cont}*} \subseteq \mathcal{J}^{\mathcal{K}\sqcup\mathcal{L}}_{\mathrm{cont}*}$, $\mathcal{J}^{\mathcal{K}}_{\sigma} \sqcup \mathcal{J}^{\mathcal{L}}_{\sigma} \subseteq \mathcal{J}^{\mathcal{K}\sqcup\mathcal{L}}_{\sigma}$, and $\operatorname{Fin} \subseteq \mathcal{J}^{\mathcal{K}}_{\mathrm{cont}} \subseteq \mathcal{J}^{\mathcal{K}}_{\mathrm{cont}*}$.*
2. *Each one of $\mathcal{J}^{\mathcal{K}}_{\mathrm{cont}}$, $\mathcal{J}^{\mathcal{K}}_{\mathrm{cont}*}$, and $\mathcal{J}^{\mathcal{K}}_{\sigma}$ is closed under finite changes of its elements.*

Proof. (1) Observe that if $\Theta$ is a $\mathcal{K}$-approximation to $\Phi$ on $\mathcal{P}(A)$ and $\Upsilon$ is an $\mathcal{L}$-approximation to $\Phi$ on $\mathcal{P}(B)$, then

$$X \mapsto \Theta(X \cap A) \cup \Upsilon((X \setminus A) \cap B)$$

is a $\mathcal{K} \sqcup \mathcal{L}$-approximation to $\Phi$ on $A \cup B$.

(2) would have been trivial had we only assumed that $\ker(\Phi) \supseteq \operatorname{Fin}$. If $A$ is in $\mathcal{J}^{\mathcal{K}}_{\mathrm{cont}}$ as witnessed by $\Theta$ and $s \Subset \mathbb{N} \setminus A$, fix a lifting $F$ of $\Phi$ on $\mathcal{P}(s)$. Then $X \mapsto F(X \cap s) \cup \Theta(X \cap A)$ is a $\mathcal{K}$-approximation to $\Phi$ on $\mathcal{P}(A \cup s)$. Proofs in case of $\mathcal{J}^{\mathcal{K}}_{\mathrm{cont}*}$ and $\mathcal{J}^{\mathcal{K}}_{\sigma}$ are analogous. □

The following is an immediate consequence of Corollary 6.2.1.

**Lemma 6.2.4.** *Suppose $\mathcal{I}$ is an ideal, $\Phi\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})/\mathcal{I}$ is a homomorphism, and $\mathcal{K}$ is an approximation to $\mathcal{I}$.*

1. *$\{A : \Phi \restriction \mathcal{P}(A)$ has a Baire measurable $\mathcal{K}$-approximation$\} \subseteq \mathcal{J}^{\mathcal{K}^2}_{\mathrm{cont}}$.*
2. *$\{A : \Phi \restriction \mathcal{P}(A)$ has a $\mathcal{K}$-approximation whose graph can be covered by graphs of countably many Baire measurable functions$\} \subseteq \mathcal{J}^{\mathcal{K}^2}_{\sigma}$.* □

## 6.3. Local liftings

A proof of the following local version of Theorem 6.1.2 occupies this entire section; see §3.3.1 for (perfect) tree-like almost disjoint families.

**Proposition 6.3.1.** *Assume $\mathrm{OCA_T}$. If an ideal $\mathcal{I}$ is countably 16-determined and $\Phi\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})/\mathcal{I}$ is a homomorphism, then $\mathcal{J}_{\mathrm{cont}}$ intersects every perfect tree-like almost disjoint family nontrivially. If in addition $\mathrm{MA}(\sigma$-linked$)$ holds and $\mathcal{I}$ is countably 32-determined, then $\mathcal{J}_{\mathrm{cont}}$ is ccc over $\operatorname{Fin}$.*

The following lemma is used in the proof of Proposition 6.3.1.

**Lemma 6.3.2.** *Assume $\mathrm{OCA_T}$, that $\mathcal{I}$ is an ideal on $\mathbb{N}$, $\Phi\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})/\mathcal{I}$ is a homomorphism, and $\mathcal{K}$ is a closed approximation to $\mathcal{I}$.*

1. *The hereditary set $\mathcal{J}^{\mathcal{K}^8}_{\sigma}(\Phi)$ has nonempty intersection with every uncountable tree-like almost disjoint family.*
2. *The hereditary set $\mathcal{J}^{\mathcal{K}^{16}}_{\mathrm{cont}}(\Phi)$ has nonempty intersection with every perfect tree-like almost disjoint family.*
3. *If in addition $\mathrm{MA}(\sigma$-linked$)$ holds, then $\mathcal{J}^{\mathcal{K}^{32}}_{\mathrm{cont}}(\Phi)$ has nonempty intersection with every uncountable almost disjoint family.*

The proof of this lemma is given after some preliminaries. Definition 6.3.3 has a long history, starting with [162]. The function $\Phi_*$ will typically be a lifting of a homomorphism from $\mathcal{P}(\mathbb{N})$ into $\mathcal{P}(\mathbb{N})/\mathcal{I}$.

**Definition 6.3.3.** Suppose that $\mathcal{A}$ is a tree-like almost disjoint family. By $\hat{\mathcal{A}}$ we denote the hereditary closure of $\mathcal{A}$,

$$\hat{\mathcal{A}} = \{B \subseteq \mathbb{N} : (\exists A \in \mathcal{A}) B \subseteq A\},$$

and for an infinite $B \in \hat{\mathcal{A}}$ write $A(B)$ for the unique element of $\mathcal{A}$ that includes $B$. Let

$$\mathcal{X}_{\mathcal{A}} = \{(C, B) : B \in \hat{\mathcal{A}}, C \subseteq B\}.$$

For $x = (C, B)$ in $\mathcal{X}_{\mathcal{A}}$ we write $C = C(x)$, $B = B(x)$, and $A(B(x)) = A(x)$.

If in addition $\Phi_* \colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ and $\mathcal{K}$ is a closed hereditary subset of $\mathcal{P}(\mathbb{N})$, then we define a partition

$$[\mathcal{X}_{\mathcal{A}}]^2 = K_0^{\Phi_*,\mathcal{A},\mathcal{K}} \cup K_1^{\Phi_*,\mathcal{A},\mathcal{K}}$$

by setting $\{x, y\} \in K_0^{\Phi_*,\mathcal{A},\mathcal{K}}$ if the following three conditions hold.

$K_0$(i) $A(x) \neq A(y)$.

$K_0$(ii) $B(x) \cap C(y) = C(x) \cap B(y)$.

$K_0$(iii) $(\Phi_*(B(x)) \cap \Phi_*(C(y)))\Delta(\Phi_*(C(x)) \cap \Phi_*(B(y))) \notin \mathcal{K}^2$.

Endow $\mathcal{X}_{\mathcal{A}}$ with a separable metric topology $\tau^{\Phi_*,\mathcal{A}}$ by identifying $x \in \mathcal{X}_{\mathcal{A}}$ with $(C(x), B(x), A(x), \Phi_*(C(x)), \Phi_*(B(x))$ in the compact metric space $\mathcal{P}(\mathbb{N})^5$.

**Lemma 6.3.4.** *The set $K_0^{\Phi_*,\mathcal{A},\mathcal{K}}$ is a $\tau^{\Phi_*,\mathcal{A}}$-open subset of $[\mathcal{X}_{\mathcal{A}}]^2$.*

Proof. Conditions ($K_0$(i)) and ($K_0$(iii)) are clearly open. The symmetric difference of $b' \cap a$ and $b \cap a'$ is included in $B_a \cap B_{a'}$, but since the family $\mathcal{A}_0$ is tree-like, this is a finite set determined by the witness for (K1). [More precisely, if $m \in B_a \Delta B_{a'}$, then $B_a \cap B_{a'}$ is included in the finite set of points that are below $m$ in the tree ordering on $\mathbb{N}$ that witnesses $\mathcal{A}$ is tree-like.] In other words, ($K_0$(ii)) is open relative to ($K_0$(i)), and this proves that the conjunction of all three conditions defines an open partition. □

Proof of Lemma 6.3.2. (1) Assume $\mathrm{OCA_T}$, let $\Phi \colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})/\mathcal{I}$ be a homomorphism, and let $\mathcal{K}$ be a closed approximation to $\mathcal{I}$. Fix a lifting $\Phi_*$ of $\Phi$. We will first prove that the hereditary set $(\mathcal{J}_\sigma^{\mathcal{K}^4})^2$ has nonempty intersection with every uncountable tree-like almost disjoint family. Fix an uncountable tree-like almost disjoint family $\mathcal{A}$. For $n \in \mathbb{N}$ let

$$\mathcal{K}_n = \mathcal{K} \mathbin{\underline{\cup}} [\mathbb{N}]^n.$$

Thus $\mathcal{I} \subseteq \bigcup_n \mathcal{K}_n$. Write $K_0^n = K_0^{\Phi_*,\mathcal{A},\mathcal{K}_n}$. When $\mathcal{X}_{\mathcal{A}}$ is endowed with the topology $\tau^{\Phi_*,\mathcal{A}}$ this is an open partition by Lemma 6.3.4.

**Claim 6.3.5.** *There are no uncountable $Z \subseteq \{0,1\}^{\mathbb{N}}$ and function $f \colon Z \to \mathcal{X}_{\mathcal{A}}$ such that $\{f(z), f(z')\} \in K_0^{\Delta(z,z')}$ for all distinct $z, z'$ in $Z$.*

Proof. Assume otherwise and fix $Z$ and $f$. By ($K_0$(i)) and ($K_0$(ii)), for all distinct $z$ and $z'$ in $Z$ we have $A(f(z)) \neq A(f(z'))$ and

$$B(f(z)) \cap C(f(z')) = C(f(z)) \cap B(f(z')).$$

Let $C = \bigcup_{z \in Z} C(f(z))$. Then $C \cap B(f(z)) = C(f(z))$ for all $z \in Z$ and since $\Phi_*$ is a lifting of $\Phi$ we have

$$(\Phi_*(C) \cap \Phi_*(B(f(z))))\Delta\Phi_*(C(f(z))) \in \textstyle\bigcup_n \mathcal{K}_n.$$

Since $Z$ is uncountable, there is $n \in \mathbb{N}$ such that the set $Z'$ of all of $z \in Z$ satisfying $(\Phi_*(C) \cap \Phi_*(C(f(z))))\Delta\Phi_*(B(f(z))) \in \mathcal{K}_n$ is uncountable.

This implies that all $z$ and $z'$ in $Z'$ satisfy

$$\begin{aligned}\Phi_*(B(f(z))) \cap \Phi_*(C(f(z'))) &=^{\mathcal{K}_n} \Phi_*(B(f(z))) \cap \Phi_*(B(f(z'))) \cap \Phi_*(C) \\ &=^{\mathcal{K}_n} \Phi_*(C(f(z))) \cap \Phi_*(B(f(z')))\end{aligned}$$

and therefore $\Phi_*(B(f(z)) \cap \Phi_*(C(f(z')))) =^{\mathcal{K}^2_n} \Phi_*(C(f(z))) \cap \Phi_*(B(f(z')))$. Since $Z'$ is uncountable, there are $z$ and $z'$ in $Z'$ such that $\Delta(z, z') > n$. Therefore ($K_0$(iii)) fails and $\{f(z), f(z')\} \notin K_0^{\Delta(z,z')}$; contradiction. □

Since $\mathrm{OCA}_\infty$ is equivalent to $\mathrm{OCA_T}$ (Theorem A.3.5), Claim 6.3.5 implies that there are $\mathcal{X}_n$, for $n \in \mathbb{N}$, such that $\mathcal{X}_\mathcal{A} = \bigcup_n \mathcal{X}_n$ and $[\mathcal{X}_n]^2 \subseteq K^n_1$ for all $n$. Let $\mathcal{D}_n \subseteq \mathcal{X}_n$ be a countable $\tau^{\Phi_*,\mathcal{A}}$-dense set. Since $\mathcal{A}$ is uncountable, there is $\tilde{A}$ in $\mathcal{A} \setminus \{A(x) : x \in \bigcup_n \mathcal{D}_n\}$.

**Definition 6.3.6.** If $n \geq 1$, $\bar{x} = (x_0, \dots x_{n-1})$ and $\bar{y} = (y_0, \dots y_{n-1})$ belong to $\mathcal{P}(\mathbb{N})^m$, and $k \in \mathbb{N}$, then we write

$$\bar{x} =^{\restriction k} \bar{y} \text{ if and only if } x_i \cap k = y_i \cap k \text{ for all } i < n.$$
$$\bar{x} =^{k} \bar{y} \text{ if and only if } \min(x_i \Delta y_i) \geq k \text{ for all } i < n.$$

Note that the condition $\bar{x} =^{\restriction k} \bar{y}$ is complementary to $\bar{x} =^{k} \bar{y}$, and that their conjunction is equivalent to $\bar{x} = \bar{y}$. For $m \in \mathbb{N}$ define

$$\begin{aligned}m^+ = \min\{l > m :& (\forall n \leq m)(\forall x \in \mathcal{X}_n) A(x) = \tilde{A} \\ &\Rightarrow (\exists d \in \mathcal{D}_n)(C(d), B(d))) =^{\restriction m} (C(x), B(x)) \text{ and } B(d) \cap \tilde{A} \subseteq l\}.\end{aligned}$$

Because $\mathcal{D}_n$ is dense in $\mathcal{X}_n$ for every $n$, $\tilde{A} \neq A(x)$ for all $x \in \bigcup_n \mathcal{D}_n$, and $\mathcal{P}(m)^2$ is finite, $m^+$ is finite for every $m$. Recursively define $m(j)$ for $j \in \mathbb{N}$ by

$$m(0) = 0 \text{ and } m(i+1) = m(i)^+ + 1 \text{ for all } i.$$

This is a strictly increasing sequence, and we let

$$B_0 = \textstyle\bigcup_i [m(2i), m(2i+1)) \cap \tilde{A}, \qquad B_1 = \tilde{A} \setminus B_0$$

so that $B_0 \sqcup B_1 = \tilde{A}$. We will prove that $B_j \in \mathcal{J}^{\mathcal{K}^2}_\sigma(\Phi)$ for $j = 0, 1$.

For each $n$ let

$$\begin{aligned}\mathcal{Z}(n) = \{(X, Y) :& X \subseteq B_0, Y \subseteq \mathbb{N}, (\forall j \geq n)(\exists d \in \mathcal{D}_n) B(d) \cap \tilde{A} \subseteq m(2j+2) \\ &\text{and } (C(d), B(d)) =^{\restriction m(2j+1)} (X, B_0)\}.\end{aligned}$$

**Claim 6.3.7.** *Suppose that $x \in \mathcal{X}_n$ and $B(x) = B_0$. Then the following holds.*

*(1)* $(C(x), \Phi_*(C(x))) \in \mathcal{Z}(n)$.

*(2) If* $(C(x), Y) \in \mathcal{Z}(n)$ *then* $\Phi_*(B_0) \cap Y =^{\mathcal{K}^2_n} \Phi_*(C(x)) \cap \Phi_*(B_0)$.

Proof. If $x \in \mathcal{X}_n$, $B(x) = B_0$, and $m(2j+1) \geq n$ then since $\mathcal{D}_n$ is $\tau^{\mathcal{A},\Phi_*}$-dense in $\mathcal{X}_n$, some $d \in \mathcal{D}_n$ satisfies $B(d) \cap \tilde{A} \subseteq m(2j+2)$ and

$$(C(d), B(d)) =^{\restriction m(2j+1)} (C(x), B_0).$$

Since $j \geq n$ was arbitrary, $(C(x), \Phi_*(C(x)) \in \mathcal{Z}(n)$ follows.

To prove the second part of the claim, towards contradiction suppose that $x$ is in $\mathcal{X}_n$, $B(x) = B_0$, $Y \subseteq \mathbb{N}$, $(C(x), Y) \in \mathcal{Z}(n)$, but

$$(\Phi_*(B_0) \cap Y) \Delta (\Phi_*(C(x)) \cap \Phi_*(B_0)) \notin \mathcal{K}^2_n.$$

Since $\mathcal{K}_n$ is hereditary, we can fix $j \geq n$ large enough to have

$$((\Phi_*(B_0) \cap Y)\Delta(\Phi_*(C(x)) \cap \Phi_*(B_0))) \cap m(2j+1) \notin \mathcal{K}_n^2. \tag{6.1}$$

Since $(C(x), Y) \in \mathcal{Z}_n$, some $d \in \mathcal{D}_n$ satisfies $B(d) \cap \tilde{A} \subseteq m(2j+2)$ and

$$(C(d), B(d)) =^{\restriction m(2j+1)} (C(x), B_0). \tag{6.2}$$

Since $B_0$ is disjoint from $[m(2j+1), m(2j+2))$, $B_0 \cap C(d) = C(x) \cap B(d)$. As $\{x, d\} \in K_1^n$ and $A(d) \neq A(x)$, we have

$$(\Phi_*(B_0) \cap \Phi_*(C(d)))\Delta(\Phi_*(C(x)) \cap \Phi_*(B(d))) \in \mathcal{K}^2.$$

Together with (6.2) this implies

$$((\Phi_*(B_0) \cap Y)\Delta(\Phi_*(C(x)) \cap \Phi_*(B_0))) \cap m(2j+1) \in \mathcal{K}^2,$$

contradicting (6.1). ∎

We claim that each $\mathcal{Z}(n)$ is Borel. For a fixed $d \in \mathcal{D}_n$ and $j \in \mathbb{N}$ the set

$$\mathcal{Z}(n,d,j) = \{(X,Y) : X \subseteq B_0, Y \subseteq \mathbb{N}, B(d) \cap \tilde{A} \subseteq m(2j+2) \text{ and } (C(d), B(d)) =^{\restriction m(2j+1)} (X, B_0)\}$$

is closed. Thus $\mathcal{Z}(n) = \bigcap_j \bigcup_{d \in \mathcal{D}_n} \mathcal{Z}(n,d,j)$ is an $F_{\sigma\delta}$ set. By the Jankov, von Neumann theorem ([**105**, 18.A]) there is a C-measurable function $\Theta_n$ whose domain includes $\{C(x) : x \in \mathcal{X}_n, B(x) = B_0\}$ such that $(X, \Theta_n(X)) \in \mathcal{Z}(n)$ for all $X$ in the domain of $\Theta_n$. Since $\mathcal{X}_\mathcal{A} = \bigcup_n \mathcal{X}_n$, Claim 6.3.7 implies that for every $X \subseteq B_0$ there is $n$ such that $\Phi_*(X) =^{\mathcal{K}_n^2} \Theta_n(X)$. Therefore $B_0$ belongs to $\mathcal{J}_\sigma^{\mathcal{K}^2}(\Phi)$, as required. By Lemma 6.2.4, $B_0$ is in $\mathcal{J}_\sigma^{\mathcal{K}^4}$.

Analogous argument shows that $B_1 \in \mathcal{J}_\sigma^{\mathcal{K}^4}$ and therefore $\tilde{A} \in (\mathcal{J}_\sigma^{\mathcal{K}^4})^2$, as promised. By Lemma 6.2.3, $\tilde{A} \in \mathcal{J}_\sigma^{\mathcal{K}^8}$. Since $\tilde{A} \in \mathcal{A} \setminus \{A(d) : d \in \bigcup_n \mathcal{D}_n\}$ was arbitrary, for every uncountable tree-like almost disjoint family $\mathcal{A}$ all but countably many elements of $\mathcal{A}$ belong to $\mathcal{J}_\sigma^{\mathcal{K}^8}$. This completes the proof of (1).

(2) Fix a perfect tree-like almost disjoint family $\mathcal{B}$. By Lemma 3.3.7, there is a perfect tree-like almost disjoint family $\mathcal{A}$ such that every $A \in \mathcal{A}$ includes infinitely many elements of $\mathcal{B}$. By (1), all but countably many elements of $\mathcal{A}$ belong to $\mathcal{J}_\sigma^{\mathcal{K}^8}$. Every $A \in \mathcal{A}$ includes infinitely many disjoint elements of $\mathcal{B}$, and by Proposition 6.4.1 all but finitely many of them belong to $\mathcal{J}_\sigma^{\mathcal{K}^{16}}$. Therefore $\mathcal{J}_\sigma^{\mathcal{K}^{16}}$ intersects every perfect tree-like almost disjoint family nontrivially.

(3) Fix an uncountable almost disjoint family $\mathcal{B}$. MA($\sigma$-linked) and Lemma A.5.4 imply that there is an uncountable almost disjoint family $\mathcal{A}$ such that for every $A \in \mathcal{A}$ the set $\{B \in \mathcal{B} : B \subseteq^* A\}$ is infinite and there is a partition $A = A_0 \cup A_1$ such that each one of $\mathcal{A}_0 = \{A_0 : A \in \mathcal{A}\}$ and $\mathcal{A}_1 = \{A_1 : A \in \mathcal{A}\}$ is a tree-like almost disjoint family. By (1) all but countably many elements of $\mathcal{A}_0 \cup \mathcal{A}_1$ belong to $\mathcal{J}_\sigma^{\mathcal{K}^8}$. Lemma 6.2.3 (1) implies that $(\mathcal{J}_\sigma^{\mathcal{K}^8})^2 \subseteq \mathcal{J}_\sigma^{\mathcal{K}^{16}}$, hence all but countably many $A \in \mathcal{A}$ belong to $\mathcal{J}_\sigma^{\mathcal{K}^{16}}$. Every $A \in \mathcal{A}$ includes infinitely many elements of $\mathcal{B}$, and by Proposition 6.4.1 all but finitely many of them belong to $\mathcal{J}_\sigma^{\mathcal{K}^{32}}$. Since $\mathcal{A}$ was arbitrary, $\mathcal{J}_\sigma^{\mathcal{K}^{32}}$ intersects every uncountable almost disjoint family nontrivially. ∎

Proof of Proposition 6.3.1 for $F_\sigma$ ideals. Suppose $\Phi \colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})/\mathcal{I}$ is a homomorphism for an $F_\sigma$ ideal $\mathcal{I}$. By Theorem 1.10.2, there is a closed approximation $\mathcal{K}$ to $\mathcal{I}$ such that $\mathcal{I} = \mathcal{K} \mathbin{\underline{\cup}} \text{Fin}$. Also, $\mathcal{K}^{16} \subseteq \mathcal{I}$ hence $\mathcal{I} = \mathcal{K}^{16} \mathbin{\underline{\cup}} \text{Fin}$, thus

$\Phi$ and $\mathcal{K}$ satisfy the assumptions of Lemma 6.3.2 (2). Therefore the ideal $\mathcal{J}_{\mathrm{cont}}^{\mathcal{K}^{16}}(\Phi)$ includes $\mathcal{J}_{\mathrm{cont}}$, and $\mathcal{J}_{\mathrm{cont}}$ intersects every perfect tree-like almost disjoint family nontrivially. If MA($\sigma$-linked) holds and $\mathcal{I}$ is countably 32-determined, then the analogous proof using Lemma 6.3.2 (3) implies that $\mathcal{J}_{\mathrm{cont}}$ is ccc over Fin. □

In order to extend the conclusion of Proposition 6.3.1 to countably determined ideals we need the following lemma.

**Lemma 6.3.8.** *Suppose that ideal $\mathcal{I}$ is an intersection of a sequence of Borel sets, $\mathcal{I} = \bigcap_n \mathcal{B}_n$. Then for every homomorphism $\Phi\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})/\mathcal{I}$ the following are equivalent.*

*(1) $\Phi$ has a continuous lifting.*
*(2) $\Phi$ has a Borel-measurable $\mathcal{B}_n$-approximation $\Theta_n$ for every $n$.*

Proof. Only the converse implication requires a proof. Let

$$\mathcal{X} = \{(a,b) : \Theta_n(a)\Delta b \in \mathcal{B}_n \uplus \mathrm{Fin}\, \text{for all } n\}.$$

Since all $\mathcal{B}_n$ and $\Theta_n$ are Borel, so is $\mathcal{X}$. By the Jankov, von Neumann uniformisation theorem ([105, 18.A]) there is a C-measurable function $\Theta\colon \mathcal{P}(B) \to \mathcal{P}(\mathbb{N})$ such that $(a, \Theta(a)) \in \mathcal{X}$ for all $a$, and therefore $\Theta$ is a lifting of a homomorphism $\Phi$ on $\mathcal{P}(B)$. By Corollary 6.2.1, $\Phi$ has a continuous lifting. □

Proof of the general case of Proposition 6.3.1. Suppose that $\mathcal{I}$ is a countably 16-determined ideal and $\Phi\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})/\mathcal{I}$ is a homomorphism. Fix a perfect tree-like almost disjoint family $\mathcal{A}$. Let $\mathcal{K}_n$, for $n \in \mathbb{N}$, be closed approximations to $\mathcal{I}$ such that $\mathcal{I} = \bigcap_n(\mathcal{K}_n^{16} \uplus \mathrm{Fin})$. Lemma 6.3.2 (2) implies that $\mathcal{J}_{\mathrm{cont}}^{\mathcal{K}_n^{16}}$ intersects $\mathcal{A}$ nontrivially for every $n$. This implies that $\mathcal{J}_{\mathrm{cont}}^{\mathcal{K}_n^{16}}$ contains all but countably many elements of $\mathcal{A}$. Then all but countably many elements of $\mathcal{A}$ belong to $\bigcap_n \mathcal{J}_{\mathrm{cont}}^{\mathcal{K}_n^{16}}$. By Corollary 6.2.1, each of these elements belongs to $\mathcal{J}_{\mathrm{cont}}$.

If MA($\sigma$-linked) holds and $\mathcal{I}$ is countably 32-determined then analogous proof using Lemma 6.3.2 (3) implies that $\mathcal{J}_{\mathrm{cont}}$ is ccc over Fin. □

### 6.4. From $\sigma$-Borel to continuous

The main result of this section, Proposition 6.4.1, is well-known (see also [69] for its version that uses measure insteaed of category) and it does not use $\mathrm{OCA_T}$, MA($\sigma$-linked), or any other additional set-theoretic axioms. To a homomorphism $\Phi\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})/\mathrm{Fin}$ one associates ideals $\mathcal{J}_\sigma$ and $\mathcal{J}_{\mathrm{cont}}$ (Definition 6.2.2). The following asserts that if a set in $\mathcal{J}_\sigma$ is partitioned into countably many sets, then all but finitely many of them belong to $\mathcal{J}_{\mathrm{cont}}$.

**Proposition 6.4.1.** *Suppose that $\Phi\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})/\mathcal{I}$ is a homomorphism, $\mathcal{K}$ is a closed approximation to $\mathcal{I}$, $\mathbb{N} = \bigsqcup_n A_n$, and there are Borel-measurable functions $F_n\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$, for $n \in \mathbb{N}$, whose graphs cover the graph of a $\mathcal{K}$-approximation to $\Phi$. Then for all but finitely many $n$ the restriction of $\Phi$ to $\mathcal{P}(A_n)$ has a continuous $\mathcal{K}^4$-approximation to $\Phi$.*

The proof of Proposition 6.4.1 is analogous to the proof of its special case when $\mathcal{I} = \mathrm{Fin}$ and $\mathcal{K} = \emptyset$, as presented in [21, Proposition A.2]. It proceeds by a recursive construction that halts at $n$ such that the restriction of $\Phi$ to $\mathcal{P}(A_n)$ has a continuous lifting. The recursive construction is facilitated by Lemma 6.4.2 below, and this lemma is preceded by an obligatory notation-introducing paragraph.

In the following we identify $\mathcal{P}(\mathbb{N})$ and $\{0,1\}^{\mathbb{N}}$, by associating a subset of $\mathbb{N}$ with its characteristic function. If $A = B \sqcup C$ then we $\mathcal{P}(A)$ naturally corresponds to $\mathcal{P}(B) \times \mathcal{P}(C)$. In this situation, if $\mathcal{B} \subseteq \mathcal{P}(B)$ and $\mathcal{C} \subseteq \mathcal{P}(C)$ it will then be convenient to write

$$\mathcal{B} \oplus \mathcal{C} = \{b \cup c : b \in \mathcal{B}, c \in \mathcal{C}\}. \tag{6.3}$$

If $s$ is a function from a finite subset of $\mathbb{N}$ into $\{0,1\}$ then we write

$$[s] = \{x \in \{0,1\}^{\mathbb{N}} : x(i) = s(i) \text{ for all } i \in \operatorname{dom}(s)\}. \tag{6.4}$$

When $\mathcal{K}$ is a hereditary subset of $\mathcal{P}(\mathbb{N})$ (such as a closed approximation to an ideal) it will be convenient to write

$$A =^{\mathcal{K}} B \Leftrightarrow A \Delta B \in \mathcal{K}.$$

If $\mathcal{K} = \mathcal{P}(n)$ for some $n \in \mathbb{N}$, then $A =^{\mathcal{K}} B$ is equivalent to $(A\Delta B) \cap n = \emptyset$, denoted $A =^n B$ (Definition 6.3.6).

**Lemma 6.4.2.** *Suppose $\Phi\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})/\mathcal{I}$ is a homomorphism with lifting $\Phi_*$, $\mathcal{K}$ is a closed approximation to $\mathcal{I}$, $\mathbb{N} = A \sqcup B$, $[s] \cap \mathcal{P}(B)$ is a relatively clopen subset of $\mathcal{P}(B)$, and $F\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ is Borel-measurable.*

1. *Then there is a C-measurable $\mathcal{K}^2$-approximation to the restriction of $\Phi$ to the set $\mathcal{T}$ of all $a \subseteq A$ such that the set*

   $$\mathcal{Z}(a) = \{b \in [s] \cap \mathcal{P}(B) : F(a \cup b) \cap \Phi_*(A) =^{\mathcal{K}} \Phi_*(a)\}$$

   *is comeagre in $[s] \cap \mathcal{P}(B)$.*
2. *If $\mathcal{T}$ is relatively comeagre in some clopen subset of $\mathcal{P}(A)$, then there is a continuous $\mathcal{K}^4$-approximation to $\Phi$ on $\mathcal{P}(A)$.*

Proof. (1) For simplicity of notation, we may assume $[s] \cap \mathcal{P}(B) = \mathcal{P}(B)$. Since Boolean operations $\cup$, $\cap$ and $\Delta$ are continuous, the function

$$(a, b, c) \mapsto (F(a \cup b) \cap \Phi_*(A))\Delta c \tag{6.5}$$

is Borel, and therefore the set

$$\mathcal{X} = \{(a, b, c) \in \mathcal{P}(A) \times \mathcal{P}(B) \times \mathcal{P}(\mathbb{N}) : F(a \cup b) \cap \Phi_*(A) =^{\mathcal{K}} c\}$$

is, being the preimage of $\mathcal{K}$ by the Borel function in (6.5), itself Borel. The set

$$\mathcal{Y} = \{(a, c) \in \mathcal{P}(A) \times \mathcal{P}(\mathbb{N}) : \{b \subseteq B : (a, b, c) \in \mathcal{X}\} \text{ is comeagre}\}$$

is, by Novikov's theorem (Theorem A.1.4), analytic. By our assumption, for every $a \in \mathcal{T}$ the set $\mathcal{Z}(a) = \{b \subseteq B : (a, b, \Phi_*(a)) \in \mathcal{X}\}$ is comeagre in $\mathcal{P}(B)$, in particular the section $\mathcal{Y}_a$ is nonempty for all $a \in \mathcal{T}$.

Therefore the Jankov–von Neumann uniformisation theorem (Theorem A.1.2) implies that there exists a C-measurable function

$$G_0\colon \mathcal{P}(A) \to \mathcal{P}(\mathbb{N})$$

such that for all $a \in \mathcal{T}$ the set $\mathcal{X}(a)$ consisting of all $b \subseteq B$ that satisfy

$$F(a \cup b) \cap \Phi_*(A) =^{\mathcal{K}} G_0(a) \tag{6.6}$$

is comeagre. In addition, for every $a \in \mathcal{T}$ the set $\mathcal{Z}(a)$ consisting of all $b \subseteq B$ such that

$$F(a \cup b) \cap \Phi_*(A) =^{\mathcal{K}} \Phi_*(a) \tag{6.7}$$

is comeagre. Hence for each $a \in \mathcal{T}$ there is $b \in \mathcal{X}(a) \cap \mathcal{Z}(a)$. For such $b$ both (6.6) and (6.7) hold and therefore $G_0(a) =^{\mathcal{K}^2} \Phi_*(a)$.

(2) Assume that $\mathcal{T}$ is relatively comeagre in some clopen subset of $\mathcal{P}(A)$. By the already proven part of this lemma, there is a C-measurable $\mathcal{K}^2$-approximation to $\Phi$ on $\mathcal{T}$. By the Baire measurability of analytic sets and Lemma 3.5.3 (2), there is a continuous $\mathcal{K}^4$ approximation to $\Phi$ on $\mathcal{P}(A)$. □

Proof of Proposition 6.4.1. Fix a partition $\mathbb{N} = \bigsqcup_n A_n$ and Borel-measurable functions $F_n \colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ whose graphs cover the graph of a lifting $\Phi_*$ of $\Phi$.

It suffices to prove that the restriction of $\Phi$ to $\mathcal{P}(A_n)$ has a continuous $\mathcal{K}^4$-approximation for some $n$. Assume this is not the case. Since the ideal $\mathcal{J}_{\mathrm{cont}}$ is closed under finite changes of its elements, this implies that for every $n$ and every nonempty clopen subset $[t] \cap \mathcal{P}(A_n)$ of $\mathcal{P}(A_n)$, the restriction of $\Phi$ to $[t] \cap \mathcal{P}(A_n)$ has no continuous $\mathcal{K}^4$-approximation.

It will be convenient to write $C_n = \bigcup_{j>n} A_j$.

We will recursively choose sets $a_n \subseteq A_n$ and $\mathcal{X}_n \subseteq \mathcal{P}(C_n)$, along with clopen subsets $[s_n] \cap \mathcal{P}(C_n)$ of $\mathcal{P}(C_n)$ and decreasing sequences $(U_{ni})_i$ of open subsets of $\mathcal{P}(C_n)$ such that the following conditions hold for all $n$.

(1) $F_n(\bigcup_{j \le n} a_j \cup b) \cap \Phi_*(A_n) \neq^{\mathcal{K}} \Phi_*(a_n)$ for all $b \in \mathcal{X}_n$.
(2) $\mathcal{X}_n \supseteq \bigcap_i U_{ni}$ and $U_{ni}$ is open dense in $[s_n] \cap \mathcal{P}(C_n)$ for all $i \in \mathbb{N}$.
(3) $\{a_{n+1}\} \oplus \mathcal{X}_{n+1} \subseteq \mathcal{X}_n$.
(4) For every $k < n$, all $b \subseteq C_k$ such that both $b \cap \bigcup_{k<j\le n} A_j = \bigcup_{k<j\le n} a_j$ and $b \cap C_n \in [s_n]$ hold belong to $U_{kn}$.

We will describe the recursive selection of $a_n$, $\mathcal{X}_n$, $(U_{ni})_i$ and $[s_n]$.

For $n = 0$, our assumption that $\Phi$ has no continuous $\mathcal{K}^4$-approximation on $\mathcal{P}(A_0)$ and Lemma 6.4.2 together imply that for some $a_0 \subseteq A_0$ the set $\mathcal{X}_0$ of all $x \subseteq C_0$ such that $F_0(a_0 \cup x) \cap \Phi_*(A_0) \neq^{\mathcal{K}} \Phi_*(a_0)$ is nonmeagre. Since this set is, as a preimage of a Borel set by a Borel-measurable function, Borel, there is a clopen set $[s_0] \cap \mathcal{P}(C_0) \subseteq \mathcal{P}(C_0)$ such that $[s_0] \cap \mathcal{X}_0$ is relatively comeagre in $[s_0] \cap \mathcal{P}(C_0)$. Choose a decreasing sequence $(U_{0i})_i$ of dense open subsets of $[s_0] \cap \mathcal{P}(C_0)$ whose intersection is contained in $[s_0] \cap \mathcal{X}_0$.

This describes the construction of $a_0$, $\mathcal{X}_0$, $(U_{0i})_i$ and $[s_0]$.

If $a_n$, $\mathcal{X}_n$, $(U_{ni})_i$ and $[s_n]$ as required had been chosen. Using the notation introduced in (6.3) and (6.4), there are clopen sets $[t_n] \cap \mathcal{P}(A_{n+1}) \subseteq \mathcal{P}(A_{n+1})$ and $[u_n] \cap \mathcal{P}(C_{n+1}) \subseteq \mathcal{P}(C_{n+1})$ such that

$$[s_n] \cap \mathcal{P}(C_n) = [t_n] \cap \mathcal{P}(A_{n+1}) \oplus [u_n] \cap \mathcal{P}(C_{n+1}).$$

By the Kuratowski–Ulam theorem (Theorem A.1.5), the set

$$\mathcal{T}_n = \{a \in [t_n] \cap \mathcal{P}(A_{n+1}) : \text{the set } \{b \in [u_n] \cap \mathcal{P}(C_{n+1}) : a \cup b \in \mathcal{X}_n\} \text{ is relatively comeagre in } [u_n] \cap \mathcal{P}(C_{n+1})\}$$

is relatively comeagre in $[t_n] \cap \mathcal{P}(A_{n+1})$. Since the intersection of comeagre sets is comeagre and since $\Phi$ has no continuous lifting on $[t_n] \cap \mathcal{P}(A_n)$, by Lemma 6.4.2 applied to $F_{n+1}$, $\mathcal{T}_n$, and $[u_n]$, we can find $a_{n+1} \in [t_n] \cap \mathcal{P}(A_{n+1})$ such that the set

$$\mathcal{X}_{n+1} = \{b \in [u_n] \cap \mathcal{P}(C_{n+1}) : a_{n+1} \cup b \in \mathcal{X}_n, F_{n+1}(\textstyle\bigcup_{j \le n} a_j \cup a_{n+1} \cup b) \cap \Phi_*(A_{n+1}) \neq^{\mathcal{K}} \Phi_*(a_{n+1})\}$$

is relatively nonmeagre in $[u_n] \cap \mathcal{P}(C_{n+1})$. Being Borel, it is relatively comeagre in some relatively clopen $[s_{n+1}] \cap \mathcal{P}(C_{n+1}) \subseteq [u_n] \cap \mathcal{P}(C_{n+1})$. We can choose $[s_{n+1}] \cap \mathcal{P}(C_{n+1})$ sufficiently small so that for all $k \leq n$ and $b \in [s_{n+1}] \cap \mathcal{P}(C_{n+1})$ the set $\bigcup_{k<j\leq n+1} a_j \cup b$ belongs to $U_{kn}$. Finally, choose a decreasing sequence $(U_{n+1i})_i$ of dense open subsets of $[s_{n+1}] \cap \mathcal{P}(C_{n+1})$ whose intersection is contained in $[s_{n+1}] \cap \mathcal{X}_{n+1}$.

Then $\{a_{n+1}\} \oplus \mathcal{X}_{n+1} \subseteq \mathcal{X}_n$, and the sets $a_{n+1}$, $\mathcal{X}_{n+1}$, $(U_{n+1i})_i$ and $[s_{n+1}]$ satisfy the requirements. This describes the recursive construction.

Given $a_n$, for $n \in \mathbb{N}$, let $a = \bigcup_n a_n$. By the assumption, $F_n(a) =^{\mathcal{K}} \Phi_*(a)$ for some $n$ and (4) implies $a \cap C_n = \bigcup_{j>n} a_j \in \bigcap_j U_{nj} \subseteq \mathcal{X}_n$. Since $a \cap A_n = a_n$, we have $F_n(a) \cap \Phi_*(A_n) =^{\mathcal{K}} \Phi_*(a_n)$, but this contradicts 1. □

### 6.5. The proof of the lifting theorem for countably generated ideals, Theorem 6.1.3

Fix a countably generated ideal $\mathcal{I}$ on $\mathbb{N}$ and a homomorphism $\Phi\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})/\mathcal{I}$. We need to prove that $\Phi$ has a completely additive lifting on a nonmeagre ideal. By Theorem 3.2.2, if $\mathcal{J}_{\text{cont}}$ were meagre then there would be a perfect almost disjoint family disjoint from $\mathcal{J}_{\text{cont}}$. However, $\mathcal{J}_{\text{cont}}$ intersects every perfect tree-like almost disjoint family by Proposition 6.3.1 and is therefore nonmeagre. If $X = \{n : \{n\} \notin \mathcal{I}\}$, then the homomorphism $A \mapsto \Phi(A) \cap X$ is already completely additive. It therefore suffices to prove the theorem for the restriction of $\Phi$ to $\mathbb{N} \setminus X$. This amounts to assuming $\mathcal{I} \supseteq \text{Fin}$ and finding a completely additive lifting on $\mathcal{J}_{\text{cont}}$.

*Proof of Theorem 6.1.3 in case when* $\mathcal{I} = \text{Fin}$. Fix $\Phi\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})/\text{Fin}$. For each $A \in \mathcal{J}_{\text{cont}}(\Phi)$ the restriction of $\Phi$ to $\mathcal{P}(A)$ has a continuous lifting, and by Theorem 4.1.2 it has a completely additive one. This lifting is of the form $B \mapsto h_A^{-1}(B)$ for a function

$$h_A\colon \Phi_*(A) \to A.$$

We fix these functions for a moment, but reserve all rights to modify then and re-evaluate the open partitions used in the proof as convenient.

The first step will be to verify that the family $h_A$, for $A \in \mathcal{J}_{\text{cont}}$, has an appropriate coherence property .

**Claim 6.5.1.** *For all $A$ and $B$ in $\mathcal{J}_{\text{cont}}$ the set*

$$\text{Diff}(A,B) = \{n \in \Phi_*(A) \cap \Phi_*(B), h_A(n) \neq h_B(n)\}$$

*is finite.*

Proof. Assume otherwise and fix $A$ and $B$ such that $C = \text{Diff}(A,B)$ is infinite. Define $c\colon [C]^2 \to \{0,1,2\}$ as follows.

$$c(\{m,n\}_<) = \begin{cases} 0 & \text{if } h_A(m) = h_B(n) \\ 1 & \text{if } h_B(m) = h_A(n) \\ 2 & \text{if } h_A(m) \neq h_B(n) \text{ and } h_B(m) \neq h_A(n). \end{cases}$$

Let $C_0 \subseteq C$ be an infinite homogeneous set. If it is 2-homogeneous then $A_0 = h_A[C_0]$ and $B_0 = h_B[C_0]$ are disjoint sets such that $\Phi_*(A_0) \cap \Phi_*(B_0)$ is infinite[2] (hence Fin-positive); contradiction.

[2]Note that we are not assuming $\ker(\Phi) \supseteq \text{Fin}$.

If $C_0$ is 0-homogeneous, then for all $m < m' < n$ in $C_0$ we have $h_B(m') = h_A(m) = h_B(n) = h_A(m')$, contradicting the choice of $C$. The case when $C_0$ is 1-homogeneous leads to a contradiction by an analogous argument. □

Extend each $h_A$ to a function $h_A^+$ from $\mathbb{N}$ into $\mathbb{N}_* = \mathbb{N} \cup \{\infty\}$ by $h_A^+(m) = \infty$ for $m \notin \Phi_*(A)$. Identify $A \in \mathcal{J}_{\mathrm{cont}}$ with the pair $(A, h_A^+) \in \mathcal{J}_{\mathrm{cont}} \times {\mathbb{N}_*}^{\mathbb{N}}$. The right-hand side is a subspace of the Polish space $\mathcal{P}(\mathbb{N}) \times {\mathbb{N}_*}^{\mathbb{N}}$ and we use this identification to equip $\mathcal{J}_{\mathrm{cont}}$ with a separable metric topology $\tau$.

For $t \Subset \mathbb{N}$ and $A, B$ in $\mathcal{J}_{\mathrm{cont}}$ we say that $A$ and $B$ *conflict on* $t$ if $\mathrm{Diff}(A, B) \supseteq t$. For a fixed $t \Subset \mathbb{N}$, let

$$U_t = \{\{A, B\} : A \text{ and } B \text{ conflict on } t\}.$$

This is a symmetric and $\tau$-open subset of $[\mathcal{J}_{\mathrm{cont}}]^2$.

This may be a good moment to take a look at the statement of OCA$^{\#}$ (Definition A.3.4). For $m \geq 1$ let

$$\mathcal{V}_m = \{U_s : s \in [\mathbb{N}]^n, \text{ with } n = m + (4^{m+1} - 1)/3\}.$$

Since $t \subseteq t'$ implies $U_t \supseteq U_{t'}$, we have $\mathcal{V}_m \supseteq \mathcal{V}_{m+1}$ for all $m$.

**Claim 6.5.2.** *There are no $(Z, f, \rho)$ with $Z \subseteq \{0,1\}^{\mathbb{N}}$ uncountable, $f\colon Z \to \mathcal{J}_{\mathrm{cont}}$, and $\rho\colon \Delta(Z) \to \bigcup_m \mathcal{V}_m$ such that $\rho(s) \in \mathcal{V}_{|s|}$ for all $s$ and $\{f(x), f(y)\} \in \rho(x \wedge y)$ for all distinct $x, y$ in $Z$.*

Proof. Assume otherwise and fix $Z$, $f$, and $\rho$. We may assume that $Z$ has no isolated points, in which case $(\Delta(Z), \sqsubseteq)$ is a perfect tree. (It is not necessarily downwards closed in $\{0,1\}^{<\mathbb{N}}$.) If $\rho(s) = U$ then $U = U_t$ for some $t$ of cardinality $|s| + (4^{|s|+1} - 1)/3$; we write $A(s) = t$. By Lemma A.6.1 there are pairwise disjoint $B(s) \subseteq A(s)$ such that $|B(s)| = 2^{|s|}$ for all $s \in \Delta(Z)$. Let $\mathsf{S}_m$ denote the $m$-th level of $\Delta(Z)$ and note that $s \in \mathsf{S}_m$ implies $|B(s)| \geq 2^m$. There are therefore disjoint sets $J_t$, for $t \in \{0,1\}^m$, such that $J_t \cap B(s) \neq \emptyset$ for all $s \in \mathsf{S}_m$ and $\bigcup\{B(s) : s \in \mathsf{S}_m\} = \bigcup\{J_t : t \in \{0,1\}^m\}$. For $g \in \{0,1\}^{\mathbb{N}}$ let

$$D(g) = \bigcup_n J_{g \restriction n}.$$

Since the finite sets $J_t$ are nonempty and disjoint, $\{D(g)\}$ is a perfect tree-like almost disjoint family. Therefore OCA$_{\mathrm{T}}$ and Proposition 6.3.1 together imply that $D(g) \in \mathcal{J}_{\mathrm{cont}}$ for some $g$. The salient property of $D(g)$ is that $D(g) \cap A(s) \neq \emptyset$ for all $s \in \Delta(Z)$. By Claim 6.5.1, for every $x \in Z$ there is $n(x)$ such that all $j \in (f(x) \cap D(g)) \setminus n(x)$ satisfy $h_{D(g)}(j) = h_{f(x)}(j)$.

Fix $n$ such that $Z' = \{x \in Z : n(x) = n\}$ is uncountable. As the sets $A(s)$ are nonempty and disjoint, the set $\{s \in \Delta(Z) : A(s) \cap n \neq \emptyset\}$ is finite. Hence we can choose distinct $x$ and $y$ in $Z'$ such that $A(x \wedge y) \cap n = \emptyset$. Therefore $D(g) \cap A(x \wedge y) \neq \emptyset$, and $f(x)$ and $f(y)$ conflict on $D(g) \cap A(x \wedge y) \neq \emptyset$. However, each one of $h_{f(x)}$ and $h_{f(y)}$ agrees with $h_{D(g)}$ on this set; contradiction. □

Since OCA$^{\#}$ is equivalent to OCA$_{\mathrm{T}}$ (Theorem A.3.5), by Claim 6.5.2 and OCA$^{\#}$ there are $\mathcal{X}_n$, for $n \in \mathbb{N}$, such that $\mathcal{J}_{\mathrm{cont}} = \bigcup_n \mathcal{X}_n$ and $[\mathcal{X}_n]^2 \cap \mathcal{V}_n = \emptyset$ for all $n$. Since $\mathcal{J}_{\mathrm{cont}}(\Phi)$ is nonmeagre, we can fix $n$ such that $\mathcal{X}_n$ is nonmeagre.

Next, we attempt to recursively choose an increasing sequence $n_i$ and $k_i \neq l_i$ for $i \in \mathbb{N}$ such that the following holds for all $m$.[3]

[3] The remaining part of the proof is Biba's trick.

(1) The set $\mathcal{F}_{0,m} = \{A \in \mathcal{X}_n : h_A(n_i) = k_i$ for all $i < m\}$ is nonmeagre.
(2) The set $\mathcal{F}_{1,m} = \{B \in \mathcal{X}_n : h_B(n_i) = l_i$ for all $i < m\}$ is nonmeagre.

Since $[\mathcal{X}_n]^2 \cap \bigcup \mathcal{V}_n = \emptyset$, a recursive construction of such sequences has to stop at a finite stage (more precisely, before the $n+(4^{n+1}-1)/3$-th stage). We therefore have $m$ (possibly $m = 0$, with $n_{-1} = 0$), $n_i, k_i, l_i$, for $i < m$ such that for all $n > n_{m-1}$ and all $k \neq l$ at least one of the sets

$$\{A \in \mathcal{F}_{0,m} : h_A(n) = k\} \text{ or } \{B \in \mathcal{F}_{1,m} : h_B(n) = l\}$$

is meagre.

Let $D$ be the set of $n > n_{m-1}$ such that both $\mathcal{F}_0 = \{A \in \mathcal{F}_{0,m} : n \in \Phi_*(A)\}$ and $\mathcal{F}_1 = \{B \in \mathcal{F}_{1,m} : n \in \Phi_*(B)\}$ are nonmeagre. Fix $n \in D$. If $k$ and $l$ are such that $\{A \in \mathcal{F}_0 : h_A(n) = k\}$ and $\{B \in \mathcal{F}_1 : h_B(n) = l\}$ are nonmeagre, then we have $k = l$ (otherwise we would have set $n_m = n$, $k_m = k$, and $l_m = l$). Therefore there is $k = h(n)$ such that $h_A(n) = k = h_B(n)$ for a relatively comeagre (in $\mathcal{F}_0$) set of $A \in \mathcal{F}_0$ and a relatively comeagre (in $\mathcal{F}_1$) set of $B \in \mathcal{F}_1$. This defines $h\colon D \to \mathbb{N}$. Moreover, it shows that for every $n \in D$ and $l \neq h(n)$ each of the sets $\{A \in \mathcal{F}_0 : h_A(n) \neq h(n)\}$ and $\{B \in \mathcal{F}_1 : h_B(n) = l\}$ is meagre. Therefore each of the following two sets is nonmeagre.

$$\tilde{\mathcal{F}}_0 = \mathcal{F}_0 \setminus \textstyle\bigcup_{n \in D}\{A \in \mathcal{F}_0 : h_A(n) \neq h(n)\},$$
$$\tilde{\mathcal{F}}_1 = \mathcal{F}_1 \setminus \textstyle\bigcup_{n \in D}\{B \in \mathcal{F}_1 : h_B(n) \neq h(n)\}.$$

**Claim 6.5.3.** *For all $A$ in $\mathcal{J}_{\mathrm{cont}}$ the set*

$$\mathrm{Diff}(A, h) = \{n \in \Phi_*(A) \cap D : h_A(n) \neq h(n)\}$$

*is finite.*

PROOF. Assume otherwise and fix $A$ such that $C = \mathrm{Diff}(A, h)$ is infinite. Since $\tilde{\mathcal{F}}_1$ is nonmeagre, there is $B \in \tilde{\mathcal{F}}_1$ such that $B \cap C$ is infinite. Then $h_B$ agrees with $h$ on this set, hence $h_A$ and $h_B$ differ on an infinite set, contradicting Claim 6.5.1. □

**Claim 6.5.4.** *The set $\mathbb{N} \setminus h[D]$ is finite.*

PROOF. Assume otherwise. Since $\mathcal{F}_1$ is nonmeagre, there is $B \in \mathcal{F}_1$ such that $(\mathbb{N} \setminus h[D]) \cap B$ is infinite. The function $h_B\colon \Phi_*(B) \to B$ has cofinite range, therefore $h_B(j) \in A$ for some $j > n$; contradiction. □

If $A \in \ker(\Phi)$, then $B \mapsto h^{-1}(B)$ is a lifting of $\Phi$ on $\mathcal{J}_{\mathrm{cont}}$. We need to consider the case when $A \notin \ker(\Phi)$. Since $A$ is finite by Claim 6.5.4, it is straightforward to extend $h$ to $\Phi_*(A)$ so that $h^{-1}(\{j\}) =^* \Phi_*(\{j\})$ for all $j \in A$. Then $B \mapsto h^{-1}(B)$ is a lifting of $\Phi$ on $\mathcal{J}_{\mathrm{cont}}$, as required. We have proven that the function $X \mapsto h^{-1}(X)$ is a continuous lifting of $\Phi$ on the nonmeagre ideal $\mathcal{J}_{\mathrm{cont}}$. This completes the proof in case when $\mathcal{I} = \mathrm{Fin}$.

*Proof of Theorem 6.1.3 for an arbitrary countably generated ideal.* Suppose $\mathcal{I}$ is generated by an increasing sequence of subsets of $\mathbb{N}$, $A_n$, for $n \in \mathbb{N}$. If there is $m$ such that $A_{n+1} \setminus A_n$ is finite for all $n \geq m$, then the range of $\Phi$ can be identified with $\mathcal{P}(\mathbb{N} \setminus A_m)/\mathrm{Fin}$ and the conclusion follows from the first part of the proof. We may therefore assume that $A_{n+1} \setminus A_n$ is infinite for infinitely many $n$. By passing to a subsequence, we may assume $A_{n+1} \setminus A_n$ is infinite for all $n$. Therefore $\mathcal{I}$ is isomorphic to the ideal

$$\mathrm{Fin} \times \emptyset = \{B \subseteq \mathbb{N}^2 \mid (\exists m) B \subseteq m \times \mathbb{N}\}.$$

The remaining part of the proof borrows some of the ideas from the proof of [40, Theorem 1.9.2]. It will be convenient to use Greek letters for the elements of $\mathbb{N}^\mathbb{N}$. For $\alpha$ and $\beta$ in $\mathbb{N}^\mathbb{N}$ we write $\alpha \leq^* \beta$ if $(\forall^\infty j)\alpha(j) \leq \beta(j)$. For $\alpha \in \mathbb{N}^\mathbb{N}$ let

$$\Gamma_\alpha = \{(m,n) : n < \alpha(m)\}.$$

Then $\mathcal{I} \cap \mathcal{P}(\Gamma_\alpha)$ is the ideal of finite subsets of $\Gamma_\alpha$. Therefore, by the first part of the proof there are $A_\alpha \subseteq \Gamma_\alpha$, $h_\alpha \colon A_\alpha \to \mathbb{N}$, and an ideal $\mathcal{J}_\alpha$ on $\mathbb{N}$ such that $\mathcal{J}_\alpha$ intersects every perfect tree-like almost disjoint family nontrivially and the function

$$\Psi_\alpha(X) = h_\alpha^{-1}(X)$$

satisfies (with $\Phi_*$ denoting a lifting of $\Phi$ and abusing notation to denote the ideal of finite subsets of $\mathbb{N}^2$ by Fin)

$$(\Psi_\alpha(X)\Delta\Phi_*(X)) \cap \Gamma_\alpha \in \mathrm{Fin}$$

for all $X \in \mathcal{J}_\alpha$.

Define a partition of $[\mathbb{N}^\mathbb{N}]^2 = K_0 \cup K_1$ by setting $\{\alpha,\beta\} \in K_0$ if and only if

$$h_\alpha((i,j)) \neq h_\beta((i,j)) \text{ for some } i \text{ and } j < \min(\alpha(i), \beta(i).$$

We identify $\alpha$ with $(\alpha, h_\alpha)$ and $\mathbb{N}^\mathbb{N}$ with a subset of the Polish space $\mathbb{N}^\mathbb{N} \times \mathbb{N}^{\mathbb{N}^2}$, and use this identification to topologise $\mathbb{N}^\mathbb{N}$. Then $K_0$ is symmetric and open in this topology.

**Claim 6.5.5.** *There is no uncountable $K_0$-homogeneous $\mathcal{X} \subseteq \mathbb{N}^\mathbb{N}$ .*

Proof. Assume otherwise. Then Lemma A.4.1 implies that there is $\alpha \in \mathbb{N}^\mathbb{N}$ such that $\{\beta \in \mathbb{N}^\mathbb{N} : \beta \leq \alpha\}$ includes an uncountable $K_0$-homogeneous set $\mathcal{H}$.

Every $\beta \in \mathcal{H}$ satisfies $h_\alpha \restriction \Gamma_\beta =^* h_\beta$, hence there is $m = m(\beta)$ such that $h_\alpha((i,j)) = h_\beta(i,j)$ for all $i \geq m$ and $j < \min(\alpha(i), \beta(i))$. Let $m$ be such that the set $Z' = \{x \in Z : m(x) = m\}$ is uncountable, and choose $\beta$ and $\gamma$ in $Z'$ such that $h_\beta(i,j) = f_\gamma(i,j)$ for all $i < m$ (this is possible since there are only finitely many possibilities). Then $\{\beta,\gamma\} \in K_1^m$; contradiction. □

By Claim 6.5.5, $\mathrm{OCA_T}$ implies that $\mathbb{N}^\mathbb{N} = \bigcup \mathcal{X}_n$ and $[\mathcal{X}_n]^2 \subseteq K_1$ for all $n$. By [111, Lemma 1], there are $n$ and $m$ such that for every $\alpha \in \mathbb{N}^\mathbb{N}$ some $\beta \in \mathcal{X}_n$ satisfies $\alpha(i) \leq \beta(i)$ for all $i \geq m$. By $K_1$-homogeneity, $h = \bigcup_{\beta \in \mathcal{X}_n} h_\beta$ is a function. Its domain includes $[m, \infty) \times \mathbb{N}$.

We claim that $\{A \subseteq \mathbb{N} : \Phi_*(A)\Delta h^{-1}(A) \in \mathrm{Fin} \times \emptyset\}$ includes an ideal that intersects every perfect tree-like almost disjoint family. Fix a perfect tree-like almost disjoint family $\mathcal{A}\{J_s\}$, for a family of disjoint finite sets $J_s$, for $s \in \{0,1\}^{<\mathbb{N}}$. Since $\mathrm{Fin} \times \emptyset$ is an $F_\sigma$ ideal, by Proposition 6.3.1 there is $f \in \{0,1\}^\mathbb{N}$ such that $A(f) = \bigcup_n J_{f \restriction n}$ belongs to $\mathcal{J}_{\mathrm{cont}}$. Let $\Theta \colon A(f) \to \mathcal{P}(\mathbb{N}^2)$ be a continuous lifting of $\Phi$ on $\mathcal{P}(A(f))$. We will use the following characterisation of $\mathrm{Fin} \times \emptyset$: Some $C \subseteq \mathbb{N}^2$ belongs to $\mathrm{Fin} \times \emptyset$ if and only if $C \cap \Gamma_\alpha$ is finite for all $\alpha \in \mathbb{N}^\mathbb{N}$. By Lemma 3.4.2, for every $\alpha \in \mathbb{N}^\mathbb{N}$ and for every $B \subseteq A(f)$ the set $\Theta(B)\Delta h_\alpha^{-1}(B)) \cap \Gamma_\alpha$ is finite. Therefore $\Theta(B)\Delta h^{-1}(B)) \cap \Gamma_\alpha$ is finite for all $\alpha$, and $\Theta(B)\Delta h^{-1}(B) \in \mathrm{Fin} \times \emptyset$.

Thus $B \mapsto h^{-1}(B)$ is a lifting of $\Phi$ on an ideal that intersects an arbitrary perfect tree-like almost disjoint family, as required.

## 6.6. Uniformisation modulo $\mathcal{I}$

The main result of this section is Proposition 6.6.2 which together with Proposition 6.3.1 completes the proof of Theorem 6.1.2.

If $\mathcal{K}$ is a closed hereditary set, $F$, $G$ are functions whose domains are subsets of $\mathcal{P}(\mathbb{N})$, and $A \subseteq \mathbb{N}$ is such that $\mathcal{P}(A) \subseteq \mathrm{dom}(F) \cap \mathrm{dom}(G)$, then we write

$$F \restriction \mathcal{P}(A) =^{\mathcal{K}} G \restriction \mathcal{P}(A) \tag{6.8}$$

if all $B \subseteq A$ satisfy $F(B) \Delta G(B) \in \mathcal{K}$ and $F \restriction \mathcal{P}(A) =^{\mathcal{K} \sqcup \mathrm{Fin}} G \restriction \mathcal{P}(A)$ if all $B \subseteq A$ satisfy $F(B)\Delta G(B) \in \mathcal{K} \sqcup \mathrm{Fin}$.

**Lemma 6.6.1.** *Assume that $\Phi\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})/\mathcal{I}$ is a homomorphism, $\mathcal{K}$ is a closed approximation to $\mathcal{I}$, and that $\mathcal{Y}_n$, for $n \in \mathbb{N}$, are hereditary sets such that $\bigcup_n \mathcal{Y}_n$ is nonmeagre and there is a C-measurable $\mathcal{K}$-approximation $\Theta_n$ to $\Phi$ on $(\mathcal{Y}_n)^2$ for all $n$. Then there is a continuous $\mathcal{K}^4$-approximation to $\Phi$ on a relatively comeagre hereditary subset of $\bigcup_n \mathcal{Y}_n$.*

Proof. Our first task is to prove that we may assume each $\mathcal{Y}_n$ is closed under finite changes. Fix $n$ such that $\mathcal{Y}_n$ is nonmeagre. By Lemma 3.2.7, there is $k(n)$ such that for every $s \Subset [k(n), \infty)]$ the set $\tilde{\mathcal{Y}}_n = \{A \subseteq \mathbb{N} : s \cup A \in \mathcal{Y}_n\}$ is hereditary and nonmeagre.

Since we are not assuming that $\ker(\Phi)$ includes Fin, there is some extra work to do. Fix a lifting $F_n\colon \mathcal{P}(k(n)) \to \mathcal{P}(\mathbb{N})$ of the restriction of $\Phi$ to $\mathcal{P}(k(n))$. Then define $\tilde{\Theta}_n\colon \tilde{\mathcal{Y}}_n \to \mathcal{P}(\mathbb{N})$ by

$$\tilde{\Theta}_n(A) = F_n(A \cap k(n)) \cup \Theta_n(A \setminus k(n)).$$

Since $\mathcal{P}(k(n))$ is finite, this function is C-measurable and it is a lifting of $\Phi$ on $\tilde{\mathcal{Y}}_n$.

Let $X = \{n : \tilde{\mathcal{Y}}_n$ is nonmeagre$\}$. Since $\bigcup_n \mathcal{Y}_n$ is nonmeagre, $X$ is nonempty.

By Theorem 3.2.2, $\mathcal{H} = \bigcap_{n \in X} \tilde{\mathcal{Y}}_n$ is nonmeagre, and it is clearly closed under finite changes of its elements.

An argument similar to the proofs of Corollary 3.2.6 and Corollary 6.2.1 follows. Let $\mathcal{G} \subseteq \mathcal{P}(\mathbb{N})$ be a dense $G_\delta$ set such that $\tilde{\Theta}_n \restriction G$ is continuous for all $n \in X$ and $G \cap \tilde{\mathcal{Y}}_n = \emptyset$ for all $n \notin X$.

Choose disjoint $J_i \Subset \mathbb{N}$ and $t_i \subseteq J_i$ such that $\{A : (\exists^\infty i) A \cap J_i = t_i\} \subseteq G$. Since $\mathcal{H}$ is hereditary and nonmeagre, there is an infinite set $Y \subseteq \mathbb{N}$ such that $\bigcup_{i \in Y} J_i \in \mathcal{H}$. Let $Y = Y_0 \sqcup Y_1$ be a partition into two infinite sets, let

$$C_0 = \textstyle\bigcup_{i \in Y_0} t_i, \quad C_1 = \bigcup_{i \in Y_1} t_i, \quad B_0 = \bigcup_{i \in Y_0} J_i, \quad \text{and} \quad B_1 = \mathbb{N} \setminus B_0.$$

Then $A \cap B_j = C_j$ for $j = 0$ or for $j = 1$ implies $A \in \mathcal{H}$. If in addition $A \in \mathcal{Y}_n$ for some $n$, then $(A \setminus B_j) \cup (A \cap B_j) \in (\mathcal{Y}_n)^2$ for $j = 0$ and for $j = 1$. For $n \in X$ let (using a lifting $\Phi_*$ of $\Phi$)

$$\Upsilon_n(A) = (\Phi_*(B_1) \cap \Theta_n((A \setminus B_0) \cup (A \cap B_0))) \cup (\Phi_*(B_0) \cap \Theta_n((A \setminus B_1) \cup (A \cap B_1))).$$

Since the arguments of $\Theta_n$ in the definition belong to $G$, $\Upsilon_n$ is continuous. Since $\Theta_n$ is a $\mathcal{K}$-approximation to $\Phi$ on $\mathcal{Y}_n \sqcup \mathcal{Y}_n$, $\Upsilon_n$ is a $\mathcal{K}^2$-approximation to $\Phi$ on a relatively comeagre subset of $\mathcal{Y}_n$.

Fix $m \in X$ and let $\Upsilon = \Upsilon_m$. We claim that $\Upsilon$ is a $\mathcal{K}^4$-approximation to $\Phi$ on $\mathcal{Y}_n$ for every $n \in X$. This follows from Lemma 3.4.1 and completes the proof. □

**Proposition 6.6.2.** *Assume $\mathrm{OCA_T}$ and MA($\sigma$-linked). Suppose that $\mathcal{I}$ is an ideal on $\mathbb{N}$ with closed approximation $\mathcal{K}$, $\Phi\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})/\mathcal{I}$ is a homomorphism, and* *the hereditary set $\mathcal{J}^{\mathcal{K}}_{\text{cont}}(\Phi)$ intersects every uncountable tree-like almost disjoint family. Then $\Phi$ has a continuous $\mathcal{K}^{80}$-approximation on $\mathcal{J}^{\mathcal{K}}_{\text{cont}}(\Phi)$.*

Proof. For $A$ and $B$ in $\mathcal{J}^{\mathcal{K}}_{\text{cont}}(\Phi)$ and $D \subseteq \mathbb{N}$ we say that $F^A$ and $F^B$ *conflict* on $D$ if there is $s \subseteq A \cap B$ such that

$$F^A(s)\Delta F^B(s) \cap D \notin \mathcal{K}^{20}.$$

Since all $F^A$ and $F^B$ are continuous, $A$ and $B$ conflict on $D$ if and only if there is $s \Subset A \cap B$ such that $F^A(s)\Delta F^B(s) \notin \mathcal{K}^{20}$. Identify $A \in \mathcal{J}^{\mathcal{K}}_{\text{cont}}$ with the pair $(A, \{(s, F^A(s)) : s \Subset A\})$. This identifies $\mathcal{J}^{\mathcal{K}}_{\text{cont}}$ with a subset of $\mathcal{P}(\mathbb{N}) \times \mathcal{P}(\mathbb{N})^{\text{Fin}}$, and endows $\mathcal{J}^{\mathcal{K}}_{\text{cont}}$ with the subspace topology, denoted $\tau$.

**Claim 6.6.3.** *For every $D \subseteq \mathbb{N}$, the set of all $\{A, B\}$ that conflict on $D$ is symmetric and $\tau$-open.*

Proof. Symmetry is obvious from the definition. Fix open subsets $U_0$ and $U_1$ of $\mathcal{P}(\mathbb{N})$ such that $F^A(s) \cap D \in U_0$, $F^B(s) \cap D \in U_1$, and for all $X_0 \in U_0$ and $X_1 \in U_1$ we have $X_0 \Delta X_1 \notin \mathcal{K}^{20}$. Then the sets $V_j = \{C : s \Subset C, F^C(s) \in U_j\}$ for $j = 0, 1$ are $\tau$-open neighbourhoods of $A$ and $B$ and $A'$ and $B'$ conflict on $D$ for all $A' \in U_0$ and $B' \in V_1$. □

By Lemma 3.4.2, for all $A$ and $B$ in $\mathcal{J}^{\mathcal{K}}_{\text{cont}}$ there is $k = k(A, B) \in \mathbb{N}$ such that for all $s \subseteq (A \cap B) \setminus k$ we have $(F^A(s)\Delta F^B(s)) \setminus k \in \mathcal{K}^{10}$. For $m \in \mathbb{N}$, let $\mathcal{V}_m$ be the set of all $U \subseteq [\mathcal{J}^{\mathcal{K}}_{\text{cont}}]^2$ such that there are $m \leq n^U_0 < n^U_1 < \dots < n^U_{2^{2m}}$ for which $U$ is the set of all pairs $\{A.B\} \in [\mathcal{J}^{\mathcal{K}}_{\text{cont}}]$ which conflict on $[n^U_i, n^U_{i+1})$ for all $i < 2^{2m}$. By Claim 6.6.3, each $\mathcal{V}_m$ is a union of symmetric open subsets of $[\mathcal{J}^{\mathcal{K}}_{\text{cont}}]^2$ and clearly $\mathcal{V}_m \supseteq \mathcal{V}_{m+1}$, hence these sets are as in the statement of OCA$^\#$.

Next we verify that one of the alternatives of OCA$^\#$ cannot hold.

**Claim 6.6.4.** *There is no triple $(Z, f, \rho)$ such that $Z \subseteq \{0,1\}^{\mathbb{N}}$ is uncountable, $f\colon Z \to \mathcal{J}^{\mathcal{K}}_{\text{cont}}$, and $\rho\colon \Delta(Z) \to \bigcup_m \mathcal{V}_m$ such that $\rho(s) \in \mathcal{V}_{|s|}$ for all $s$ and*

$$\{f(x), f(y)\} \in \rho(x \wedge y)$$

*for all distinct $x, y$ in $Z$.*

Proof. Assume otherwise and fix $Z$, $f$, and $\rho$. For $s \in \Delta(Z)$ let

$$I(s) = [|s|, n^{\rho(s)}_{2^{2|s|}}).$$

By MA($\sigma$-linked) and Lemma A.6.5 there are an uncountable $Z' \subseteq Z$ and an increasing sequence $\{k_i\}$ such that for every $s \in \Delta(Z')$ some $m(s)$ satisfies

$$I(s) \subseteq [k_{m(s)}, k_{m(s)+1})$$

and $\mathsf{S}_m = \{s \in \Delta(Z') : m(s) = m\}$ is the $m$-th level of the tree $(\Delta(Z'), \sqsubseteq)$.

Fix $m$. For each $s \in \mathsf{S}_m$ we have $|s| \leq m$ and therefore $\rho(s)$ is given by $m \leq n^s_0 < n^s_1 < \dots < n^s_{l(s)}$ for $l(s) \geq 2^{2m}$. By Lemma A.6.2 there are disjoint $A(t) \subseteq [k_m, k_{m+1})$, for $t \in \{0,1\}^m$, such that for all $s \in \mathsf{S}_m$ and $t \in \{0,1\}^m$, for some $i$ we have $A(t) \supseteq [n^s_i, n^s_{i+1})$. The sets

$$A(h) = \bigcup_m A(h \restriction m) \qquad \text{for } h \in \{0,1\}^{\mathbb{N}}$$

satisfy $A(h) \cap A(h') \subseteq k_{\Delta(h,h')+1}$ for all distinct $h$ and $h'$. Therefore $A(h)$, for $h \in \{0,1\}^{\mathbb{N}}$, is an uncountable tree-like almost disjoint family. By the assumption, $A(h) \in \mathcal{J}^{\mathcal{K}}_{\text{cont}}$ for some $h$. By Lemma 3.4.2, for every $A \in \mathcal{J}^{\mathcal{K}}_{\text{cont}}$ there is $k(A)$ such that for all $s \subseteq (A \cap A(h)) \setminus k$ we have $(F^A(s) \Delta F^{A(h)}(s)) \setminus k \in \mathcal{K}^{10}$. Let $k$ be such that $Z'' = \{z \in Z' : k(f(z)) = k\}$ is uncountable. Choose $x$ and $y$ in $Z''$ such that $\Delta(x, y) > k$. Then for all $s \subseteq (A(h) \cap f(x) \cap f(y)) \setminus k$ we have (see (6.8))

$$F^{f(x)}(s) \setminus k =^{\mathcal{K}^{10}} F^{A(h)}(s) \setminus k =^{\mathcal{K}^{10}} F^{f(y)}(s) \setminus k.$$

However, $A(h) \cap \rho(x \wedge y)$ includes an interval of the form $[n_i^s, n_{i+1}^s)$, for $s = \rho(x \wedge y)$ on which $F^{f(x)}$ and $F^{f(y)}$ conflict. The minimum of this interval is at least $|s| \geq k$; contradiction. □

Since $\mathrm{OCA_T}$ implies $\mathrm{OCA}^\#$ (Theorem A.3.5), by Claim 6.6.4 we conclude that there are sets $\mathcal{X}_n$, for $n \in \mathbb{N}$, such that $\mathcal{J}^{\mathcal{K}}_{\mathrm{cont}} = \bigcup_n \mathcal{X}_n$ and $[\mathcal{X}_n]^2 \cap \mathcal{V}_n = \emptyset$ for all $n$.

We will use a version of Biba's trick as in the final part of the proof of Theorem 6.1.3. Let $n$ be such that $\mathcal{X}_n$ is nonmeagre in the original topology on $\mathcal{P}(\mathbb{N})$. We attempt to recursively choose an increasing sequence $n_i$, $s_i \subseteq [n_{i-1}, n_i)$ (with $n_{-1} = 0$), $u_i$, and $v_i$ for $i \in \mathbb{N}$ such that the following holds for all $m$.

(1) The set $\mathcal{F}_{0,m} = \{A \in \mathcal{X}_n : F^A(s_i) = u_i \text{ for all } i < m\}$ is nonmeagre.
(2) The set $\mathcal{F}_{1,m} = \{B \in \mathcal{X}_n : F^B(s_i) = v_i \text{ for all } i < m\}$ is nonmeagre.
(3) $u_i \Delta v_i \notin \mathcal{K}^{20}$ for all $i < m$.

Since $[\mathcal{X}_n]^2 \cap \bigcup \mathcal{V}_n = \emptyset$, a recursive construction of such sequences has to stop at a finite stage (more precisely, before the $2^{2n}$-th stage). We therefore have $m$ (possibly $m = 0$), $n_i, s_i, u_i, v_i$, for $i < m$ such that for all $s \subseteq [n_{m-1}, \infty)$ the set

$$\{(A, B) \in \mathcal{F}_{0,m} \times \mathcal{F}_{1,m} : s \subseteq A \cap B, F^A(s) \Delta F^B(s) \notin \mathcal{K}_n^{20}\}$$

is meagre. By increasing $n_{m-1}$ if needed, we can assure that for every $s \subseteq [n_{m-1}, \infty)$ both sets $\{A \in \mathcal{F}_{0,m} : s \subseteq A\}$ and $\{B \in \mathcal{F}_{1,m} : s \subseteq B\}$ are nonmeagre.

By Lemma 3.2.7, for a large enough $k \geq n$, for every interval of the form $[k, l)$ for $l > k$ there is an $A(l) \in \mathcal{X}_n$ such that $[k, l) \subseteq A(l)$. Identify $\mathcal{X}_n$ with a subset of $\mathcal{P}(\mathbb{N}) \times \mathcal{P}(\mathbb{N})^{\mathrm{Fin}}$ as in the definition of the topology $\tau$ (see the paragraph preceding Claim 6.6.3). Let

$$\mathcal{Y} = \{(X, Y) \in \mathcal{P}(\mathbb{N})^2 \mid (\forall j)(\forall^\infty l) \min(Y \Delta F^{A(l)}(X \cap [k, l)])) \cap j \in \mathcal{K}_n^{20}\}.$$

This is a Borel (more precisely, $F_{\sigma\delta}$) set. We claim that the section $\mathcal{Y}_X$ is nonempty for every $X \in \mathcal{X}_n$. Fix $X$. Since $\mathcal{P}(j)$ is finite, for every $j$ the set

$$T_{X,j} = \{s \subseteq j : (\forall^\infty l) F^{A(l)}(X \cap [k, l)) \cap j = s\}$$

is nonempty. Then $T_X = \bigcup_j T_{X,j}$ is a finitely branching tree with respect to the end-extension, and $T_{X,j}$ is its $j$-th level. Since each $T_{X,j}$ is nonempty, $T_X$ has an infinite branch $Y$. Clearly $Y$ belongs to $\mathcal{Y}_X$.

Because $[\mathcal{X}_n] \cap \mathcal{V}_n = \emptyset$ and because $k \geq n$, for all $(X, Y) \in \mathcal{Y}$ such that $X \in \hat{\mathcal{X}}_n$ and $A \in \mathcal{X}_n$ satisfies $X \subseteq A$ we have $F^A(X) \Delta Y \in \mathcal{K}_n^{20}$.

The set $\mathcal{Z} = \{X : \mathcal{Y}_X \neq \emptyset\}$ is analytic. By the Jankov, von Neumann uniformisation theorem (Theorem A.1.2) there is a C-measurable selection $\Theta_n \colon \mathcal{Z} \to \mathcal{P}(\mathbb{N})$ for $\mathcal{Y}$. By the previous paragraph, for every $X \in \mathcal{Z} \cap \hat{\mathcal{X}}_n$ and $A \in \mathcal{X}_n$ such that $X \subseteq A$ we have that $\Theta_n(X) \Delta F^A(X) \in \mathcal{K}_n^{20}$.

By Lemma 6.6.1 applied to $\mathcal{K}_n^{20}$, there is a continuous $\mathcal{K}_n^{80}$-approximation to $\Phi$ on a relatively comeagre, hereditary subset of $\mathcal{J}_{\mathrm{cont}}$. □

## 6.7. Proof of the OCA lifting theorem, Theorem 6.1.2

Suppose that $\mathcal{I}$ is countably 80-determined by closed approximations $\mathcal{K}_n$, for $n \in \mathbb{N}$, and that $\Phi\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})/\mathcal{I}$ is a homomorphism. By MA($\sigma$-linked) and Proposition 6.3.1, the ideal $\mathcal{J}_{\text{cont}}$ is ccc over Fin. By Proposition 6.6.2, $\Phi$ has a continuous $\mathcal{K}_n^{80}$ approximation on $\mathcal{J}_{\text{cont}}$ for every $n$. Lemma 6.3.8, applied with $\mathcal{B}_n = \mathcal{K}_n^{80}$, implies that $\Phi$ has a continuous lifting on a relatively comeagre subset $\mathcal{X}$ of $\mathcal{J}_{\text{cont}}$. Since $\mathcal{J}_{\text{cont}}$ is nonmeagre, so is $\mathcal{X}$. By Corollary 6.2.1 $\Phi$ has a continuous lifting on $\mathcal{J}^2_{\text{cont}}$. Since $\mathcal{J}_{\text{cont}}$ is an ideal, $\mathcal{J}^2_{\text{cont}} = \mathcal{J}_{\text{cont}}$ and this completes the proof that $\Phi$ has a continuous almost lifting. Proposition 3.5.4 implies that it is decomposable.

If in addition $\mathcal{I}$ has the Fubini property, then it has the Radon–Nikodym property by Theorem 4.1.2, and by applying it to the summand of $\Phi$ with a continuous lifting we obtain a completely additive almost lifting of $\Phi$.

## 6.8. Concluding remarks

The conclusion of Theorem 6.1.2 (perhaps under stronger forcing axioms) ought to hold for all analytic ideals. At present it applies only to certain $F_{\sigma\delta}$ ideals. Whether all $F_{\sigma\delta}$ ideals are (strongly) countably determined is a nice question, but a positive answer would extend Theorem 6.1.2 only to the realm of all $F_{\sigma\delta}$-ideals. The simplest ideal for which it is not known whether the conclusion of Theorem 6.1.2 applies to homomorphisms into its quotient is $\text{Fin} \times \text{Fin}$ (also known as the ordinal ideal $\mathcal{O}_{\omega^2}$). The current stalemate is similar to the one preceding the breakthrough of [**141**], fuelled by the realisation that gaps in $\mathcal{P}(\mathbb{N})/\,\text{Fin}$ can be 'frozen'. At present, we only know how to freeze gaps in countably 2-determined ideals.

Two ideas that may be relevant to extending the conclusion of Theorem 6.1.2 to all Borel ideals are the following (this is far-fetched, but who knows?). For every analytic ideal $\mathcal{I}$ there is a $G_\delta$ dense subset of $\mathcal{I}_+$ ([**148**]). Also, the results of Debs and Saint-Reymond may be relevant ([**24**]).

In Just's oracle-cc forcing extension a local version of the $\text{OCA}_{\text{T}}$ lifting theorem applies to homomorphisms from $\mathcal{P}(\mathbb{N})$ into $\mathcal{P}(\mathbb{N})/\mathcal{I}$ for every analytic ideal $\mathcal{I}$ ([**93**, Theorem A]). In [**60**] and [**73**] it was shown that in some forcing extension all isomorphisms between quotients over analytic ideals have continuous liftings. It is not known whether this implies that they are trivial, or whether this conclusion follows from forcing axioms.

CHAPTER 7

# Applications of lifting theorems, II

Together with Theorem 4.1.2, Theorem 6.1.2 has a number of consequences. For example, under $\mathrm{OCA_T}$ and MA($\sigma$-linked) the ideals Fin, Fin $\times\emptyset$, $\emptyset\times$ Fin, $\mathcal{I}_{1/n}$, $\mathcal{I}_{1/\sqrt{n}}$, $\mathcal{Z}_0$ and $\mathcal{Z}_{\log}$ have pairwisenonisomorphic quotients (Corollary 7.1.2). More generally, in the class of countably 80-determined ideals with the Fubini property our basic question has a satisfactory answer: Assuming $\mathrm{OCA_T}$ and MA($\sigma$-linked), two quotients over ideals in this class are isomorphic if and only if the corresponding ideals are Rudin–Keisler isomorphic and every isomorphism is implemented by a Rudin–Keisler isomorphism. Moreover, a version of the basic question stated in terms of embeddability (instead of isomorphism) relation between quotients has a satisfactory answer as well: $\mathcal{P}(\mathbb{N})/\mathcal{I}$ embeds into $\mathcal{P}(\mathbb{N})/\mathcal{J}$ if and only if there is a reduction of $\mathcal{I}$ to $\mathcal{J}$ (Corollary 7.2.2).

Assuming $\mathrm{OCA_T}$ and MA($\sigma$-linked), similar rigidity phenomena apply to arbitrary ideals whose quotients embed into a quotient over a countably 80-determined ideal. If $\mathcal{P}(\mathbb{N})/\mathcal{I}$ embeds into $\mathcal{P}(\mathbb{N})/\mathcal{J}$, and $\mathcal{J}$ is an analytic P-ideal, then $\mathcal{I}$ is an amalgamation of an analytic P-ideal reducible to $\mathcal{J}$ and a nonmeagre ideal (Corollary 7.2.2). This implies that there are many quotients $\mathcal{P}(\mathbb{N})/\mathcal{I}$ that are not embeddable into any quotient over an analytic P-ideal (Corollary 7.2.5). Also, if $\mathcal{I}$ is any ideal such that $\mathcal{P}(\mathbb{N})/\mathcal{I}$ embeds into $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$, then $\mathcal{I}$ and its orthogonal, $\mathcal{I}^\perp$, do not form a gap (Corollary 7.2.5).

Another consequence of the $\mathrm{OCA_T}$ lifting theorem is that all automorphisms of a quotient over a non-pathological analytic P-ideal are induced by almost permutations of the integers, extending Shelah'seminal result for $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ that started all this ([**139**]).

We also investigate which quotients over analytic P-ideals are homogeneous under $\mathrm{OCA_T}$ and MA($\sigma$-linked), and prove results suggesting that $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ may be the only one (§7.5). This stands in sharp contrast to the fact that under CH all quotients over $F_\sigma$-ideals that include Fin, all quotients over EU-ideals, and all quotients over LV-ideals are homogeneous (Corollary 11.1.11, Theorem 11.2.3, and Corollary 11.2.7). More consequences of the $\mathrm{OCA_T}$ lifting theorem and its applications to topology can be found in Chapter 9.

## 7.1. Rigidity gained

After all this hard work, it is time to list some easy corollaries.

**Theorem 7.1.1.** *Assume* $\mathrm{OCA_T}$ *and* MA($\sigma$*-linked*). *Suppose that* $\mathcal{I}$ *is a countably 80-determined ideal with the Fubini Property. If* $\mathcal{I}'$ *is an analytic ideal and* $\Phi\colon \mathcal{P}(\mathbb{N})/\mathcal{I}'\to\mathcal{P}(\mathbb{N})/\mathcal{I}$ *is an isomorphism, then it has a completely additive lifting. In particular,* $\mathcal{I}$ *and* $\mathcal{I}'$ *are Rudin–Keisler isomorphic.*

Proof. Suppose that $\Phi\colon \mathcal{P}(\mathbb{N})/\mathcal{I}' \to \mathcal{P}(\mathbb{N})/\mathcal{I}$ is an isomorphism between analytic quotients. Therefore Theorem 6.1.2 implies that $\Phi$ has a continuous lifting on an ideal that intersects every perfect tree-like almost disjoint family. Such ideal is by Theorem 3.2.2 (and Definition 3.3.6) nonmeagre. Since the restriction of an analytic ideal to a positive set $A$ is a relatively meagre subset of $\mathcal{P}(A)$ and $\Phi$ is an isomorphism, this implies that $\Phi$ has a continuous lifting. By the Fubini property and Theorem 4.3.1, $\Phi$ has a completely additive lifting. □

**Corollary 7.1.2.** $\mathrm{OCA_T}$ *and* $\mathrm{MA}(\sigma$*-linked*$)$ *impliy that* $\mathrm{Fin}$, $\mathrm{Fin}\times\emptyset$, $\emptyset\times\mathrm{Fin}$, $\mathcal{I}_{1/n}$, $\mathcal{I}_{1/\sqrt{n}}$, $\mathcal{Z}_0$ *and* $\mathcal{Z}_{\log}$ *have nonisomorphic quotients.*

The result that $\mathcal{Z}_0$ and $\mathcal{Z}_{\log}$ have consistently nonisomorphic quotients answering a question of Erdös and Ulam, was proved by Just ([92]). The result that $\mathcal{I}_{1/n}$ and $\mathcal{I}_{1/\sqrt{n}}$ have consistently nonisomorphic quotients, answerin a question of Koppelberg ([110]), is taken from [40].

Proof. By Theorem 7.1.1 it suffices to prove that these ideals are pairwise not Rudin–Keisler isomorphic. Neither Fin not $\mathrm{Fin}\times\emptyset$ can be isomorphic to any of the other ideals, since Fin is singly generated, $\mathrm{Fin}\times\emptyset$ is countably generated, and the others are not because every countably generated P-ideal is generated by a single set over Fin. The ideal $\emptyset\times\mathrm{Fin}$ is the only one of the remaining ideals whose orthogonal is isomorphic to $\mathrm{Fin}\times\emptyset$. By Proposition 2.8.4, no dense summable ideal is isomorphic to a density ideal, therefore neither of $\mathcal{I}_{1/n}$ and $\mathcal{I}_{1/\sqrt{n}}$ is isomorphic to either one of $\mathcal{Z}_0$ or $\mathcal{Z}_{\log}$. (This can also be proved by observing that every summable ideal is $F_\sigma$, while nontrivial density ideals are complete $F_{\sigma\delta}$ subsets of $\mathcal{P}(\mathbb{N})$.)

By the Radon–Nikodym property of $\mathcal{I}_{1/n}$, $\mathcal{I}_{1/\sqrt{n}}$, $\mathcal{Z}_0$, and $\mathcal{Z}_{\log}$ (Theorem 4.1.2), it suffices to prove that these ideals are pairwise RK-nonisomorphic. For $\mathcal{Z}_{\log}$ and $\mathcal{Z}_0$ this is Corollary 2.7.17 and for $\mathcal{I}_{1/n}$ and $\mathcal{I}_{1/\sqrt{n}}$ it is Corollary 2.6.7 . □

By Proposition 5.2.5, quotients over ideals nwd and null are pairwise nonisomorphic (and nonisomorphic to quotients over all analytic P-ideals) even in ZFC.

As pointed out earlier, the question of triviality of automorphisms of $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ was the starting point of the line of study of analytic quotients to which this work belongs. An automorphism of $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is called *trivial* if it has a completely additive lifting $\Phi_h$ for an injective function $h$ from a subset of $\mathbb{N}$ into $\mathbb{N}$. Theorem 7.1.1 implies the following (see §2.9 for the RK-automorphism group $\mathrm{Aut_{RK}}(\mathcal{I})$ of an ideal $\mathcal{I}$).

**Corollary 7.1.3.** $\mathrm{OCA_T}$ *and* $\mathrm{MA}(\sigma$*-linked*$)$ *imply that if* $\mathcal{I}$ *is a countably 80-determined ideal with the Fubini property, then all automorphisms of* $\mathcal{P}(\mathbb{N})/\mathcal{I}$ *are trivial. In other words, the natural embedding* $\mathrm{Aut_{RK}}(\mathcal{I})$ *into* $\mathrm{Aut}(\mathcal{P}(\mathbb{N})/\mathcal{I})$ *is surjective.* □

### 7.2. Embeddability of analytic quotients under $\mathrm{OCA_T}$ and $\mathrm{MA}(\sigma$-linked$)$

More easy corollaries ahead.

**Definition 7.2.1.** For ideals $\mathcal{I}$ and $\mathcal{J}$ on $\mathbb{N}$ we write $\mathcal{I}\leq_{\mathrm{EM}}\mathcal{J}$ if $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is embeddable into $\mathcal{P}(\mathbb{N})/\mathcal{J}$, or $\mathcal{P}(\mathbb{N})/\mathcal{I}\hookrightarrow\mathcal{P}(\mathbb{N})/\mathcal{J}$ in symbols.

Unlike preorders defined in §1.4, $\leq_{\mathrm{EM}}$ is sensitive to the choice of additional set-theoretic axioms (compare Corollary 11.1.9 to the ensuing results). Moreover,

$\mathcal{I} \leq^+_{\mathrm{BE}} \mathcal{J}$ (see §1.4) implies $\mathcal{I} \leq_{\mathrm{EM}} \mathcal{J}$ (Proposition 2.3.4), and therefore Example 5.4.1 shows that $\mathcal{I} \leq_{\mathrm{EM}} \mathcal{J}$ is in general not equivalent to $\mathcal{I} \leq_{\mathrm{BE}} \mathcal{J}$, provably in ZFC. In particular, every quotient over a proper summable ideal $\mathcal{I}_f$ is embeddable into every quotient over a proper dense summable ideal $\mathcal{I}_g$, because Lemma 2.6.9 implies $\mathrm{Fin} \oplus \mathcal{I}_f \leq_{\mathrm{RB}} \mathcal{I}_g$. However, if the conclusions of Theorem 4.1.2 and Theorem 6.1.2 apply to $\mathcal{I}$ then $\mathrm{OCA_T}$ and MA($\sigma$-linked) together imply that $\mathcal{J} \leq_{\mathrm{EM}} \mathcal{I}$ if and only if $\mathcal{J} \leq^+_{\mathrm{BE}} \mathcal{I}$. The first result of this form is due to Just ([**93**], [**92**]), and it says that $\mathrm{OCA_T}$ and MA, together imply that if $\mathcal{I}$ is an analytic ideal then $\mathcal{I} \leq_{\mathrm{EM}} \mathrm{Fin}$ if and only if $\mathcal{I}$ is generated by a single set over Fin. ($\mathrm{OCA_T}$ alone suffices for this conclusion by [**155**].)

**Corollary 7.2.2.** *Assume* $\mathrm{OCA_T}$ *and* MA($\sigma$-*linked*).

*(1) If $\mathcal{I}$ is an analytic ideal and $\mathcal{J}$ is countably 80-determined, then $\mathcal{I} \leq_{\mathrm{EM}} \mathcal{J}$ if and only if $\mathcal{I} \leq^+_{\mathrm{BE}} \mathcal{J}$.*

*(2) If $\mathcal{J}$ is countably 80-determined and $\mathcal{I}$ is an arbitrary ideal on $\mathbb{N}$ such that $\mathcal{P}(\mathbb{N})/\mathcal{I} \hookrightarrow \mathcal{P}(\mathbb{N})/\mathcal{J}$, then $\mathcal{I}$ is of the form $\mathcal{I}_0$ or $\mathcal{I}_0 \oplus \mathcal{I}_1$ for some Borel ideal $\mathcal{I}_0$ such that $\mathcal{I}_0 \leq^+_{\mathrm{BE}} \mathcal{J}$ and an ideal $\mathcal{I}_1$ which is ccc over* Fin.

Proof. (1) Proposition 5.4.2 implies that $\mathcal{I} \leq^+_{\mathrm{BE}} \mathcal{J}$ implies $\mathcal{I} \leq_{\mathrm{EM}} \mathcal{J}$. To prove the converse, assume $\mathcal{I} \leq_{\mathrm{EM}} \mathcal{J}$, and let $\Phi\colon \mathcal{P}(\mathbb{N})/\mathcal{I} \to \mathcal{P}(\mathbb{N})/\mathcal{J}$ be a monomorphism. By Theorem 7.1.1, $\Phi$ decomposes as $\Phi_1 \oplus \Phi_2$, where $\Phi_1$ has a Baire measurable lifting, $\Phi_{1*}$. The set $A = \Phi_{1*}(\mathbb{N})$ is not in $\mathcal{J}$ because $\ker(\Phi) \neq \ker(\Phi_2)$. We claim that $\ker(\Phi) = \ker(\Phi_1)$. Otherwise, since $\ker(\Phi) \subseteq \ker(\Phi_1)$, we have $B \in \ker(\Phi) \setminus \ker(\Phi_2)$. Therefore the ideal $\mathcal{I} \restriction B$ is ccc over Fin, which is impossible because it is analytic. So we have $\mathcal{I} = \ker(\Phi) = \ker(\Phi_1)$ and $\mathcal{I} \leq_{\mathrm{BE}} \mathcal{J} \restriction A$, as required.

(2) is an easy consequence of (1). □

Recall that $\mathcal{I} \leq_{\mathrm{EM}} \mathcal{J}$ if $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is embeddable into $\mathcal{P}(\mathbb{N})/\mathcal{J}$ (Definition 7.2.1) and that $\mathcal{I} \leq_{\mathrm{BE}} \mathcal{J}$ if $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is embeddable into $\mathcal{P}(\mathbb{N})'\mathcal{J}$ via a homomorphism that has a Baire measurable lifting (Definition 2.3.2 ).

**Corollary 7.2.3.** *There are summable ideals $\mathcal{I}_f$ and $\mathcal{I}_g$ such that $\mathcal{I}_f \leq_{\mathrm{EM}} \mathcal{I}_g$ and $\mathcal{I} \not\leq_{\mathrm{BE}} \mathcal{J}$.*

Proof. By Corollary 2.6.13, there are $\mathcal{I}_f$ and $\mathcal{I}_g$ such that $\mathcal{I}_f \not\leq_{\mathrm{RB}} \mathcal{I}_g$ but $\mathcal{I}_f \leq_{\mathrm{RB}} \mathcal{I}_g \restriction A$ for some $\mathcal{I}_g$-positive set $A$. The former relation implies $\mathcal{I}_f \not\leq_{\mathrm{BE}} \mathcal{I}_g$, while the latter immediately implies $\mathcal{I}_f \leq^+_{\mathrm{BE}} \mathcal{I}_g$ (Definition 2.3.2), which in turn implies $\mathcal{I}_f \leq_{\mathrm{EM}} \mathcal{I}_g$ by Proposition 5.4.2. □

The orderings $\leq_{\mathrm{EM}}$ and $\leq_{\mathrm{BE}}$ behave similarly on some ideals close to Fin. For example, since $\mathrm{Fin} \restriction A$ is isomorphic to Fin for every positive set $A$, $\mathcal{I} \leq_{\mathrm{RB}} \mathrm{Fin}$ if and only if $\mathcal{I} \leq^+_{\mathrm{BE}} \mathrm{Fin}$. Similarly, $\mathcal{I} \leq_{\mathrm{RB}} \mathrm{Fin} \times \emptyset$ if and only if $\mathcal{I} \leq^+_{\mathrm{BE}} \mathrm{Fin} \times \emptyset$ and $\mathcal{I} \leq_{\mathrm{RB}} \emptyset \times \mathrm{Fin}$ if and only if $\mathcal{I} \leq^+_{\mathrm{BE}} \emptyset \times \mathrm{Fin}$.

The *orthogonal* of an ideal $\mathcal{I}$ is defined as

$$\mathcal{I}^\perp = \{B : B \text{ is almost disjoint from all } A \in \mathcal{I}\}.$$

**Definition 7.2.4.** Two families $\mathcal{A}$, $\mathbb{B}$ of subsets of $\mathbb{N}$ are *orthogonal* if $A \cap B$ is finite for all $A \in \mathcal{A}$ and all $B \in \mathcal{B}$. They are *countably separated* if there are $C_n$ ($n \in \mathbb{N}$) such that for every pair $A \in \mathcal{A}$, $B \in \mathbb{B}$ some $n$ satisfies $A \subseteq^* C_n$ and $C_n \cap B$ is finite. They are *separated* (or they do not form a *gap*) if there is $C \subseteq \mathbb{N}$ such that $A \subseteq^* C$ almost includes for all $A \in \mathcal{A}$ and $C \cap B$ is finite for all $B \in \mathcal{B}$.

The following result was suggested by Todorcevic and it is related to his results about analytic gaps ([153]).

**Corollary 7.2.5.** *Assume* $\mathrm{OCA_T}$ *and* MA($\sigma$-*linked*).

(1) *If* $\mathcal{I}$ *is an ideal on* $\mathbb{N}$ *such that* $\mathcal{P}(\mathbb{N})/\mathcal{I} \hookrightarrow \mathcal{P}(\mathbb{N})/\operatorname{Fin}$, *then* $\mathcal{I}$ *can be separated from its orthogonal.*

(2) *If* $\mathcal{I}$ *is an arbitrary ideal on the integers such that* $\mathcal{P}(\mathbb{N})/\mathcal{I} \hookrightarrow \mathcal{P}(\mathbb{N})/\mathcal{J}$ *for* $\mathcal{J}$ *an analytic P-ideal or* $\operatorname{Fin}\times\emptyset$, *then* $\mathcal{I}$ *can be countably separated from its orthogonal.*

(3) *There is an ideal* $\mathcal{I}$ *such that* $\mathcal{P}(\mathbb{N})/\mathcal{I}$ *is not embeddable into any quotient over an analytic P-ideal or* $\operatorname{Fin}\times\emptyset$.

Proof. For the first part, let $\Phi\colon \mathcal{P}(\mathbb{N})/\mathcal{I} \to \mathcal{P}(\mathbb{N})/\operatorname{Fin}$ be an isomorphic embedding. By Theorem 7.1.1 and the Radon–Nikodym property of Fin, there is a finite-to-one function $h\colon \mathbb{N}\to\mathbb{N}$ such that $\Phi_h$ is a lifting of $\Phi$ on a nonmeagre ideal $\mathcal{I}_\Phi$. Every nonmeagre ideal is dense, so $\mathcal{I}^\perp = (\mathcal{I}\cap\mathcal{I}_\Phi)^\perp$. Now note that $(\mathcal{I}\cap\mathcal{I}_\Phi)^\perp$ is an ideal generated by $\mathbb{N}\setminus h''\mathbb{N}$ and Fin, and therefore the set $h''\mathbb{N}$ separates $\mathcal{I}$ from $\mathcal{I}^\perp$.

For the second part, Corollary 7.2.2 implies $\mathcal{I} = \mathcal{I}_0 \oplus \mathcal{I}_1$ where $\mathcal{I}_1$ is nonmeagre, and therefore dense, and $\mathcal{I}_1$ is an analytic P-ideal or $\operatorname{Fin}\times\emptyset$. Therefore by Lemma 2.4.2, $\mathcal{I}$ is countably separated from $\mathcal{I}^\perp$.

Fix an $\aleph_1,\aleph_1$-gap in $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ (see e.g., [54, Theorem 9.3.1]), i.e., two $\subseteq^*$-increasing $\aleph_1$-chains of sets of integers which cannot be separated by a single subset of $\mathbb{N}$. Note that this implies these two chains cannot be countably separated. Let $\mathcal{I}$ be an ideal generated by one of these two chains. Then $\mathcal{I}$ cannot be countably separated from its orthogonal, and therefore, Corollary 7.2.5 implies that $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is not embeddable into any quotient over an analytic P-ideal. □

Another application of Corollary 7.2.2 is its topological reformulation. All results about the structure of homomorphisms of quotients $\mathcal{P}(\mathbb{N})/\mathcal{I}$ have their topological duals which talk about the continuous maps of closed subsets of the Čech–Stone remainder $\mathbb{N}^*$ of the integers (equivalently, the Stone space of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$; see [127]). Recall that $A\subseteq\mathbb{N}^*$ is a *P-set* if for every sequence of open neighborhoods $\{U_n\}$ of $A$ set $\bigcap_n U_n$ includes an open neighborhood of $A$. In the case when $A$ is closed, this reduces to fact that the ideal on $\mathbb{N}$ corresponding to $A$ is a P-ideal. Just ([90], [94]) proved that $\mathrm{OCA_T}$ and MA together imply that no nowhere dense P-subset of $\mathbb{N}^*$ is homeomorphic to $\mathbb{N}^*$ itself, answering a question of van Mill ([127, p. 537]). We shall now give a slight strengthening of this result. Note that Example 5.4.3 shows that the assumption of $A$ being a P-set cannot be dropped from the following Corollary.

**Corollary 7.2.6.** *Assume* $\mathrm{OCA_T}$ *and* MA($\sigma$-*linked*).

(1) *If a subset of* $\mathbb{N}^*$ *is a continuous image of* $\mathbb{N}^*$, *then it is equal to the disjoint union of a clopen set and a nowhere dense set.*

(2) *Every P-subset of* $\mathbb{N}^*$ *homeomorphic to* $\mathbb{N}^*$ *must be clopen.*

Proof. The first part the topological dual of Corollary 7.2.2 (see [127]).

We now prove the dual of the second part: If $\mathcal{I}$ is a P-ideal and $\mathcal{P}(\mathbb{N})/\mathcal{I}$ embeds into $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$, then $\mathcal{I} = \operatorname{Fin}\oplus\mathcal{I}_1$ so that the quotient $\mathcal{P}(\mathbb{N})/\mathcal{I}_1$ has ccc. By Corollary 7.2.2, $\mathcal{I} = \mathcal{I}_0\oplus\mathcal{I}_1$ where $\mathcal{I}_0 \leq_{\mathrm{BE}} \operatorname{Fin}$ and $\mathcal{I}_1$ is ccc over Fin, so $\mathcal{I}_0$ is isomorphic to Fin. Since a P-ideal is ccc over Fin if and only if the corresponding quotient is ccc (see e.g., [90]), this ends the proof. □

Additional applications to the strucure of Čech–Stone remainders can be found in Chapter 9.

## 7.3. Permanence properties

The following should be contrasted with the fact that under CH all quotients over analytic ideals that include Fin are embeddable into one another (Proposition 11.1.9).

**Corollary 7.3.1.** *Assume* $\mathrm{OCA_T}$ *and* $\mathrm{MA}(\sigma\text{-linked})$ *and let* $\mathcal{I}$ *and* $\mathcal{J}$ *be analytic ideals such that* $\mathcal{P}(\mathbb{N})/\mathcal{I}$ *embeds into* $\mathcal{P}(\mathbb{N})/\mathcal{J}$.

*(1) If* $\mathcal{J}$ *is* $F_\sigma$*, then so is* $\mathcal{J}$.
*(2) If* $\mathcal{I}$ *is summable, then so is* $\mathcal{J}$.
*(3) If* $\mathcal{J}$ *has the Fubini property, then so does* $\mathcal{I}$.
*(4) If* $\mathcal{J}$ *is a non-pathological analytic P-ideal, then so is* $\mathcal{I}$.
*(5) If* $\mathcal{J}$ *is an analytic P-ideal, then so is* $\mathcal{I}$.

Proof. In each case, $\mathcal{J}$ is countably 80-determined and therefore by Corollary 7.2.2 (1) we have $\mathcal{I} \leq_{\mathrm{BA}} \mathcal{J} \restriction A$ for some $\mathcal{J}$-positive set $A$.

(1) If $\mathcal{J}$ is $F_\sigma$ then so are $\mathcal{J} \restriction A$ and its continuous preimage $\mathcal{I}$.

(3) If $\mathcal{J}$ has the Fubini property then so does $\mathcal{J} \restriction A$ (Theorem 4.2.3), and the conclusion follows by Theorem 4.5.2.

(2) Since $\mathcal{J} \restriction A$ is summable, by Theorem 4.1.2 it has the Radon–Nikodym property. Thus $\mathcal{I} \leq_{\mathrm{BE}} \mathcal{J} \restriction A$ implies $\mathcal{I} \leq_{\mathrm{RB}} \mathcal{J} \restriction A$. Let $f$ be a function such that $\mathcal{J} \restriction A$ is $\mathcal{I}_f$. If $h$ is a finite-to-one function such that $B \in \mathcal{I}$ if and only if $h^{-1}(A) \in \mathcal{I}_f$, then $g(n) = \sum_{h(i)=n} f(i)$ satisfies $\mathcal{I} = \mathcal{I}_g$.

(4) Suppose $\mathcal{J}$ is a non-pathological analytic P-ideal. Then so is $\mathcal{J} \restriction A$, and by Theorem 4.1.2 it has the Radon–Nikodym property, and therefore $\mathcal{I} \leq_{\mathrm{RB}} \mathcal{J} \restriction A$. Fix a finite-to-one $h$ that implements ths reduction. Let $\varphi$ be a lower semicontinuous non-pathological submeasure such that $\mathcal{J} \restriction A = \mathrm{Exh}(\varphi)$. Then the pull-back submeasure $\psi$ defined by $\psi(s) = \varphi(h^{-1}(s))$ is lower semicontinuous (because $h$ is finite-to-one). It is non-pathological, because pullbacks of measures dominated by $\varphi$ are measures dominated by $\psi$.

(5) By [40, Theorem 1.9.1], the embedding of $\mathcal{P}(\mathbb{N})/\mathcal{I}$ into $\mathcal{P}(\mathbb{N})/\mathcal{J}$ has an asymptotically additive lifting ([40, Definition 1.5.1]). [1] This easily implies that $\mathcal{I}$ is an analytic P-ideal. □

Part (5) was first published in [155, Theorem 8].

## 7.4. Automorphism groups

The group $\mathrm{Aut_{RK}}(\mathcal{I})$ of Rudin–Keisler automorphisms of an ideal $\mathcal{I}$ was defined in §2.9, and by Corollary 7.1.3 $\mathrm{OCA_T}$ and $\mathrm{MA}(\sigma\text{-linked})$ imply that under our standard assumptions its natural embedding into $\mathrm{Aut}(\mathcal{P}(\mathbb{N})/\mathcal{I})$ is surjective. The following is immediate from Theorem 2.9.5 (see the paragraph preceding it for the notation).

[1] This notion was central in the old proof of the OCA Lifting Theorem and it is not used in the present approach. For more on it see [59, §5.2].

**Theorem 7.4.1.** *Assume* $\mathrm{OCA_T}$ *and* $\mathrm{MA}(\sigma$*-linked*$)$. *If* $\mathcal{I}_f$ *is a summable ideal not RK-isomorphjic to* Fin, *then* $\mathrm{Aut}(\mathcal{P}(\mathbb{N})/\mathcal{I}_f)$ *is isomorphic to the quotient* $G_f/H_f$. □

Boolean algebras such that every surjective endomorphism is an isomorphism are called *Hopfian* (see [26]). The following is an immediate consequence of Theorem 2.9.6.

**Theorem 7.4.2.** *Assume* $\mathrm{OCA_T}$ *and* $\mathrm{MA}(\sigma$*-linked*$)$. *There is a summable ideal* $\mathcal{I}_f$ *with the following properties.*

*(1)* $\mathrm{Aut}(\mathcal{P}(\mathbb{N})/\mathcal{I}_f)$ *is isomorphic to a quotient of the group* $\prod_{n=1}^{\infty} S_{n!}$.
*(2) Every surjective homomorphism* $\Phi\colon \mathcal{P}(\mathbb{N})/\mathcal{I}_f \to \mathcal{P}(\mathbb{N})/\mathcal{I}_f$ *is automatically an isomorphism.*

*In particular,* $\mathcal{P}(\mathbb{N})/\mathcal{I}_f$ *is a Hopfian Boolean algebra.* □

CH implies that no quotient over an $F_\sigma$ ideal that includes Fin is Hopfian, because $\mathcal{P}(\mathbb{N})/\,\mathrm{Fin}$ is not Hopfian and all quotients over $F_\sigma$ ideals are isomorphic under CH (Corollary 11.1.10).

Not all quotients over dense summable ideals are Hopfian,

**Proposition 7.4.3.** *The quotient* $\mathcal{P}(\mathbb{N})/\mathcal{I}_{1/n}$ *is not Hopfian (provably in* ZFC*).*

Proof. Consider the ideal $\mathcal{J}$ generated by $\mathcal{I}_{1/n}$ and the even numbers. Then $\mathcal{J} \supsetneq \mathcal{I}_{1/n}$ and $n \mapsto 2n$ is an RK-isomorphism between the ideals. It corresponds to a surjective endomorphism of $\mathcal{P}(\mathbb{N})/\mathcal{I}_{1/n}$ whose kernel is equal to $\mathcal{J}$. □

A Boolean algebra $\mathbb{B}$ is called *dual Hopfian* (see [26]) if every monomorphism $\Phi\colon \mathcal{P}(\mathbb{N})/\mathcal{I}_f \to \mathcal{P}(\mathbb{N})/\mathcal{I}_f$ is automatically an automorphism. So the following statement (which should be compared to Proposition 2.6.6).

**Proposition 7.4.4.** *No quotient over a summable ideal is dual Hopfian, provably in* ZFC.

Proof. Since this is true for Fin it suffices to prove it for dense summable ideals $\mathcal{I}_f$. We may therefore assume $f$ is nonincreasing. Let $n_i \geq i$, for $i \in \mathbb{N}$, be such that $n_i < n_{i+1}$ and all $i \geq 0$ satisfy $2f(i) \leq \mu_f([n_i, n_{i+1})) \leq 3f(i)$. This is possible because $n_i \geq i$ implies $f(n_i) \leq f(i)$ for all $i$. This inequality also implies $n_{i+1} - n_i \geq 2$ for all $i$. Fix $h\colon \mathbb{N} \to \mathbb{N}$ that collapses $[n_i, n_{i+1})$ to $i$ for all $i$. Then $A \mapsto h^{-1}(A)$ is a lifting of a monomorphism $\Phi\colon \mathcal{P}(\mathbb{N})/\mathcal{I}_f \to \mathcal{P}(\mathbb{N})/\mathcal{I}_f$. We claim that this is not an isomorphism. Otherwise, for some $A \in \mathcal{I}_f^*$ the restriction of $h$ to $A$ is injective; contradiction. □

## 7.5. Homogeneity of quotients

Boolean algebra $\mathbb{B}$ is *homogeneous* if for all $a, b \in \mathbb{B}$ distinct from $0_\mathbb{B}$ and $1_\mathbb{B}$ there is an automorphism of $\mathbb{B}$ sending $a$ to $b$. It is *weakly homogeneous* if for all $a, b \in \mathbb{B}$ distinct from $0_\mathbb{B}$ there is an automorphism $\Phi$ of $\mathbb{B}$ such that $\Phi(a) \cap b \neq 0_\mathbb{B}$.

Clearly if an ideal is (weakly) RK-homogeneous (§2.5) then so is its quotient. Assuming CH, the quotient over every layered ideal (Definition 11.1.1) such as any $F_\sigma$ ideal, $\mathcal{O}_\alpha$, or $\mathcal{W}_\alpha$ is isomorphic to $\mathcal{P}(\mathbb{N})/\,\mathrm{Fin}$ (Corollary 11.1.10) and therefore homogeneous. Also, CH implies that quotients over EU-ideals (Theorem 11.2.3) and dense LV-ideals (Theorem 11.2.6) are homogeneous.

**Proposition 7.5.1.** *Assume* $\mathrm{OCA_T}$ *and MA. If* $\mathcal{I}_f$ *is a summable ideal other than* Fin*, then the quotient* $\mathcal{P}(\mathbb{N})/\mathcal{I}$ *is not weakly homogeneous.*

Proof. Since $\mathcal{I}_f$ is not isomorphic to Fin, there is a positive set $C$ such that $\mathcal{I}_f \restriction C$ is a dense summable ideal and we may assume $\mathcal{I}_f$ has this property. Let $s_i$ be disjoint finite sets of integers such that for all $i < k$ we have

$$1 \leq \mu_f(s_i) \leq 2 \ \ \& \ \min_{j \in s_i} f(j) > i \cdot \max_{j \in s_k} f(j).$$

Let $A = \bigcup_i s_{2i}$ and $B = \bigcup_i s_{2i+1}$. Then Lemma 2.6.8 implies that $\mathcal{I}_f \restriction A'$ and $\mathcal{I}_f \restriction B'$ are not RK-isomorphic whenever $A' \subseteq A$ and $B' \subseteq B$ are $\mathcal{I}_f$-positive sets. By Theorem 6.1.2 and the Radon–Nikodym property of summable ideals (Theorem 4.1.2), the corresponding quotients are not isomorphic either. □

**Proposition 7.5.2.** *Assume* $\mathrm{OCA_T}$ *and MA. If* $\mathcal{Z}_\mu$ *is a density ideal not isomorphic to* Fin *or to* $\emptyset \times \mathrm{Fin}$*, then the algebra* $\mathcal{P}(\mathbb{N})/\mathcal{Z}_\mu$ *is not weakly homogeneous.*

Proof. The proof is very similar to that of Proposition 7.5.1. Since Theorem 6.1.2 and Theorem 4.1.2 apply to density ideals it suffices to prove that for every $\mathcal{Z}_\mu$-positive set has positive subsets with the property that the restrictions of $\mathcal{Z}_\mu$ to them are not RK-isomorphic. This follows from Theorem 2.7.16. □

There is an analytic P-ideal different from Fin whose quotient is weakly homogeneous; the ideal $\emptyset \times \mathrm{Fin}$ verifies this assertion. One can say more (recall that an ideal $\mathcal{I}$ has the *Fréchet property* if $(\mathcal{I}^\perp)^\perp = \mathcal{I}$).

**Lemma 7.5.3.** *If* $\mathcal{I}$ *has the Fréchet property, then its quotient is weakly homogeneous.*

Proof. If $\mathcal{I}$ has the Fréchet property, then every positive set $A$ has a subset $B$ such that $\mathcal{I} \restriction B$ is isomorphic to Fin. Therefore if $C, D$ are $\mathcal{I}$-positive sets, then we can easily define a bijection $h\colon \mathbb{N} \to \mathbb{N}$ (we can assume that $\mathcal{I} \neq \mathrm{Fin}$) which sends $\mathcal{I}$ into itself and such that $h''C \cap D$ is infinite. Then $\Phi_h$ is an automorphism of the quotient algebra as required. □

We do not know whether there are analytic P-ideals other than Fin and $\emptyset \times \mathrm{Fin}$ whose quotients are weakly homogeneous (the second part of Corollary 2.4.3 suggests that a variation of our proof of Proposition 7.5.1 may apply to give a negative answer).

**Proposition 7.5.4.** *Assume* $\mathrm{OCA_T}$ *and* MA($\sigma$*-linked*)*. If* $\mathcal{I}$ *is a non-pathological analytic P-ideal different from* Fin*, then the algebra* $\mathcal{P}(\mathbb{N})/\mathcal{I}$ *is not homogeneous.*

Proof. The restriction of $\mathcal{I}$ to a positive set is a non-pathological analytic P-ideal, hence Theorem 6.1.2 and Theorem 4.1.2 together imply that $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is homogeneous if and only if $\mathcal{I}$ is RK-homogeneous. By Proposition 2.5.2, this is not the case. □

Since $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is homogeneous, there is an isomorphism $\Phi\colon \mathcal{P}(B)/\mathcal{I} \to \mathcal{P}(A)/\mathcal{I}$. By Theorem 7.1.1, there is a bijection $h\colon A \to B$ which defines an isomorphism between $\mathcal{I} \restriction B$ and $\mathcal{I} \restriction A$. (Recall that, although in general we cannot assume that $h$ is a bijection, in this case we can since both $\mathcal{I} \restriction A$ and $\mathcal{I} \restriction B$ are dense.)

CHAPTER 8

# Dimension phenomena for Čech–Stone remainders

This chapter is based on [**45**], [**47**], [**48**], and [**56**]. The first three papers were written shortly after the completion of [**40**], and the last one is a thoroughly revised version of the first that was intended for inclusion in this text rather than independent publication.

The notion of $\beta\mathbb{N}$-space due to van Douwen ([**25**]) is introduced in §8.1. These spaces include the Čech–Stone remainder $X^*$ of every non-compact subspace $X$ of a Polish space (Lemma 8.1.2). The main result of this chapter is that if $f$ a continuous function from a product of compact Hausdorff spaces into a $\beta\mathbb{N}$-space then its domain can be partitioned into finitely many clopen pieces such that the restriction of $f$ to each one of them depends on at most one coordinate (Theorem 8.1.3). The reduction from an arbitrary product into a finite one is Theorem 8.2.1. The finite case uses a finitary combinatorial result (Theorem 8.3.1) on dependence of functions from products on their variables. Proposition 8.3.6, not used elsewhere in this text, shows that the conclusion of Theorem 8.3.1 can fail in choiceless models (more precisely, if there is a Dedekind-finite, infinite set). In §8.4 a topological result of van Douwen used in the proof of Theorem 8.1.3 is reproduced, and the following two sections contain additional lemmas towards the proof of Theorem 8.1.3. In Theorem 8.7.1 we prove that a compact Hausdorff space $X$ maps onto $X^2$ if and only if it maps onto $X^{\mathbb{N}}$. This is applied to powers of the Čech–Stone remainder $\mathbb{N}^*$ of $\mathbb{N}$ in §8.8.

## 8.1. $\beta\mathbb{N}$-spaces and prime mappings

For a topological space $Z$ its Čech–Stone compactification is denoted $\beta Z$. The following definition was introduced by van Douwen in [**25**].

**Definition 8.1.1.** A Hausdorff topological space $Z$ is a $\beta\mathbb{N}$-*space* if for every countably infinite relatively discrete subset $D$ of $Z$ with compact closure, this closure is homeomorphic to $\beta D$.

The following is [**25**, Lemma 4.1] (recall that a space is *Lindelöf* if every open cover has a countable subcover).

**Lemma 8.1.2.** *If $X$ is a regular Lindelöf space then $\beta X \setminus X$ is a nontrivial $\beta\mathbb{N}$-space. In particular, if $X$ is a non-compact subspace of a Polish space (or any second countable space) then $\beta X$ is a $\beta\mathbb{N}$-space.*

Proof. Let $D \subseteq \beta X \setminus X$ be a countably infinite discrete subset with compact closure. In the subspace $D \cup X$ of $\beta X$, $D$ is then a closed subspace. Since $X$ is Lindelöf and $D$ is countable, $D \cup X$ is Lindelöf. Being regular and Lindelöf, $D \cup X$ is normal. Therefore Tietze extension theorem shows that every bounded continuous real-valued function on $D$ has a continuous extension to $D \cup X$. Therefore $\beta D$ is a subspace of $\beta(D \cup X) = \beta X$. Since $D$ was arbitrary, this implies that $X$ is a $\beta\mathbb{N}$-space. Since $X$ is not compact, $\beta X \setminus X$ is nontrivial.

As every second countable space is Lindelöf, the second claim follows. $\square$

**Theorem 8.1.3.** *Assume $Z$ is a $\beta\mathbb{N}$-space, $\mathbb{I}$ is an index set, $X_i$ for $i \in \mathbb{I}$ is compact, and $f\colon \prod_i X_i \to Z$ is continuous. Then $\prod_i X_i$ can be covered by finitely many clopen rectangles such that $f$ depends on at most one coordinate on each one of them.*

Corollary 8.1.5 below is an immediate consequence of Theorem 8.1.3 and it confirms [**25**, Conjecture 8.4] (reproduced as [**80**, Question 42]).

**Definition 8.1.4.** For $m, n \geq 1$ and topological spaces $X_i$ for $i < m$ and $Y_j$, for $j < n$, a continuous function $\Phi\colon \prod_{i<m} X_i \to \prod_{j<n} Y_j$ is *elementary* if there is an injection $\alpha\colon n \to m$ and continuous functions $\Phi_j\colon X_{\alpha(j)}\colon X_{\alpha(j)} \to Y_j$, for $j < n$, such that $\Phi(\bar{x})(j) = \Phi_j(x_{\alpha(j)})$ for all $j < n$. It is *piecewise elementary* if its domain can be partitioned into finitely many clopen pieces such that the restriction of $\Phi$ to each of the pieces is elementary.

**Corollary 8.1.5.** *Assume $m \geq 1$ and $X_i$, for $i < m$, are $\beta\mathbb{N}$-spaces.*

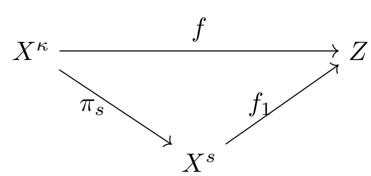
*(1) Every autohomeomorphism of $\prod_{i<m} X_i$ is piecewise elementary.*
*(2) If each $X_i$ is in addition connected, then every autohomeomorphism of $\prod_{i<m} X_i$ is elementary even for infinite $m$.* $\square$

The conclusion of Corollary 8.1.5 is true even when the Continuum Hypothesis holds, although then $Z$ possibly has many nontrivial autohomeomorphisms; compare with the situation under forcing axioms studied in §9.

## 8.2. Reduction to finitely many coordinates

Towards proving Theorem 8.1.3we first prove its weaker version. If $s \subseteq \mathbb{I}$ then $\pi_s\colon \prod_{i\in\mathbb{I}} X_i \to \prod_{i\in s} X_i$ is the projection map.

**Theorem 8.2.1.** *If $Z$ is a $\beta\mathbb{N}$-space, $\mathbb{I}$ is an index set, $X_i$, for $i \in I$, are compact Hausdorff spaces, and $f\colon \prod_i X_i \to Z$, then there are a finite $s \subseteq \mathbb{I}$ and a continuous $f_1\colon \prod_{i\in s} X_i \to Z$ such that the diagram in Fig. 8.2.1 commutes.*




This theorem will be proved after some lemmas. The first one is [**25**, Fact on page 29].

**Lemma 8.2.2.** *If $Z$ is a compact $\beta\mathbb{N}$-space and $D, E$ are countably infinite disjoint subsets of $Z$, then there is an infinite $D_2 \subseteq D$ such that $\overline{D_2} \cap E = \emptyset$.*

Proof. First find an infinite relatively discrete $D_1 \subseteq D$. Let $E_1 = E \cap \overline{D_1}$. Since $\overline{D_1}$ is homeomorphic to $\beta\mathbb{N}$, the set $E_1$ is nowhere dense in $\overline{D_1}$, thus there is an infinite $D_2 \subseteq D$ such that $\overline{D_2} \cap E_1 = \overline{D_2} \cap E = \emptyset$, and $D_2$ is as required. $\square$

The following is [25, p. 28], where its case when $Z = \beta\mathbb{N}$ was attributed to Hušek.

**Lemma 8.2.3.** *Assume $Z$ is a compact $\beta\mathbb{N}$-space and $a_i$, $b_i$ ($i \in \mathbb{N}$) are elements of $Z$ such that $a_i \neq b_i$ for all $i$. Then there is an infinite $I \subseteq \mathbb{N}$ such that*

$$\overline{\{a_i : i \in I\}} \cap \overline{\{b_i : i \in I\}} = \emptyset.$$

Proof. For $C \subseteq \mathbb{N}$ let $A_C = \{a_i : i \in C\}$ and $B_C = \{b_i : i \in C\}$. First we obtain an infinite $C \subseteq \mathbb{N}$ such that $A_C \cap B_C = \emptyset$.

By $[\mathbb{N}]^2$ we denote the set of all unordered pairs of natural numbers, and by $\{i, j\}_<$ we denote a pair in $[\mathbb{N}]^2$ such that $i < j$. Define a partition $[\mathbb{N}]^2 = K_0 \cup K_1 \cup K_2$ by

$$\{i, j\}_< \in \begin{cases} K_0, & \text{if } a_i = b_j, \\ K_1, & \text{if } a_j = b_i, \\ K_2, & \text{if } a_i \neq b_j \text{ and } a_j \neq b_i. \end{cases}$$

If $\{i, j, k\}_<$ is a $K_0$-homogeneous triple, then $a_j = b_k = a_i = b_j$, a contradiction. Thus there are no $K_0$-homogeneous triples. One similarly proves that there are no $K_1$-homogeneous triples either. Thus there is an infinite $C \subseteq \mathbb{N}$ such that $\{i, j\}_< \in K_2$ (and thus $a_i \neq b_j$) for all $i, j \in C$, as required.

Next, we find an infinite $I \subseteq C$ such that $\overline{A_I} \cap B_I = \emptyset$ and $A_I \cap \overline{B_I} = \emptyset$. Apply Lemma 8.2.2 with $D = A_C$ and $E = B_C$ to get $C_1 \subseteq C$ so that $\overline{A_{C_1}} \cap B_C = \emptyset$. Applying Lemma 8.2.2 with $D = B_{C_1}$ and $E = A_{C_1}$ we obtain $I \subseteq C_1$ such that $A_{C_1} \cap \overline{B_I} = \emptyset$. Then $I$ is as required.

Finally, we prove that $\overline{A_I} \cap \overline{B_I} = \emptyset$.

The sets $A_I$ and $B_I$ are both relatively discrete. Moreover, by the choice of $I$ no point of $A_I$ is an accumulation point of $B_I$ and no point of $B_I$ is an accumulation point of $A_I$. Thus $A_I \cup B_I$ is relatively discrete, and since $Z$ is a compact $\beta\mathbb{N}$-space, the closures of $A_I$ and $B_I$ are disjoint.

This concludes the proof. □

Proof of Theorem 8.2.1. Fix a $\beta\mathbb{N}$-space $Z$, and index set $\mathbb{I}$, compact Hausdorff spaces $X_i$, for $i \in I$, is compact, and a continuous $f\colon \prod_i X_i \to Z$. We need to find a finite $s \subseteq \mathbb{I}$ such that $f$ depends only on coordinates in $s$. Since a closed subset of a $\beta\mathbb{N}$-space is a $\beta\mathbb{N}$-space, we may assume that $f$ is onto and therefore that $Z$ is compact. The proof proceeds by transfinite induction on the cardinality of $\mathbb{I}$ (for all $X_i$ and $Z$). The statement is vacuously true when $\mathbb{I}$ is finite. Assume that $\mathbb{I}$ is infinite and that the assertion is true for all index sets of smaller cardinality.

First consider the case when the cofinality of $|\mathbb{I}|$ is countable. Fix a strictly increasing sequence $\mathbb{I}(n)$, for $n \in \mathbb{N}$, of subsets of $\mathbb{I}$ such that $|\mathbb{I}(n)|| < |\mathbb{I}|$ for all $n$ and $\mathbb{I} = \bigcup_n \mathbb{I}(n)$. We will attempt to construct sequences $x_i$, $y_i$ ($i \in \mathbb{N}$) such that for all $i$ the following holds.

(1) $x_i, y_i \in \prod_{i \in \mathbb{I}(i)}$,
(2) $x_i \restriction \mathbb{I}(i) = y_i \restriction \mathbb{I}(i)$, i.e., $x_i(j) = y_i(j)$, for all $j \in \mathbb{I}(i)$, and
(3) $f(x_i) \neq f(y_i)$.

Assume that we have constructed $x_i, y_i$ for $i < n$ (possibly $n = 0$). If we cannot find $x_n$ and $y_n$ satisfying the above conditions, then define $f_0\colon \prod_{i \in \mathbb{I}(n)} X_i \to Z$ by

$$f_0(x) = f(x'),$$

where $x'$ is any element of $\prod_{i\in\mathbb{I}} X_i$ that satisfies $x' \restriction t = x$. By our assumptions, if $x'$ and $x''$ are such that $x' \restriction t = x'' \restriction t$ then $f(x') = f(x'')$. Therefore $f_0$ is well-defined; it is clearly continuous and satisfies $f = f_0 \circ \pi_t$. Since $|\mathbb{I}(n)| < |\mathbb{I}|$, by the inductive assumption there are a finite $s \subseteq \mathbb{I}(n)$ and $f_1\colon \prod_{i\in s} X_i \to Z$ such that $f_1 = f_0 \circ \pi_s^{\kappa_n}$, and $f = f_1 \circ \pi_s^{\kappa}$, as required.

Now assume that the recursive construction does not stop and there are $x_i, y_i$ ($i \in \mathbb{N}$) satisfying (1)–(3).

Back to the proof of Theorem. Apply Lemma 8.2.3 with $a_i = f(x_i)$ and $b_i = f(y_i)$ to obtain an infinite $\mathbb{J} \subseteq \mathbb{N}$ such that the sets $\{f(x_i) : i \in \mathbb{J}\}$ and $\{f(y_i) : i \in \mathbb{J}\}$ have disjoint closures.

Let $X = \{x_i : i \in \mathbb{J}\}$ is an accumulation point of $Y = \{y_i : i \in \mathbb{J}\}$. We claim that the set $X \cup Y$ is infinite. Assume otherwise, let $n$ be large enough to have $X \cup Y \subseteq \{x_i, y_i : i < n\}$. For $i, j$ in $\mathbb{N}$ let $m = m(i, j)$ be such that $x_i \restriction \mathbb{I}(m) \neq y_j \restriction \mathbb{I}(m)$ if such $m$ exists, and $n$ otherwise. Let $m = \max_{i,j<n} m(i, j)$. However for $i \geq m$ we have $m(x_i, y_i) > m$, contradiction.

Some $a \in \prod_{\mathbb{I}} X_i$ is an accumulation point of $X$ if and only if for every finite $t \subseteq \mathbb{I}$ there is $i$ such that $a \restriction s = x_i \restriction s$. Since every finite subset of $\mathbb{I}$ is included in $\mathbb{I}(n)$ for all large enough $n$, every accumulation point of $X$ is an accumulation point of $Y$ and vice versa.

By compactness of $\prod_{\mathbb{I}} X_i$ and the fact that $X \cup Y$ is infinite, the closures of the sets $X$ and $Y$ have a nonempty intersection. By the choice of $\mathbb{J}$, the closures of $f[X]$ and $f[Y]$ are disjoint; contradiction. Therefore the construction of sequences $x_i, y_i$ stops at some finite stage, as required.

Now assume that $\lambda = \mathrm{cf}(|\mathbb{I}|)$ is uncountable and write $\mathbb{I}$ as the increasing union of subsets $\mathbb{I}(\xi)$, for $\xi < \lambda$, of cardinality smaller than $|\mathbb{I}|$. Fix $x \in \prod_{\mathbb{I}} X$, and for an ordinal $\xi < \lambda$ define $f_\xi\colon \prod_{i\in\mathbb{I}(\xi)} X \to Z$ by

$$f_\xi(y) = f(y^\frown(x \restriction (\mathbb{I} \setminus \mathbb{I}(\xi)))).$$

By the inductive assumption, for every $\xi$ there is a finite $s_\xi \subseteq \mathbb{I}(\xi)$ such that $f_\xi$ depends only on coordinates in $s_\xi$. By the Pressing Down Lemma there is a stationary $S \subseteq \lambda$ and $\eta < \lambda$ such that $s_\xi \subseteq \mathbb{I}(\eta)$ for all $\xi \in S$.

We claim that all $y, z \in \prod_{\mathbb{I}} X_i$ such that $y \restriction \mathbb{I}(\eta) = z \restriction \mathbb{I}(\eta)$ satisfy $f(y) = f(z)$. Otherwise we can find Tychonoff open neighborhoods $U, V$ of $y$ and $z$ such that $f[U]$ and $f[V]$ are disjoint. Fix large enough $\xi \in S \setminus \eta$ so that $U$ and $V$ depend only on coordinates in $\mathbb{I}(\xi)$. Then $f_\xi(y) \in f[U]$ and $f_\xi(z) \in f[V]$. However, $f_\xi$ depends only on coordinates in $\mathbb{I}(\eta) \subseteq \mathbb{I}(\xi)$, hence $f_\xi(y) = f_\xi(z)$; contradiction.

Since $|\mathbb{I}(\xi)| < |\mathbb{I}|$, by the induction hypothesis there are a finite $s \subseteq \mathbb{I}(\xi)$ and $f_1\colon \prod_{i\in s} X_i \to Z$ such that $f = f_1 \circ \pi_s$. This completes the induction and the proof of theorem. □

### 8.3. Dependence of functions on their variables

The key step in the proof of Theorem 8.1.3 is a combinatorial result about dependence of functions on their variables first proved in [45] Our presentation is based on unpublished, but publicly available, [56].

In the following we identify $d \in \mathbb{N}$ with the set $\{0, \dots, d-1\}$ and $\pi_j$ stands for the projection from $\prod_{i<d} X_i$ onto the $j$-th coordinate, $\pi_j(x_0, x_1, \dots x_{d-1}) = x_j$. For a partition $d = u \sqcup v$, $x \in \prod_{i\in u} X_i$ and $y \in \prod_{i\in v} X_i$ we write $x^\frown y$ for $z \in X^d$ such that $z(i) = x(i)$ for $i \in u$ and $z(i) = y(i)$ for $i \in v$.

**Theorem 8.3.1.** *For all $d \geq 1$, sets $X_i$, for $i < d$ and $Y$, every $f\colon \prod_{i<d} X_i \to Y$ satisfies exactly one of the following.*

*(1) There are $k \in \mathbb{N}$ and partitions $X_i = \bigsqcup_{j<k} U_{i,j}$ such that for every $s \in k^d$ for some $j(s) < d$ and $g_s\colon U_{i,j(s)} \to Y$ the functions $f$ and $g_s \circ \pi_{j(s)}$ agree on $\prod_{i<d} U_{i,s(i)}$.*

*(2) There are a partition $d = u \sqcup v$ into nonempty sets and $x_m \in \prod_{i \in u} X_i$ and $y_m \in \prod_{i\in v} X_i$, for $m \in \mathbb{N}$, such that for all $l$ and all $m < n$ we have*

$$f(x_l{}^\frown y_l) \neq f(x_m{}^\frown y_n).$$

**Definition 8.3.2.** If (1) of Theorem 8.3.1 applies, then we say that $f$ *depends on at most one coordinate* on $\prod_{i<d} U_{i,s(i)}$. If there is a need to be more specific, we say that $f$ *depends only on the $j(s)$-th coordinate.* By convention a constant function depends only on the $j$-th coordinate for every coordinate $j$.

The proof of Theorem 8.3.1 below uses compactness of the Čech–Stone compactification $\beta X$ of $X$ equipped with the discrete topology, a consequence of the Axiom of Choice not provable in ZF alone. In §8.3.1 we show that (assuming that ZF has a model) in some models of ZF its conclusion fails, and also show in ZF that the conclusion of Theorem 8.3.1 holds for every well-orderable set $X$. The latter improves an observation due to the referee of [45], where a proof in the case $X = \mathbb{N}$ had been sketched, while the former is new.

We will prove the case of Theorem 8.3.1 when the sets $X_i$ are equal for $i < d$. The case when all $X_i$ are of the same cardinality follows immediately, and the general case can be proven by appropriately changing the notation in the provided proof. The proof of Theorem 8.3.1 is given after a few lemmas.

**Lemma 8.3.3.** *For all $f\colon X^d \to Y$, possibilities (1) and (2) from Theorem 8.3.1 exclude each other.*

Proof. Otherwise, we can fix a partition $X = \bigsqcup_{j<k} U_j$, $j(s)$, and $g_s\colon U_{j(s)} \to Y$ such that $f$ and $g_s \circ \pi_{j(s)}$ agree on $\prod_{i<d} U_{s(i)}$ for all $s \in d^k$ as in (1). Also fix $u, v$, $x_m$, and $y_m$ as in (2). Let $s_u \in k^u$ be such that the set $Z = \{m : x_m \in \prod_{i\in u} U_{s_u(i)}\}$ is infinite, and let $s_v \in k^v$ be such that the set $Z' = \{m \in Z : y_m \in \prod_{i\in v} U_{s_v(i)}\}$ is infinite. Let $s = s_u{}^\frown s_v$ . Fix $m < n$ in $Z'$. If $j(s) \in u$, then $f(x_m{}^\frown y_m) \neq f(x_m{}^\frown y_n)$ although $\pi_{j(s)}(x_m{}^\frown y_m) = \pi_{j(s)}(x_m{}^\frown y_n)$, contradicting the assumption that on $\prod_{i<d} U_{s(i)}$ the function $f$ depends only on the $j(s)$-coordinate. Therefore $j(s) \in v$. Then $f(x_m{}^\frown y_n) \neq f(x_n{}^\frown y_n)$ and $\pi_{j(s)}(x_m{}^\frown y_m) = \pi_{j(s)}(x_m{}^\frown y_n)$, contradicting the assumption that $f$ depends only on the $j(s)$-coordinate on $\prod_{i<d} U_{s(i)}$. □

**Lemma 8.3.4.** *Assume that for $f\colon X^d \to Y$ there are a partition $d = u \sqcup v$ into nonempty sets and $x_m \in X^u$ and $y_m \in X^v$, for $m \in \mathbb{N}$, that satisfy one of the following.*

*(3) For all $l < m < n$ we have*

$$f(x_l{}^\frown y_l) \neq f(x_l{}^\frown y_m) \quad \text{and} \quad f(x_l{}^\frown y_m) = f(x_l{}^\frown y_n).$$

*(4) For all $m < n$ we have*

$$f(x_m, y_m) \neq f(x_m, y_n) \quad \text{and} \quad f(x_m, y_m) \neq f(x_n, y_m).$$

*Then (2) of Theorem 8.3.1 applies.*

A standard bit of notation will come handy in the proof of Lemma 8.3.4. $\{l,m,n\}_<$ to denote the set $\{l,m,n\}$ while asserting that $l<m<n$ (analogous notation applies for any $d\geq 2$ in place of the number 3.)

PROOF. Readers familiar with the canonical Erdös–Rado extension of Ramsey's theorem ([34]) surely don't need to read this proof and processing a 'slow' proof may take less effort on reader's behalf than parsing the former.[1] The proof in each of the two cases begins by defining a partition $h\colon [\mathbb{N}]^3\to\{0,1\}$ by

$$h(\{l,m,n\}_<)=\begin{cases}0, & \text{if } f(x_l{}^\frown y_l)=f(x_m{}^\frown y_n),\\ 1, & \text{if } f(x_l{}^\frown y_l)\neq f(x_m{}^\frown y_n).\end{cases}$$

By Ramsey's theorem, there is an infinite $h$-homogeneous $H\subseteq\mathbb{N}$. We claim that $H$ cannot be 0-homogeneous. Assume otherwise. Then 0-homogeneity of the set $H$ implies that for $l<m<n<k$ in $H$ we have

$$f(x_m{}^\frown y_m)=f(x_n{}^\frown y_k)=f(x_l{}^\frown y_l)=f(x_m{}^\frown y_n),$$

contradicting (3). Also, for $l<m<n<k$ in $H$ we have

$$f(x_m,y_n)=f(x_l,y_l)=f(x_n,y_k)=f(x_m,y_m),$$

contradicting (4).

Therefore in each of the cases (3) and (4) the set $H$ is 1-homogeneous and for all $l<m<n$ we have $f(x_l{}^\frown y_l)\neq f(x_m{}^\frown y_n)$. By Ramsey's Theorem (and passing to subsequences of $x_m$ and $y_m$) we may assume that either (i) for all $m<n$ we have $f(x_m{}^\frown y_m)=f(x_n{}^\frown y_n)$ or that (ii) for all $m<n$ we have $f(x_m{}^\frown y_m)\neq f(x_n{}^\frown y_n)$.

If (i) applies, then for all $m$ and $k<l$ we have $f(x_m{}^\frown y_m)=f(x_k{}^\frown y_k)\neq f(x_k{}^\frown y_l)$, and (2) holds. If (ii) applies, define $h'\colon[\mathbb{N}]^3\to\{0,1\}$ by

$$h'(\{l,m,n\}_<)=\begin{cases}1, & \text{if } f(x_l{}^\frown y_l)=f(x_m{}^\frown y_n),\\ 2, & \text{if } f(x_l{}^\frown y_l)=f(x_l{}^\frown y_m),\\ 3, & \text{if } f(x_m{}^\frown y_m)=f(x_l{}^\frown y_m),\\ 4, & \text{if } f(x_m{}^\frown y_m)=f(x_l{}^\frown y_n),\\ 5, & \text{if } f(x_n{}^\frown y_n)=f(x_l{}^\frown y_m),\\ 0, & \text{if none of the above cases applies.}\end{cases}$$

Since all functions $m\mapsto f(x_m{}^\frown y_m)$ and $n\mapsto f(x_m{}^\frown x_n)$ for $n>m$ are injective, $h'$ admits no infinite $j$-homogeneous sets for any $j\geq 1$. An infinite 0-homogeneous set gives family as in (2). This concludes the proof. □

In the proof of Theorem 8.3.1 we use the consequence of the Axiom of Choice, compactness of the space $\beta X$ of ultrafilters on $X$ (i.e., its Čech–Stone compactification) is compact. If $\mathcal{U}$ is an ultrafilter on $X$ and $\varphi(x)$ is a formula, then

$$(\mathcal{U}x)\varphi(x)$$

stands for 'the set of $x\in X$ for which $\varphi(x)$ holds belongs to $\mathcal{U}$'. The quantifier $(\mathcal{U}x)$ is self-dual, meaning that all $\mathcal{U}$ and $\varphi$ satisfy

$$\neg(\mathcal{U}x)\varphi(x)\Leftrightarrow(\mathcal{U}x)\neg\varphi(x),$$

as is easy to check.

[1] The unseemly definition of the function $h'$ will hopefully pique the reader's interest in [34].

**Lemma 8.3.5.** *If $d \geq 1$, $\mathcal{U}_i$ for $i < d$ is an ultrafilter on $X$, and $f \colon X^d \to Y$ then either* (2) *of Theorem 8.3.1 applies or*

*(5) there are $A_i \in \mathcal{U}_i$ for $i < d$, $j < d$, and $g \colon A_j \to Y$ such that $f$ and $g \circ \pi_j$ agree on $\prod_{i<d} A_i$.*

Proof. By induction on $d$. The case $d = 1$ is trivial (take $j = 0$, $A_0 = X$, and $g = f$). We now prove the case when $d = 2$ because it will be used in the proof of the inductive step.

Fix $f \colon X^2 \to Y$ and ultrafilters $\mathcal{U}$ and $\mathcal{V}$ on $X$. If at least one of $\mathcal{U}$ and $\mathcal{V}$ is principal, then the assertion follows from the case $d = 1$. We can therefore assume that both $\mathcal{U}$ and $\mathcal{V}$ are nonprincipal. The proof splits into three cases.

Assume for a moment that $(\mathcal{U}x)(\exists g(x) \in Y)(\mathcal{V}y)f(x, y) = g(x)$. If in addition there is $A_1 \in \mathcal{V}$ such that $A_0 = \{x : (\exists g(x) \in Y)(\mathcal{V}y)f(x, y) = g(x)\}$ satisfies $(\forall x \in A_0)(\exists g(x) \in Y)(\forall y \in A_1)f(x, y) = g(x)$, then $A_0, A_1$, $j = 0$, and $g$ witness that (5) holds. Otherwise, there is $B \in \mathcal{U}$ such that for all $x \in B$ and every $A_1 \in \mathcal{V}$ some $y = y(x, A_1) \in A_1$ satisfies $f(x, y) \neq g(x)$. Let $u = \{0\}$ and $v = \{1\}$. We will find $x_m \in X^u$, $y_m \in X^v$, and $C_m \in \mathcal{V}$ such that all $m \geq 0$ satisfy

(i) $C_m \supseteq C_{m+1}$,
(ii) For all $y \in C_m$ we have $g(x_m) = f(x_m, y)$ and and $g(x_m) \neq f(x_m, y_m)$.
(iii) $y_{m+1} \in C_m$.

Take $x_0 \in B$, let $C_0 = \{y \in X : f(x, y) = g(x)\}$, and let $y_0 = y(x_0, C_0)$ so that $f(x_0, y_0) \neq g(x_0)$. If $x_n, y_n, C_n$, had been chosen for $n < m$ and satisfy the conditions, then take $x_m \in B \setminus \{x_n : n < m\}$, let

$$C_m = \{y \in C_{m-1} : f(x, y) = g(x)\}$$

and $y_m = y(x_m, C_m)$, so that $f(x_m, y_m) \neq g(x_m)$. This describes the construction.

For all $l < m < n$ we have $f(x_l, y_l) \neq f(x_l, y_m)$ and $f(x_l, y_m) = f(x_l, y_n)$ and by Lemma 8.3.4, (2) of Theorem 8.3.1 follows.

This concludes the discussion of the first case.

The second case is when $(\mathcal{V}y)(\exists g(y) \in Y)(\mathcal{U}y)f(x, y) = g(y)$. Then the function $f'(x, y) = f(y, x)$ satisfies the assumptions of the first case, and the conclusion follows.

We may therefore assume that neither of the first two cases applies, hence

$$(\mathcal{U}x)(\forall c)(\mathcal{V}y)f(x, y) \neq c,$$
$$(\mathcal{V}y)(\forall c)(\mathcal{U}x)f(x, y) \neq c.$$

Let $u = \{0\}$, $v = \{1\}$. We will find $x_m \in X^u$ and $y_m \in X^v$ such that (4) of Lemma 8.3.4 holds, hence all $0 \leq m < n$ satisfy

(6) $f(x_m, y_m) \neq f(x_m, y_n)$ and $f(x_m, y_m) \neq f(x_n, y_m)$.

Towards this, we will also find $B_m \in \mathcal{U}$, $C_m \in \mathcal{V}$, $x_m \in B_m$, and $y_m \in C_m$ for $m \geq 0$ such that the following conditions hold for all $m$.

$$(\forall x \in B_{m+1})f(x, y_m) \neq f(x_m, y_m),$$
$$(\forall y \in C_{m+1})f(x_m, y) \neq f(x_m, y_m),$$
$$C_{m+1} \subseteq C_m, \qquad B_{m+1} \subseteq B_m.$$

Let $B_0 = \{x : (\forall c)(\mathcal{V}y)f(x, y) \neq c\}$ and $C_0 = \{y : (\forall c)(\mathcal{U}x)f(x, y) \neq c\}$. These sets belong to $\mathcal{U}$ and $\mathcal{V}$, respectively, Choose $x_0 \in B_0$ and $y_0 \in C_0$. Then $B_1 = \{x \in B_0 : f(x, y_0) \neq f(x_0, y_0)\}$ belongs to $\mathcal{U}$ and $C_1 = \{y \in C_0 : f(x_0, y) \neq f(x_0, y_0)\}$ belongs to $\mathcal{V}$. If $x_m, y_m, B_m$, and $C_m$ had been chosen, pick $x_{m+1} \in B_m$ and $y_{m+1} \in C_m$. Then the sets

$$B_{m+1} = \{x \in B_m : f(x, y_{m+1}) \neq f(x_{m+1}, y_{m+1})\}$$
$$C_{m+1} = \{y \in C_m : f(x_{m+1}, y) \neq f(x_{m+1}, y_{m+1})\}$$

belong to $\mathcal{U}$ and $\mathcal{V}$, respectively. This describes the construction of sequences that satisfy (6).

Lemma 8.3.4 implies that (2) of Theorem 8.3.1 applies. This completes the proof of Lemma 8.3.5 in case when $d = 2$.

Assume that the conclusion holds for $d \geq 2$. Fix $f\colon \prod_{i<d+1} X \to Y$ and ultrafilters $\mathcal{U}_i$, for $i < d+1$, on $X$. We may assume that each $\mathcal{U}_i$ is nonprincipal. For $x \in X$ let $f^x\colon \prod_{i<d} X \to Y$ be defined by

$$f^x(x_0, \dots, x_{d-1}) = f(x_0, \dots, x_{d-1}, x).$$

By the inductive hypothesis, for every $x \in X$ the function $f^x$ satisfies one of the alternatives given by Theorem 8.3.1. If for some $x \in X$ the function $f^x$ satisfies 2 with $d = \bar{u} \sqcup \bar{v}$,$\bar{x}_m \in X^{\bar{u}}$, $\bar{y}_m \in X^{\bar{v}}$ then let $u = \bar{u} \cup \{d\}$, $v = \bar{v}$, $x_m = \bar{x}_m{}^\frown x$, and $y_m = \bar{y}_m$. These objects witness that $f$ satisfies (2).

We may therefore assume that (1) of Theorem 8.3.1 holds for $f^x$ for all $x \in X$. For each $x \in X$ fix $A_i^x \in \mathcal{U}_i$ for $i < d$, $j(x) < d$, and $g^x\colon X \to Y$ such that

$$f^x(y) = (g^x \circ \pi_{j(x)})(y) \text{ for all } y \in \textstyle\prod_{i<d} A_i^x.$$

Let $j < d$ be such that $(\mathcal{U}_d x) j(x) = j$. As in the case when $d = 2$, we consider cases.

Assume for a moment that there are $A_i \in \mathcal{U}_i$ for $i < d+1$ such that for every $x \in A_d$ the function $g^x$ is defined on $A_j$ and $f^x$ and $g^x \circ \pi_j$ agree on $\prod_{i<d} A_i$. Let $\bar{f}(x, y) = g^x(y)$. Denoting the projection from $\prod_{i<d+1} A_i$ to $A_j \times A_d$ by $\pi_{j,d}$, the restriction of $f$ to $\prod_{i<d+1} A_i$ agrees with $\bar{f} \circ \pi_{j,d}$. By the already proven case $d = 2$, the function $\bar{f}$ satisfies (5) of Lemma 8.3.5 and it is straightforward to see that in each of the two cases this conclusion carries to $f$.

We may therefore assume that for all $\bar{A} = (A_i)_{i<d}$ in $\prod_{i<d} \mathcal{U}_i$ there is $B(\bar{A}) \in \mathcal{U}_d$ such that for every $x \in B(\bar{A})$ some $z = z(x, \bar{A})$ in $A_j$ and $y = y(x, \bar{A})$ in $\prod_{i<d, i \neq j} A_i$ satisfy

$$f^x(y) \neq (g^x \circ \pi_{j(x)})(y). \tag{8.1}$$

Let $u = \{j, d\}$ and $v = (d+1) \setminus u$. We will proceed to choose sequences $x_m \in X^u$, $y_m \in X^v$, and $\bar{A}_m = (A_{i,m})_{i<d}$ in $\prod_{i<d} \mathcal{U}_i$ so that these objects satisfy the following (it will be convenient to present $x_m$ as $x_m(j)^\frown x_m(d)$).

(i) $A_{i,0} = X$ for $i < d$.
(ii) $x_0(d) \in B(\bar{A}_0)$, $x_0(j) = z(x_0(d), \bar{A}_0)$, and $y_0 = y(x, \bar{A}_0)$.
(iii) $A_{i,m+1} = A_i^{x_m(0)} \cap A_{i,m}$ for all $i < d$.
(iv) $x_{m+1}(d) \in B(\bar{A}_{m+1})$, $x_{m+1}(j) = z(x_{m+1}(d), \bar{A}_{m+1})$.
(v) $y_m = y(x_m, \bar{A}_m)$.

Clause (iii) implies $A_{i,m+1} \in \mathcal{U}_i$ for all $i < d$ and $f^{x_m}(y) = g^{x_m(d)}(x_m(j))$ for all $y \in \prod_{i \in v} A_{i,m+1}$. As (v) implies that all $y \in \prod_{i<d, i\neq j} A_{i,m-1}$ satisfy $f(x_m{}^\frown y_m) \neq g^{x_m(d)}(x_m(j))$, we have $f(x_m{}^\frown y_m) \neq f(x_m{}^\frown y_n)$ and $f(x_m{}^\frown y_n) = f(x_m{}^\frown y_k)$ for all $m < n < k$. Lemma 8.3.4 (3) implies that (2) of Theorem 8.3.1 holds and completes the proof. □

Proof of Theorem 8.3.1. Fix $f\colon \prod_{i<d} X_i \to Y$. Suppose for a moment that for every $\bar{\mathcal{U}} = (\mathcal{U}_i)_{i<d}$ in $(\beta X)^d$ for $i<d$ there are sets $A_i^{\bar{\mathcal{U}}} \in \mathcal{U}_i$, for $i<d$, $j<d$, and $g^{\bar{\mathcal{U}}}\colon U_j \to Y$ such that the restriction of $f$ to $\prod_{i<d} A_i^{\bar{\mathcal{U}}}$ agrees with $g\circ\pi_j$. These sets correspond to an open neighbourhood of $\bar{\mathcal{U}}$ hence compactness of $(\beta X)^d$ implies 1 of Theorem 8.3.1.

We may therefore assume that for some $\bar{\mathcal{U}}$ there are no such objects, and Lemma 8.3.5 implies that (2) of Theorem 8.3.1 holds. □

**8.3.1. On the necessity of the Axiom of Choice.** We do not know what fragment of the Axiom of Choice is needed to prove Theorem 8.3.1, but we do know that it cannot be proven in ZF. Recall that a set is *Dedekind-finite* if it admits no injection into a proper subset of itself. It is well-known that ZF is relatively consistent with the existence of an infinite, Dedekind–finite set (see e.g., [88]).

**Proposition 8.3.6.** *Suppose that $X$ is a Dedekind-finite, infinite set. Then there exists $f\colon X^2 \to \{0,1\}$ such that both alternatives of Theorem 8.3.1 fail.*

Proof. Let $f(x,y)=0$ if $x=y$ and $f(x,y)=1$ if $x\neq y$. If $X=\bigcup_{i<k} U_i$ then since $X$ is infinite there exists $j<k$ such that $X^2\cap U_j^2$ is infinite. Clearly the function $f$ depends on both variables on $U_j^2$.

Assume that (2) holds. Then each one of $u$ and $v$ is a singleton and $x_m$, $y_m$, for $m\in\mathbb{N}$, are sequences of elements of $X$. Since $X$ is Dedekind-finite, there are $m<n$ such that $y_m=y_n$. Then $f(x_m{}^\frown y_m)=f(x_m{}^\frown y_n)$; contradiction. □

The reader may object that in choiceless context a more natural analog of (2) should involve sequences $(x_m)$ and $(y_m)$ indexed by some infinite (possibly Dedekind-finite) sets instead of $\mathbb{N}$. Clearly if $X$ is Dedekind-finite then the partition into singletons is Dedekind-finite and each of the rectangles is a singleton, hence $f$ is constant on it and allowing this version of (1) leads nowhere. We may therefore consider the variant in which (1) of Theorem 8.3.1 is unchanged but (2) is modified by allowing the sequences to be indexed by a Dedekind-finite set. A stronger assumption refutes this alternative. A set $X$ is a *Russell set* if $X=\bigsqcup_{n\in\mathbb{N}} X_n$ where each $X_n$ has two elements but for every $Y\subseteq X$ the set $\{n : |Y\cap X_n|=1\}$ is finite. A Russell set is Dedekind-finite and it is well-known that if ZF is relatively consistent with the existence of Russell set (see e.g., [88]).

**Proposition 8.3.7.** *If $X$ is a Russell set then there is $f\colon X^2\to\{0,1\}$ such that for every $k\in\mathbb{N}$ and partition $X=\bigsqcup_{i<k} U_i$ for some $i<k$ the restriction of $f$ to $U_i^2$ depends on both coordinates and for every infinite set $Z$ and all $x_z, y_z$ in $X$ there are arbitrarily large finite sets $I\subseteq Z$ such that the restriction of $f$ to $\{x_z : z\in Z\}\times\{y_z : z\in Z\}$ is constant.*

Proof. We have that $X=\bigsqcup_{n\in\mathbb{N}} X_n$, each $X_n$ has two elements and for every $Y\subseteq X$ the set $\{n : |Y\cap X_n|=1\}$ is finite. Let $f(x,y)=0$ if $\{x,y\}=X_n$ for some $n$ and $f(x,y)=1$ otherwise. If $X=\bigsqcup_{i<k} U_i$, then for some $i$ the set $\{n : X_n\cap U_i\neq\emptyset\}$ is infinite. Thus the set $\{n : X_n\subseteq U_i\}$ is infinite, and the restriction of $f$ to $U_i^2$ is isomorphic to the restriction of $f$ to $X^2$.

Now assume that $Z$ is an infinite set and that $x_z, y_z$, for $z\in Z$ belong to $X$. Then the set of $n$ such that some $z$ satisfies $\{x_z,y_z\}=X_n$ is finite, therefore all but finitely many $z$ satisfy $f(x_z,y_z)=1$. For every $x\in X$ we have that $f(x,y)=0$

or $f(y,z)=0$ for exactly one $y\in X$. One can therefore for every $k\geq 1$ recursively choose $F\subseteq Z$ of cardinality $k$ such that $f(x_z,y_{z'})=1$ for all $z$ and $z'$ in $F$. □

We conclude with an easy self-strengthening of Theorem 8.3.1.

**Proposition 8.3.8** (ZF)**.** *For all $d\geq 1$, well-orderable sets $X_i$, for $i<d$ and $Y$, every function $f\colon \prod_{i<d} X_i \to Y$ satisfies one of the alternatives of Theorem 8.3.1.*

Proof. As in Theorem 8.3.1, it suffices to consider the case when $f\colon X^d\to Y$. Clearly $X^d$ is well-orderable, and so is the range of $f$, as an image of the well-orderable set $X^d$. Therefore we can identify $X$ and $Y$ with ordinals, and the graph of $f$ with a subset of $X^d\times Y$. The model $L[f]$ is a model of ZFC ([113, Exercise II.6.30]). Each of the alternatives of Theorem 8.3.1 is a $\Sigma_1$ statement and therefore holds in $V$. □

In [45, Question 11] I asked whether there is a finitary version of Theorem 8.3.1, without suggesting what such finitary version should look like. A more precise question (to which I do not know the answer) is whether for all $d$ and $k$ there is $N=N(d,k)$ such that for all $X$ and all $f\colon X^d\to X$, if there are no $d=u\sqcup v$, $x_m\in X^u$, $y_m\in Y^v$ for $m<n$ such that $f(x_l{}^\frown y_l)\neq f(x_m{}^\frown y_n)$ for all $l<N$ and $m<n<N$, then there is a partition $X=\bigsqcup_{j<k} U_i$ such that for every $s\in k^d$ there are $j(s)<d$ and $g_s\colon U_{j(s)}\to Y$ such that $f$ agrees with $g_s\circ\pi_{j(s)}$ on $\prod_{i<d} U_{s(i)}$.

## 8.4. An extension of van Douwen's lemma

In this section we prove Corollary 8.4.5 which is the key topological component of the proof of Theorem 8.1.3.

**Definition 8.4.1.** For $d\geq 1$ we will define three subsets of $\mathbb{N}^d$. Let $[\mathbb{N}]^d$ denote the set of all increasing elements of $\mathbb{N}^d$. By identifying such set with its range, $[\mathbb{N}]^d$ corresponds to the family of $d$-element sets of natural numbers. By

$$\{m_0,m_1,\dots,m_{d-1}\}_<$$

we denote an element of $[\mathbb{N}]^d$ such that $m_0<m_1<\dots<m_{d-1}$. Let $\langle\mathbb{N}\rangle^d$ denote the set of all nondecreasing $d$-tuples of natural numbers. Finally, let $\Delta^d\mathbb{N}$ denote the diagonal, i.e., the set of all constant $d$-tuples $(n,n,\dots,n)$ of natural numbers.

The following is [25, Fact 14.2] (recall that $[\mathbb{N}]^k$ is identified with the subset of $\mathbb{N}^k$ consisting of all increasing $k$-tuples).

**Lemma 8.4.2.** *If $s_n$ are disjoint finite subsets of $\mathbb{N}$ such that $\lim_n |s_n|=\infty$, then for every $k\geq 2$ closures of the sets $\bigcup [s_n]^k$ and $\Delta^k\bigcup s_n$ in $(\beta\mathbb{N})^2$ are not disjoint.*

Proof. Let $\mathcal{F}=\{A\subseteq\bigcup_n s_n : \sup_n |s_n\setminus A|<\infty\}$. Since $|s_n|\to\infty$ as $n\to\infty$, $\mathcal{F}$ is a filter. Let $\mathcal{U}$ be an ultrafilter extending $\mathcal{F}$. Towards proving $\mathcal{U}\in\overline{\bigcup[s_n]^k}\cap\overline{\Delta^k\bigcup s_n}$, fix an open neighbourhood of $\mathcal{U}^k$. For some $A\in\mathcal{U}$ this neighbourhood includes the set of all $(\mathcal{V}_0,\dots,\mathcal{V}_{k-1})$ such that $A\in\mathcal{V}_i$ for all $i<k$. It therefore suffices to prove that $A^k$ intersects both $\bigcup[s_n]^k$ and $\Delta^k\bigcup s_n$ nontrivially.

Since $A\subseteq\bigcup_n s_n$ there is $j\in A\cap s_n$ for some $n$, and $(j,j,\dots,j)\in\Delta^k s_n\cap A^k$. Since $\mathcal{U}\supseteq\mathcal{F}$, there exists $n$ such that $|s_n\setminus A|>k$. Therefore $[s_n]^k\cap A^k\neq\emptyset$. □

The following lemma will be generalised in Corollary 8.4.5 below.

**Lemma 8.4.3.** *If $f\colon(\beta\mathbb{N})^2\to Z$ is a continuous map such that the sets $f[[\mathbb{N}]^2]$ and $f[\Delta^2\mathbb{N}]$ are disjoint, then $Z$ is not a $\beta\mathbb{N}$-space.*

Proof. Suppose for a moment that the closure of the set

$$X = f[\Delta^2\mathbb{N}] \tag{8.2}$$

in $Z$ is infinite. If it is countable, then it has an infinite discrete subset $D$ whose closure is too small to be $\beta D$, contradicting the assumption that $Z$ is a $\beta\mathbb{N}$-space.

We may therefore assume that the closure of the set in (8.2) is uncountable. Fix $\mathcal{U} \in \mathbb{N}^*$ such that $f(\langle\mathcal{U},\mathcal{U}\rangle) \notin f[\langle\mathbb{N}\rangle^2]$. Since $f[\langle\mathbb{N}\rangle^2]$ is countable, we can recursively choose clopen neighbourhoods $U_1 \supset \overline{U_2} \supset U_2 \supset \overline{U_3} \supset \dots$ of $f(\mathcal{U},\mathcal{U})$ whose intersection is disjoint from $f[\langle\mathbb{N}\rangle^2]$. By continuity there are $A_n \in \mathcal{U}$ such that $f[A_n^2] \subseteq U_n$ for every $n$. Since each $A_n$ is infinite, we may choose pairwise disjoint $s_n \subseteq A_n$ satisfying $|s_n| = n$ for all $n$. By Lemma 8.4.2, the closures of $Y_1 = \bigcup_n [s_n]^2$ and $Y_2 = \bigcup_n \Delta^2 s_n$ intersect nontrivially, and therefore so do the closures of $f[Y_1]$ and $f[Y_2]$.

We claim that the set $T = f[Y_1] \cup f[Y_2]$ is relatively discrete in $Z$. This is because $T \cap \bigcap_n U_n = \emptyset$, while $T \setminus \overline{U}_n$ is finite for all $n$. But the closures of $f[Y_1]$ and $f[Y_2]$ have nonempty intersection, contradicting the assumption that $Z$ is a $\beta\mathbb{N}$-space.

We can therefore assume that the closure of the set $X = f[\Delta^2\mathbb{N}]$ in $Z$ is finite. By replacing $\mathbb{N}$ with an going infinite subset, we may assume that there is $\bar{z} \in Z$ such that $f(m,m) = \bar{z}$ for all $m$. Lemma 8.4.2 implies that closures of the sets $[A]^2$ and $\Delta^2 A$ have nonempty intersection for every infinite $A \subseteq \mathbb{N}$. Therefore we can derive a contradiction by finding an infinite $A$ such that the closure of $f[[A]^2]$ does not contain $\bar{z}$.

By applying Ramsey's theorem, find an infinite $A_0 \subseteq \mathbb{N}$ such that one of the following two alternatives applies.

(1) $f(l,m) = f(l,n)$ for all $l < m < n$ in $A_0$.
(2) $f(l,m) \neq f(l,n)$ for all $l < m < n$ in $A_0$.

Applying Ramsey's theorem again, we can find an infinite $A \subseteq A_0$ such that exactly one of the following two alternatives applies.

(3) $f(l,n) = f(m,n)$ for all $l < m < n$ in $A$.
(4) $f(l,n) \neq f(m,n)$ for all $l < m < n$ in $A$.

It remains to consider the four possible cases. If (1) and (3) apply then $f$ is constant on $[A]^2$. Since the closures of the sets $f[[\mathbb{N}]^2]$ and $f[\Delta^2\mathbb{N}]$ are disjoint, the constant value of $f$ is not $\bar{z}$ and the closure of $f[[A]^2]$ is as required.

If (1) and (4) apply, let $z_m = f(m,n)$; by (1) $z_m$ does not depend on $n$ and by (4) all $z_m$ are distinct. Since $Z$ is a $\beta\mathbb{N}$-space, $\bar{z}$ has an open neighborhood $U$ such that $C = \{m : z_m \notin U\}$ is infinite. But this implies that the closure of $f[[C]^2]$ avoids $\bar{z}$, as required.

If (2) and (3) apply, define $z_n = f(m,n)$ for $m < n$ in $A$. As in the previous case, we find an infinite subsequence of $\{z_n\}$ which does not accumulate to $\bar{z}$.

Finally assume that (2) and (4) apply. Define $f_1 \colon \mathbb{N}^2 \to Z$ by

$$f_1(m,n) = f(m,n+1).$$

Then $f_1$ continuously extends to $f_2 \colon (\beta\mathbb{N})^2 \to Z$. Since $f_2[\Delta^2\mathbb{N}]$ is infinite, the first part of this proof contradicts the assumption that that $Z$ is not a $\beta\mathbb{N}$-space.

This completes the proof of Lemma 8.4.3. □

Recall that $\beta\mathbb{N}$ is characterised by the fact that every map from $\mathbb{N}$ into a compact space continuously extends to $\beta\mathbb{N}$. We can now state and prove a characterisation of when a map from $\mathbb{N}^d$ into a $\beta\mathbb{N}$-space continuously extends to $(\beta\mathbb{N})^d$.

**Theorem 8.4.4.** *Suppose that $Z$ is a $\beta\mathbb{N}$-space and $f\colon \mathbb{N}^d \to Z$ is continuous. Then the following are equivalent:*

*(1) $f$ continuously extends to $(\beta\mathbb{N})^d$.*
*(2) $\mathbb{N}^d$ can be can be covered by finitely many rectangles such that $f$ depends on at most one coordinate on each one of them.*
*(3) There is no disjoint partition $d = s \sqcup t$ for which there are $x_i \in \mathbb{N}^s$ and $y_i \in \mathbb{N}^t$, for $i \in \mathbb{N}$, such that*

$$f(x_i{}^\frown y_i) \neq f(x_i{}^\frown y_j) \text{ and } f(x_i{}^\frown y_j) \neq f(x_j{}^\frown y_j)$$

*for all $i < j$.*

*Moreover, these are equivalent even when $d$ is an arbitrary infinite cardinal.*

Proof. Assume (3) fails and let $g\colon \mathbb{N}^2 \to Z$ be

$$g(i,j) = f(\vec{x}_i{}^\frown \vec{y}_j).$$

Then $g[\Delta^2\mathbb{N}]$ and $g[[\mathbb{N}]^2]$ are disjoint, thus Lemma 8.4.3 implies that (1) fails, and (1) implies (3)

(3) implies (2) by Theorem 8.3.1.

It remains to prove that (2) implies (1). Assume (2), that $\mathbb{N}^d$ can be covered by finitely many rectangles such that $f$ depends on at most one coordinate on each one of them. These rectangles correspond to disjoint clopen subsets of $(\beta\mathbb{N})^d$. Since every function from $\mathbb{N}$ into $Z$ continuously extends to $\beta\mathbb{N}$, every $f\colon \mathbb{N}^d \to Z$ which depends on at most one coordinate continuously extends to $\beta\mathbb{N}$ as well. Since each of the sets are clopen, the union the graphs of of these functions is the graph of a continuous function and (1) follows.

The 'moreover' part follows from Theorem 8.2.1. □

As an application of Theorem 8.4.4, we have a generalisation of Lemma 8.4.3 to higher powers, also generalizing [**25**, Lemma 14.1].

**Corollary 8.4.5.** *If $f\colon (\beta\mathbb{N})^d \to Z$ is a continuous map such that the sets $f[[\mathbb{N}]^d]$ and $f[\Delta^d\mathbb{N}]$ are disjoint, then $Z$ is not a $\beta\mathbb{N}$-space.*

Proof. If $Z$ is a $\beta\mathbb{N}$-space, then by Theorem 8.4.4 we can partition $\mathbb{N}^d$ into finitely many rectangles such that $f$ depends on at most one coordinate on each one of them. One of these rectangles, say $R$, has an infinite intersection with the diagonal; let $\{n_i : i \in \mathbb{N}\}$ be the increasing enumeration of the set of $n$ such that the $d$-tuple $(n, n, \dots, n)$ is in $R$. Let $j < d$ be such that $f \restriction R$ depends only on $j$-th coordinate. Then

$$f(n_0, n_1, \dots, n_{d-1}) = f(n_j, n_j, \dots, n_j),$$

therefore $f''[\mathbb{N}]^d$ and $f''\Delta^d\mathbb{N}$ are not disjoint. □

## 8.5. Clopen decompositions

In Theorem 8.5.2 below we show that in Theorem 8.1.3 it suffices to obtain a decomposition of $\prod_{i<d} X_i$ into closed (instead of clopen) rectangles.

**Lemma 8.5.1.** *Assume $f\colon \prod_{i<d} X_i \to Z$, $n < d$, and that $P, R$ are rectangles in $\prod_{i<d} X_i$ such that both $f \upharpoonright P$ and $f \upharpoonright R$ depend only on the $n$-th coordinate. Then $f \upharpoonright (P \cup R)$ depends only on the $n$-th coordinate.*

The condition that $P \cap R \neq \emptyset$ is easily seen to be necessary—otherwise both $f \upharpoonright P$ and $f \upharpoonright P$ can be constant, with $f \upharpoonright (P \cup R)$ not constant. The condition that $P$ and $R$ are rectangles is also easily seen to be necessary.

Proof. For convenience of notation, we may assume $n = 0$. Let $P = \prod_{i=0}^{d-1} A_i$ and $R = \prod_{i=0}^{d-1} B_i$. By the assumption, there are $g_1\colon \prod_{i \neq n, i<d} A_i \to Z$ and $g_2\colon \prod_{i \neq n, i<d} B_i \to Z$ such that $f \upharpoonright P = g_1 \circ \pi_0$ and $f \upharpoonright R = g_2 \circ \pi_0$. We claim that $g_1 \cup g_2$ is a function. Pick $a \in \mathrm{dom}(g_1) \cap \mathrm{dom}(g_2) = A_0 \cap B_0$. Since $P \cap R \neq \emptyset$, we can pick $\vec{x} \in \prod_{i=1}^{d-1} A_i$. Then $a^\frown \vec{x} \in P \cap R$, and $g_1(a) = f(a^\frown \vec{x}) = g_2(a)$, thus $g = g_1 \cup g_2$ is indeed a function. Therefore $f \upharpoonright (P \cup R) = g \circ \pi_0$, and this concludes the proof. □

**Theorem 8.5.2.** *For $d \geq 2$ let $X_i$, for $i < d$, and $Z$ be arbitrary topological spaces. Assume $f\colon \prod_{i<d} X_i \to Z$ is such that $\prod_{i<d} X_i$ can be covered by finitely many* closed *rectangles such that $f$ depends on at most one coordinate on each one of them. Then $\prod_{i<d} X_i$ can be covered by finitely many* clopen *rectangles such that $f$ depends on at most one coordinate on each one of them.*

Proof. By possibly refining a given covering of $\prod_{i<d} X_i$ by rectangles, we may assume that for every $i < d$ there are $l(i) \in \mathbb{N}$ and a covering

$$X_i = A_0^i \cup A_1^i \cup \cdots \cup A_{l(i)-1}^i$$

by closed sets such that for every $h \in \prod_{i=0}^{d-1} l(i)$ the restriction of $f$ to $\prod_{i=0}^{d-1} A_{h(i)}^i$ depends on at most one coordinate. For $h \in \prod_{i=1}^{d-1} l(i)$ let

$$C_h = \prod_{i=1}^{d-1} A_{h(i)}^i.$$

**Claim 8.5.3.** *If $A_i^0 \cap A_j^0 \neq \emptyset$, then the restriction of $f$ to $(A_i^0 \cup A_j^0) \times C_h$ depends on at most one coordinate for every $h$.*

Proof. Fix an $h$. If neither $f \upharpoonright (A_i^0 \times C_h)$ nor $f \upharpoonright (A_j^0 \times C_h)$ depends on $x_n$ for any $n > 0$, then they both depend only on $x_0$ and the conclusion follows by Lemma 8.5.1.

Now assume $f \upharpoonright (A_i^0 \times C_h)$ "really depends" on $x_n$ for some $n > 0$, meaning that there are $\langle x_0, x_1, \ldots, x_{d-1}\rangle$ and $\langle y_0, y_1, \ldots, y_{d-1}\rangle$ in $A_i^0 \times C_h$ such that $x_l = y_l$ for all $l \neq n$ but $f(x_0, x_1, \ldots, x_{d-1}) \neq f(y_0, y_1, \ldots, y_{d-1})$. Pick $\bar{x}_0 \in A_i^0 \cap A_j^0$. Then $f \upharpoonright (A_i^0 \times C_h)$ does not depend on $x_0$, thus

$$\begin{aligned} f(\bar{x}_0, x_1, \ldots, x_{d-1}) &= f(x_0, x_1, \ldots, x_{d-1}) \\ &\neq f(y_0, y_1, \ldots, y_{d-1}) = f(\bar{x}_0, y_1, \ldots, y_{d-1}). \end{aligned}$$

Since $\langle \bar{x}_0, x_1, \ldots, x_{d-1}\rangle$ and $\langle \bar{x}_0, y_1, \ldots, y_{d-1}\rangle$ both belong to $A_j^0 \times C_h$, this implies that $f \upharpoonright (A_j^0 \times C_h)$ depends on $x_n$, thus by our assumption it depends only on $x_n$. By Lemma 8.5.1, the Claim follows.

The case when $f \upharpoonright (A_j^0 \times C_h)$ "really depends" on some $x_n$ for $n > 0$ is treated analogously. This concludes the proof. □

By using Claim 8.5.3 repeatedly we may continue joining $A_i^0$'s until we obtain a pairwise disjoint cover $F_0^0, \dots, F_{m(0)}^0$ of $X_0$, such that the restriction of $f$ to every $F_i^0 \times C_h$ depends on at most one coordinate. Since each $A_i^0$ is closed, each $F_j^0$ is clopen.

Repeat the above construction and replace $A_0^1, \dots, A_{d-1}^1$ with a clopen partition $F_0^1, \dots, F_{m(1)}^1$ of $X_1$ such that $f$ depends on at most one coordinate on each $F_{h(0)}^0 \times F_n^1 \times \prod_{i=2}^{d-1} A_{h(i)}^i$ for $n < m(1)$. By repeating this construction for $2, 3, \dots, d-1$, we finally obtain integers $m(j)$ $(j < d)$ and a sequence of clopen partitions $F_i^j$ $(j < d$, $i < m(j))$ of $X_j$ such that $f$ depends on at most one coordinate on each clopen rectangle $\prod_{j=0}^{d-1} F_{h(j)}^j$. This concludes the proof. □

## 8.6. Proof of Theorem 8.1.3

Proof of Theorem 8.1.3. Assume that $Z$ is a $\beta\mathbb{N}$-space, $\mathbb{I}$ is an index set, $X_i$ for $i \in \mathbb{I}$ are compact, and $f\colon \prod_i X_i \to Z$ is continuous. By Theorem 8.2.1, there is a finite subset $s$ of $\mathbb{I}$ such that $f$ depends on coordinates in $s$ only. We may therefore assume $\mathbb{I} = d$ for some $d \in \mathbb{N}$. Apply Theorem 8.3.1 to $f$, ignoring topology for a moment.

Assume that (1) of Theorem 8.3.1 applies. Then each $X_i$ can be partitioned into finitely many sets, $X_i = \bigsqcup_{j<k} U_{i,j}$, so that for every $s \in k^d$ for some $j(s) < d$ and $g_s\colon U_{i,j(s)} \to Y$ the functions $f$ and $g_s \circ \pi_{j(s)}$ agree on the rectangle $\prod_{i<d} U_{i,s(i)}$. The closures of these rectangles are still rectangles, and since $f$ is continuous, it depends on at most one coordinate on each one of these closures. By Theorem 8.5.2, we may assume that the rectangles are clopen.

Therefore it suffices to prove that the alternative (2) of Theorem 8.3.1 leads to contradiction. Suppose that there are a partition $d = u \sqcup v$ into nonempty sets and $x_m \in \prod_{i\in u} X_i$ and $y_m \in \prod_{i \in v} X_i$, for $m \in \mathbb{N}$, such that for all $l$ and all $m < n$ we have

$$f(x_l {}^\frown y_l) \neq f(x_m {}^\frown y_n).$$

Define $h_0\colon [\mathbb{N}]^2 \to \prod_{i<d} X_i$ by

$$h_0(\{m,n\}) = \vec{x}_m {}^\frown \vec{y}_n.$$

Since $\vec{x}_m \in \prod_{i\in s} X_i$ and $\vec{y}_m \in \prod_{i\in t} X_i$ for all $m$, and $\prod_{i<d} X_i$ is compact, $h_0$ can be continuously extended to $H\colon (\beta\mathbb{N})^2 \to \prod_{i\in s} X_i \times \prod_{i\in t} X_i = \prod_{i<d} X_i$. Then $g = f \circ H$ is a continuous function from $(\beta\mathbb{N})^d$ into $Z$, and it has the property that $g(m,n) \neq g(l,l)$ for all $m < n$ and $l$. Therefore $g[[\mathbb{N}]^2]$ and $g[\Delta^2\mathbb{N}]$ are disjoint and $Z$ is not a $\beta\mathbb{N}$-space by Corollary 8.4.5; contradiction. □

## 8.7. Peano curve on steroids

The following somewhat surprising result taken from [48] applies to an arbitrary compact Hausdorff space. In addition to Theorem 8.1.3, it is the main ingredient of the proof of Theorem 8.8.1.

**Theorem 8.7.1.** *A compact Hausdorff space $X$ maps onto $X^2$ if and only if it maps onto $X^{\mathbb{N}}$.*

Proof. It suffices to prove the direct implication. Let $f_i\colon X \to X$ $(i \in \{1,2\})$ be continuous maps such that $x \mapsto (f_1(x), f_2(x))$ maps $X$ onto $X^2$. Define $g_n\colon X \to$

$X$ $(n \in \mathbb{N})$ as follows:

$$\begin{aligned} g_1 &= f_1 \\ g_2 &= f_1 \circ f_2 \\ g_3 &= f_1 \circ f_2 \circ f_2 \\ g_n &= f_1 \circ \underbrace{f_2 \circ \cdots \circ f_2}_{n-1 \text{ times}}, \quad \text{for } n \in \mathbb{N}. \end{aligned}$$

We claim that $x \mapsto (g_n(x))_{n=1}^{\infty}$ maps $X$ onto $X^{\mathbb{N}}$. Since $X$ is compact, it suffices to show that the image of $X$ under this map is dense in $X^{\mathbb{N}}$, and it turn it suffices to show that for every $n \in \mathbb{N}$ and $\vec{y} = (y_i)_{i=1}^{n}$ in $X^n$ there is an $x \in X$ such that $\vec{z} = f(x)$ satisfies $z_i = y_i$ for all $i \leq n$. Equivalently, we need to prove that for every $n$-tuple $(y_1, \dots, y_n)$ there is an $x \in X$ such that $g_i(x) = y_i$ for $i \leq n$. We prove this by induction on $n$.

For $n = 2$, find $x' \in X$ such that $f_1(x') = y_2$ (possible, since $f_1$ is onto), and then, by using the fact that $x \mapsto (f_1(x), f_2(x))$ maps $X$ onto $X^2$, find $x \in X$ such that $f_1(x) = y_1$ and $f_1(x) = x'$. Then we have $g_2(x) = f_1(f_2(x)) = f_1(x') = y_2$.

Assume the claim is true for $n$ and prove it for $n + 1$. By the induction hypothesis, find $x' \in X$ such that $g_i(x') = y_{i+1}$, for $1 \leq i \leq n$. Now pick $x$ such that $f_1(x) = y_1$ and $f_2(x) = x'$. Then for $2 \leq i \leq n + 1$ we have $g_i(x) = g_{i-1}(f_2(x)) = g_{i-1}(x') = y_i$, and this completes the proof. □

By simple cardinal arithmetic, Theorem 8.7.1 implies that a compactum of cardinality strictly smaller than the continuum cannot map onto its own square.

**Theorem 8.7.2.** *Let $X$ be an arbitrary topological space, and assume $\kappa$ is a singular cardinal such that $X$ maps onto $X^{\lambda}$ for all $\lambda < \kappa$. Then $X$ maps onto $X^{\kappa}$.*

Proof. Let $\kappa_\alpha$ $(\alpha < \mathrm{cf}(\kappa))$ be a strictly increasing sequence of cardinals with supremum equal to $\kappa$. For $\eta < \kappa$ let $f_\eta \colon X \to X$ be continuous maps such that $x \mapsto (f_\eta(x))_{\eta \in [\kappa_\alpha, \kappa_{\alpha+1})}$ maps $X$ onto $X^{[\kappa_\alpha, \kappa_{\alpha+1})}$. Define $g_\eta \colon X \to X$ $(\mathrm{cf}(\kappa) \leq \eta < \kappa)$ by:

$$g_\eta = f_\eta \circ f_\alpha \text{ if } \eta \in [\kappa_\alpha, \kappa_{\alpha+1}).$$

We claim that $x \mapsto (g_\eta(x))_{\eta=\kappa_0}^{\kappa}$ maps $X$ onto $X^{[\kappa_0, \kappa)}$. Since the latter power is homeomorphic to $X^{\kappa}$, this will conclude the proof. Fix $\vec{y} = (y_\eta)_{\eta=\kappa_0}^{\kappa}$. Let $x_\alpha$ $(\alpha < \mathrm{cf}(\kappa))$ be such that $f_\eta(x_\alpha) = y_\eta$ for $\eta \in [\kappa_\alpha, \kappa_{\alpha+1})$. Now let $x \in X$ be such that $f_\alpha(x) = x_\alpha$ for all $\alpha < \mathrm{cf}(\kappa)$. Then if $\eta \in [\kappa_\alpha, \kappa_{\alpha+1})$ we have $g_\eta(x) = f_\eta(f_\alpha(x)) = f_\eta(x_\alpha) = y_\eta$, as required. □

**Remark 8.7.3.** J. van Mill pointed out that the analog of Theorem 8.7.1 in which $X^2$ and $X^{\mathbb{N}}$ are required to be homeomorphic to $X$ is false, by [**126**].

S. Solecki pointed out that the construction of the maps $g_n$ in the proof of Theorem 8.7.1 appears in [**143**]. In this paper Sierpiński proves that if $x \mapsto (f_1(x), f_2(x))$ maps $[0, 1]$ onto $[0, 1]^2$, then $x \mapsto \langle g_n(x) : n \in \mathbb{N}\rangle$ maps $[0, 1]$ onto $[0, 1]^{\mathbb{N}}$. Although [**143**] does not use assumptions other than compactness (and Hausdorffness) of $[0, 1]$, Theorem 8.7.1 was not stated in this paper.

Last, but not least, N. Hindman pointed out that the analog of Theorem 8.7.1 as stated in [**48**] is wrong. Namely, its conclusion fails for spaces that are compact, but not Hausdorff. Twenty years later, many thanks to Neil for this remark.

## 8.8. Powers of $\beta\mathbb{N}$-spaces

Structure of the Čech-Stone remainder $\mathbb{N}^* = \beta\mathbb{N}\setminus\mathbb{N}$ of $\mathbb{N}$ is very sensitive to the choice of set-theoretic axioms (see [127]). For instance, the Continuum Hypothesis implies that $\mathbb{N}^*$ maps onto $(\mathbb{N}^*)^\kappa$ for all $\kappa \leq \mathfrak{c}$ ([134]), but under different axioms $\mathbb{N}^*$ does not even map onto $(\mathbb{N}^*)^2$, and moreover $(\mathbb{N}^*)^n$ does not map onto $(\mathbb{N}^*)^{n+1}$ for any $n \in \mathbb{N}$ ([91], also [94]). Obviously, if $\mathbb{N}^*$ maps onto $(\mathbb{N}^*)^2$, then $(\mathbb{N}^*)^n$ maps onto $(\mathbb{N}^*)^{n+1}$ for all $n \in \mathbb{N}$. In [91, p. 60] Winfried Just asked whether it is relatively consistent with the usual axioms of Set Theory that there are $m > n$ in $\mathbb{N}$ such that $(\mathbb{N}^*)^m$ maps onto $(\mathbb{N}^*)^{m+1}$, but $(\mathbb{N}^*)^n$ does not map onto $(\mathbb{N}^*)^{n+1}$. The answer is, somewhat surprisingly, negative ($X \twoheadrightarrow Y$ is short for 'there is a continuous surjection from $X$ onto $Y$').

**Theorem 8.8.1.** *The following are equivalent*

*(1)* $\mathbb{N}^* \twoheadrightarrow (\mathbb{N}^*)^2$,
*(2)* $(\mathbb{N}^*)^n \twoheadrightarrow (\mathbb{N}^*)^{n+1}$ *for some* $n \in \mathbb{N}$,
*(3)* $(\mathbb{N}^*)^\kappa \twoheadrightarrow (\mathbb{N}^*)^\lambda$ *for some pair of cardinals* $\kappa < \lambda$*, finite or infinite.*
*(4)* $\mathbb{N}^* \twoheadrightarrow (\mathbb{N}^*)^{\mathbb{N}}$.

We first recall the notation from §8. If $s \subseteq \kappa$ then $\pi^\kappa_s\colon X^\kappa \to X^s$ is the *projection* of $X^\kappa$ to $X^s$, defined by

$$\pi^\kappa_s(\langle x_\xi : \xi < \kappa\rangle) = \langle x_\xi : \xi \in s\rangle.$$

When $\kappa$ is clear from the context, we write $\pi_s$ instead of $\pi^\kappa_s$. If $s = \{\xi\}$, then we write $\pi_\xi$ instead of $\pi_{\{\xi\}}$ for simplicity, although this is an abuse of notation. A map $f\colon X^\kappa \to Z$ *depends on at most one ($\alpha$-th) coordinate* if there is $g\colon X \to Z$ such that $f(\langle x_\xi\rangle_{\xi<\kappa}) = g(x_\alpha)$, i.e., $f = g \circ \pi_\alpha$.

**Lemma 8.8.2.** *The following are equivalent for every cardinal* $\lambda$ *(finite or infinite) and every* $\beta\mathbb{N}$*-space* $Z$*.*

*(1)* $Z^\kappa \twoheadrightarrow Z^\lambda$ *for some* $\kappa < \lambda$,
*(2)* $Z^n \twoheadrightarrow Z^\lambda$ *for some finite* $n < \lambda$.

Proof. We will prove only the nontrivial direction; note that we may assume $\lambda$ is infinite. Assume $f\colon Z^\kappa \to Z^\lambda$ is a surjection. By Theorem 8.1.3, for each $\xi < \lambda$ there is a finite $s_\xi \subseteq \kappa$ such that $\pi_\xi \circ f$ depends only on coordinates in $s_\xi$. By a counting argument, there is $I \subseteq \lambda$ of cardinality $\lambda$ and a finite $s \subseteq \kappa$ such that $s_\xi = s$ for all $\xi \in I$. Hence there is a map $g\colon Z^s \to Z^I$ such that $g \circ \pi^\kappa_s = \pi_I \circ f$. Since the map on the right-hand side is surjective, so is $g$. This concludes the proof. □

Proof of Theorem8.8.1. Implications (4) $\Rightarrow$ (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are trivial and the analogs of (4) and (1) are equivalent for an arbitrary compact Hausdorff space iby Theorem 8.7.1.

It remains to prove that (3) implies (1). Assume $\kappa < \lambda$ and $f\colon (\mathbb{N}^*)^\kappa \to (\mathbb{N}^*)^\lambda$ is a surjection. By Lemma 8.8.2, we may assume $\kappa$ is finite, say $\kappa = n$. We may also assume $\lambda = n+1$. By applying Theorem 8.1.3 to $\pi_i \circ f$ for all $i < n+1$ and refining the obtained clopen coverings, we can cover $(\mathbb{N}^*)^n$ by finitely many clopen sets such that $\pi_i \circ f$ depends on at most one coordinate on each one of them for each $i < n+1$. The image of at least one of these clopen sets, call it $U$, includes a nonempty clopen subset, call it $V$, of $(\mathbb{N}^*)^{n+1}$. There are $i < j < n+1$ and $k < n$ such that both $\pi_i \circ f$ and $\pi_j \circ f$ depend at most on the $k$-th coordinate. Since $V$

is clopen, its image under the projection $\vec{y} \mapsto \langle \pi_i(\vec{y}), \pi_j(\vec{y})\rangle$ is a clopen subset, call it $V'$, of $(\mathbb{N}^*)^2$. Let $h_l\colon U \to \mathbb{N}^*$ be such that $\pi_l \circ f = h_l \circ \pi_k$ for $l \in \{i, j\}$ on $U$. Pick $\langle \mathcal{U}, \mathcal{V}\rangle \in V'$, and define $h\colon \mathbb{N}^* \to V$ by

$$h(\vec{x}) = \begin{cases} \langle h_i(\vec{x}), h_j(\vec{x})\rangle, & \text{if } \langle h_i(\vec{x}), h_j(\vec{x})\rangle \in V', \\ \langle \mathcal{U}, \mathcal{V}\rangle, & \text{otherwise.} \end{cases}$$

This is a continuous surjection from $\mathbb{N}^*$ onto $V'$; but $V'$ is homeomorphic to $(\mathbb{N}^*)^2$, and this concludes the proof. ☐

**Corollary 8.8.3.** *Let $\mu$ be the smallest cardinal such that there is no continuous surjection from $\mathbb{N}^*$ onto $(\mathbb{N}^*)^\mu$. Then for every pair of cardinals $\kappa, \lambda$ the following are equivalent:*

*(1)* $(\mathbb{N}^*)^\kappa \twoheadrightarrow (\mathbb{N}^*)^\lambda$,

*(2)* $\kappa \geq \lambda$ *or* $\kappa < \lambda < \mu$.

*Also, $\mu$ is a regular cardinal and $\mu = 2$ or $\aleph_1 \leq \mu \leq \mathfrak{c}^+$.*

Proof. Clearly $\kappa < \lambda < \mu$ implies $(\mathbb{N}^*)^\kappa \twoheadrightarrow \mathbb{N}^* \twoheadrightarrow (\mathbb{N}^*)^\lambda$, hence (2) implies (1).

To prove that (1) implies (2), fix $\kappa < \lambda$ and a surjection $f\colon (\mathbb{N}^*)^\kappa \to (\mathbb{N}^*)^\lambda$. By Lemma 8.8.2, we may assume that $\kappa$ is finite. Theorem 8.8.1 implies $\mathbb{N}^* \twoheadrightarrow (\mathbb{N}^*)^\kappa$, hence by composing we have $\mathbb{N}^* \twoheadrightarrow (\mathbb{N}^*)^\lambda$.

Theorem 8.7.2 implies that $\mu$ is regular. The inequality $2 \leq \mu$ is trivial and since the weight of $\mathbb{N}^*$ is $\mathfrak{c}$, $\mu \leq \mathfrak{c}^+$. By Theorem 8.8.1, $\mu > 2$ implies $\mu \geq \aleph_1$. ☐

By [134], Continuum Hypothesis implies that $\mu = \mathfrak{c}^+$ while in [91] Just obtained a forcing extension in which $\mu = 2$. In §9 we will see that this follows from $\mathrm{OCA_T}$.

§8.7 and §8.8 are based on [48]. This paper is a coda to [47] and its results were proved shortly after the latter paper went to print. Another immediate consequence of Theorem 8.1.3 was obtained in [52]. The main result of this paper asserts that if $X$ is an infinite connected $\beta\mathbb{N}$-space then the product $X \times Y$ is not homogeneous for any compact space $Y$ (see [1] for more information). An important example of $X$ is the Čech–Stone remainder of the half-line (see [79]).

## 8.9. Nonhomogeneity in products with $\beta\mathbb{N}$-spaces

Another application of Theorem 8.1.3 to the study of large compact homogeneous spaces (see [112], [128]) and homogeneity properties of products of compact spaces was given in [52].

**Theorem 8.9.1.** *If $X$ is an infinite connected $\beta\mathbb{N}$-space then the product $X \times Y$ is not homogeneous for any compact space $Y$.* ☐

It seems plausible that the conclusion of this theorem holds for not necessarily connected $\beta\mathbb{N}$-spaces.

CHAPTER 9

# Lifting theorems III: Čech–Stone remainders

In [25] Van Douwen proved that, if $X$ is a countable locally compact space, then two finite powers of $X^*$ are homeomorphic if and only if their exponents are equal; in Chapter 8 we have seen an extension of this ZFC result. Not much else can be said about $\beta\mathbb{N}$-spaces in ZFC (see [127]). In [136], W. Rudin used CH to prove that $\mathbb{N}^*$ has $2^{\mathfrak{c}}$ nontrivial automorphisms while in [139] Shelah produced a model of ZFC in which every autohomeomorphism of $\mathbb{N}^*$ is trivial, in the sense that it is determined by an almost permutation of $\mathbb{N}$ ([139]). By the Stone duality, this is an equivalent reformulation of the fact that all automorphisms of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ are trivial[1], that started the study of rigidity of analytic quotients that resulted, among other results, in our OCA lifting theorem. In [91] Just produced a model of ZFC in which $(\mathbb{N}^*)^d$ does not even map onto $(\mathbb{N}^*)^{d+1}$, and in [94] he proved that $\mathrm{OCA_T}$ implies this conclusion. In [127, Theorem 2.2.1] van Mill proved that if all autohomeomorphisms of $\mathbb{N}^*$ are trivial, then $\mathbb{N}^*$ and the Čech–Stone remainder of the ordinal $\omega^2$ are not homeomorphic. Dow and Hart proved, assuming $\mathrm{OCA_T}$, that the Stone space of the Lebesgue measure algebra is not a continuous image of $\mathbb{N}^*$ ([31]) and that the only Čech–Stone remainder of a $\sigma$-compact, locally compact, space which is a surjective image of $\mathbb{N}^*$ is $\mathbb{N}^*$ ([30]), and in [40, §4] the author used $\mathrm{OCA_T}$ and MA to prove related results that will be discussed and extended below.

In this chapter we will use an extension of Theorem 6.1.3 together with Theorem 8.1.3 to analyze Čech–Stone remainders of locally compact, non-compact, Polish, zero-dimensional, spaces. For a completely regular topological space $X$ we write $\beta X$ for its Čech–Stone compactification and

$$X^* = \beta X \setminus X$$

for the remainder (corona).

## 9.1. The weak Extension Principle and trivial maps

The following axiom subsumes the results mentioned in the introduction to this chapter.

**Definition 9.1.1** (Weak Extension Principle, wEP)**.** Suppose that $X$ and $Y$ are completely regular topological spaces.

We say that $\mathrm{wEP}(X,Y)$ holds if for every continuous $f\colon X^* \to Y^*$ there are a partition $X^* = U \sqcup V$ into clopen sets, such that $f[V]$ is nowhere dense in $Y^*$ and $f \restriction U$ has a continuous extension to $\beta X$.

[1] A alternative and very fruitful point of view is provided by the Gelfand–Naimark duality (e.g., [54, §1.3]), see §9.4.

By wEP(Polish) we denote the assertion that $\mathrm{wEP}(X, Y)$ holds for all Polish spaces $X$ and $Y$. wEP(Polish, 0-dim) is the assertion that $\mathrm{wEP}(X, Y)$ holds for all Polish, 0-dimensional, spaces $X$ and $Y$.

By Theorem 9.1.4, $\mathrm{OCA_T}$ implies wEP(Polish, 0-dim). The assumption that the spaces be zero-dimensional is unnecessary by [125], [165], and [167]; see also [166] for a related rigidity result. See the paragraph following Definition 9.1.3.

**Definition 9.1.2.** A continuous $f\colon X^* \to \beta Y$ is *trivial* if there is a continuous $\tilde{f}\colon X \to \beta Y$ such that $(\beta\tilde{f}) \restriction X^* = f$. A continuous $f\colon \prod_{i<m} X_i^* \to \prod_{j<n} \beta Y_j$ is *trivial* if there is a continuous $\tilde{f}\colon \prod_{i<m} X_i \to \prod_{j<n} \beta Y_j$ such that $f$ has a continuous extension to a function $(\beta\tilde{f})\colon \prod_{i<m} X_i \to \prod_{j<n} \beta Y_j$ which satisfies $(\beta\tilde{f}) \restriction \prod_{i<m} X_i^* = f$.

While nontrivial continuous functions from $\mathbb{N}^*$ into $\beta\mathbb{N}$ or into $\mathbb{N}^*$ are obtained by dualising the examples from §5.4, the following 'poor man's version' is quite useful.

**Definition 9.1.3.** A continuous $f\colon \prod_{i<m} X_i^* \to \prod_{j<n} \beta Y_J$ is *almost trivial* if there is a clopen partition $\prod_{i<m} X_i^* = U \sqcup V$ such that $f \restriction U$ is trivial and $f[V] \cap \prod_{j<n} Y_j^*$ is nowhere dense.

The conclusion of case (1) of Theorem 9.1.4 below was proved from $\mathrm{OCA_T}$ and MA for countable locally compact spaces in [40, §4] and (from the same assumptions and using results from [125] and [165]) for all locally compact, non-compact, Polish spaces in [167].

**Theorem 9.1.4.** *Assume* $\mathrm{OCA_T}$. *For all $m, n$, and locally compact, non-compact, Polish, zero-dimensional spaces $X_i$, for $i < m$ and $Y_j$, for $j < n$, if $f$ is a continuous function such that*

*(1)* $f\colon \prod_{i<m} X_i^* \to \prod_{j<n} Y_j^*$,
*(2)* $f\colon \prod_{i<m} X_i^* \to \beta Y_0$, *or*
*(3)* $f\colon \prod_{i<m} X_i^* \to (\beta\mathbb{N})^n$,

*then $f$ is almost trivial.*

The implication from $\mathrm{OCA_T} + \mathrm{MA}$ to (1) was announced in [40]. In addition to using stronger assumptions, the old proof was considerably more difficult than the proof given here and will mercifully never see the light of the day. The proof of Theorem 9.1.4 for remainders of countable, locally compact spaces (i.e., countable ordinals with respect to the ordinal topology) in [40, §4] was also fairly complex, largely because neither Biba's trick nor Theorem 8.1.3 and its corollary, Proposition 9.1.7 below, were available at the time. We will need the analog of 'ccc over Fin' (Definition 3.3.1).

**Definition 9.1.5.** A subset $Z$ of $Y^*$ is *relatively ccc* if there is no uncountable family $\mathsf{U}$ of disjoint open subsets of $Y^*$ each of which intersects $Z$ nontrivially.

The following theorem, in which the smallness property of the clopen set $V$ is improved, will be proved in §9.3.7. It is a corollary to Theorem 9.1.4, and the relation between these two theorems is analogous to that between Theorem 6.1.2 and Theorem 7.1.1.

**Theorem 9.1.6.** *Assume* $\mathrm{OCA_T} + \mathrm{MA}(\sigma\text{-linked})$. *For all* $m \geq 1$ *and locally compact, non-compact, Polish, zero-dimensional spaces* $X_i$, *for* $i < m$ *and* $Y$, *if* $f\colon \prod_{i<m} X_i^* \to \beta Y$ *is a continuous function then there is a clopen partition* $\prod_{i<m} X_i^* = U \sqcup V$ *such that* $f \restriction U$ *is trivial and* $f[V]$ *is relatively ccc.*

**Proposition 9.1.7.** *The following are equivalent.*

(1) *For all locally compact, non-compact, Polish, zero-dimensional spaces* $X$ *and* $Y$, *every continuous* $f\colon X^* \to Y^*$ *is almost trivial.*

(2) *For all* $m, n$, *and locally compact, non-compact, Polish, zero-dimensional spaces* $X_i$, *for* $i < m$ *and* $Y_j$, *for* $j < n$, *every continuous* $f\colon \prod_{i<m} X_i^* \to \prod_{j<n} Y_j^*$ *is almost trivial.*

*The following are also equivalent.*

(3) *For all locally compact, non-compact, Polish, zero-dimensional spaces* $X$, *every continuous* $f\colon X^* \to \beta\mathbb{N}$ *is almost trivial.*

(4) *For all* $m, n$, *and locally compact, non-compact, Polish, zero-dimensional spaces* $X_i$, *every continuous* $f\colon \prod_{i<m} X_i^* \to (\beta\mathbb{N})^n$ *is almost trivial.*

Proof. (1) is a special case of (2).

To prove that (1) implies (2). Suppose $f\colon \prod_{i<m} X_i^* \to \prod_{j<n} Y_j^*$ is as in (2). Since $Y_j^*$ is a $\beta\mathbb{N}$-space by Lemma 8.1.2, by applying Theorem 8.1.3 to (using the notation $\pi_j$ for projections onto the factor $X_j$ as in §8.3) $\pi_j \circ f\colon n \prod_{i<m} X_i^* \to Y_j^*$, we can partition $\prod_{i<m} X_i^*$ into finitely many clopen sets, $U_{l,j}$, for $l < k(j)$, so that the restriction of $\pi_j \circ f$ to each one of them depends on at most one coordinate. By (1), for each $l < k(j)$ there is a clopen partition $U_{l,j} = U_{l,j}^0 \cup U_{l,j}^1$ so that $(\pi_j \circ f) \restriction U_{l,j}^0$ is trivial and $(\pi_j \circ f)[U_{l,j}^1]$ is nowhere dense in $Y_j^*$. Therefore $f[U_{l,j}^1]$ is nowhere dense in $\prod_{i<n} Y_i^*$. Then for every $s \in \prod_{j<n} l(j)$, the restriction of $f$ to $\bigcap_{j<n} U_{s(j),j}^0$ is trivial, hence the restriction of $f$ to the union of these clopen sets, $U$ is trivial. The $f$-image of the complement of $U$ is nowhere dense, and this completes the proof.

The equivalence of (3) and (4) is proved analogously, using the fact that $\beta\mathbb{N}$ is a $\beta\mathbb{N}$-space. □

## 9.2. Prerequisites

Although forcing will not be used in this section, it will be convenient to borrow its standard terminology. Two nonzero elements $a$ and $b$ of a Boolean algebra $\mathbb{B}$ are called *incompatible* if $a \wedge b = 0_{\mathbb{B}}$. A set of incompatible elements of $\mathbb{B}$ is an *antichain.*

**9.2.1. Standard form of algebras** $\mathrm{Clop}(X)$ **for a locally compact, non-compact, Polish, zero-dimensional space** $X$**.** Suppose that $X$ is a locally compact, non-compact, Polish, zero-dimensional space. Then $X$ is a direct sum of compact, zero-dimensional, metric spaces $X_n$, and the Boolean algebra $\mathrm{Clop}(X)$ is isomorphic to the product $\prod_n \mathbb{B}_n$ of countable (possibly finite) Boolean algebras, $\mathbb{B}_n = \mathrm{Clop}(X_n)$, for $n \in \mathbb{N}$. We say that

$$\mathbb{B} = \prod_n \mathbb{B}_n$$

is the *standard form* of $\mathrm{Clop}(X)$. This standard form is clearly not unique but this is of no concern. We identify each $\mathbb{B}_n$ with a principal ideal of $\mathbb{B}$. By $\bar{b} = (b_n)$ we denote a typical element of $\mathbb{B}$, and its *support* is

$$\mathrm{supp}(\bar{b}) = \{n : b_n \neq 0_{\mathbb{B}_n}\}.$$

Similarly, if $J \subseteq \mathbb{B}$ then

$$\mathrm{supp}(J) = \bigcup\{\mathrm{supp}(\bar{b}) : \bar{b} \in J\}.$$

The Boolean algebra $\mathbb{B} = \prod_n \mathbb{B}_n$ can be equipped with the metric

$$d(\bar{b}, \bar{c}) = 1/(\min\{n : b_n \neq c_n\} + 1).$$

Since the sequences with finite support are dense, $(\mathbb{B}, d)$ is a separable, complete, metric space.

For each $n$ let $\mathbb{B}_{n,j}$, for $j \in \mathbb{N}$, be an increasing sequence of finite Boolean subalgebras of $\mathbb{B}_n$ such that $\mathbb{B}_n = \bigcup_j \mathbb{B}_{n,j}$ and let $\mathbb{A}_{n,j}$ be the set of atoms of $\mathbb{B}_{n,j}$. For $f \colon \mathbb{N} \to \mathbb{N}$ let

$$\mathbb{B}_f = \prod_n \mathbb{B}_{n,f(n)}, \qquad \mathbb{A}_f = \bigcup_n \mathbb{A}_{n,f(n)}. \tag{9.1}$$

Then $\mathbb{B}_f$ is an atomic Boolean algebras and $\mathbb{A}_f$ is the set of its atoms. The algebra $\mathbb{B}_f$ is identified with $\mathcal{P}(\mathbb{A}_f)$, which is in turn identified with $\mathcal{P}(\mathbb{N})$, and $\mathbb{B}$ is the inductive limit of the directed system of its subalgebras $\mathbb{B}_f$, for $f \in \mathbb{N}^{\mathbb{N}}$.

If $\mathbb{B} = \prod_n \mathbb{B}_n$ is an algebra of clopen subsets of a locally compact, non-compact, Polish, zero-dimensional space in standard form, let=

$$\mathrm{Fin}(\mathbb{B}) = \{a \in \mathbb{B} : \mathrm{supp}(a) \text{ is finite}\}. \tag{9.2}$$

While standard form of $\mathbb{B}$ is not unique, $\mathrm{Fin}(\mathbb{B})$ does not depend on the choice of the standard form, since $\mathrm{Fin}(\mathbb{B}) = \{a \in \mathrm{Clop}(X) : a \text{ is compact}\}$.

**9.2.2. Tree-like families of sharply almost disjoint antichains.** The discussion of this section parallels that of §3.3.1. The following definition depends on the choice of standard form of $\mathbb{B}$, but this is of no concern.

**Definition 9.2.1.** Fix an algebra of clopen subsets of a locally compact, non-compact, Polish, zero-dimensional space in standard form $\mathbb{B}$ and consider the following properties of $A \subseteq \mathbb{B}$.

(Anti 1) $A$ is an antichain,
(Anti 2) $A \subseteq \bigcup_n \mathbb{B}_n$,
(Anti 3) For all $n$, $A \cap \mathbb{B}_n$ is finite.
(Anti 4) For all $n$, $A \cap \mathbb{B}_n$ is either a maximal antichain in $\mathbb{B}_n$ or empty.

Let

$$\mathrm{Anti}(\mathbb{B}) = \{A \subseteq \mathbb{B} : A \text{ satisfies (Anti 1)–(Anti 3)}\},$$
$$\mathrm{Anti}^+(\mathbb{B}) = \{A \subseteq \mathbb{B} : A \text{ satisfies (Anti 1)–(Anti 4)}\}.$$

Two antichains $A$ and $B$ in $\mathrm{Anti}(\mathbb{B})$ have *almost disjoint supports* if their supports have finite intersection. If in addition $A$ and $B$ agree on the intersection of their supports, then we say that $A$ and $B$ are sharply almost disjoint.)

Two antichains in $\mathrm{Anti}(\mathbb{B})$ such that $A \cap B$ is finite do not necessarily have almost disjoint supports. Lemma 9.2.3 below provides motivation for considering sharply almost disjoint antichains.

**Definition 9.2.2** (Perfect tree-like sharply almost disjoint families of antichains)**.** Suppose that $\mathbb{B} = \prod_n \mathbb{B}_n$ is an algebra of clopen subsets of a locally compact, non-compact, Polish, zero-dimensional space in standard form.

A *tree-like family of sharply almost disjoint antichains* in $\mathbb{B}$ is $\mathcal{A} \subseteq \text{Anti}(\mathbb{B})$ such that some ordering $\prec$ on $\mathbb{N}$ satisfies the following.

(TL1) $(\mathbb{N}, \prec)$ is a tree.
(TL2) For every $A \in \mathcal{A}$, $\bigcup\{\text{supp}(a) : a \in A\}$ is included in a branch of $(\mathbb{N}, \prec)$.
(TL3) For distinct $A$ and $A'$ in $\mathcal{A}$ the corresponding branches are distinct.
(TL4) For $A$ and $A'$ in $\mathcal{A}$ and all $n \in \text{supp}(\mathcal{A}) \cap \text{supp}(\mathcal{A}')$ we have $A \cap \mathbb{B}_n = A' \cap \mathbb{B}_n$.

Suppose that $\bar{J} = (J(s) : s \in \{0,1\}^{<\mathbb{N}})$, is a family of finite antichains in $\bigcup_n \mathbb{B}_n$ with disjoint and finite supports. For $f \in \{0,1\}^{\mathbb{N}}$ let

$$A(f) = \bigcup\nolimits_n J(s \restriction n)$$

If $f$ and $g$ are distinct elements of $\{0,1\}^{\mathbb{N}}$ then

$$\text{supp}(A(f)) \cap \text{supp}(A(g)) \subseteq \bigcup\nolimits_{n \leq \Delta(f,g)} \text{supp}(J(f \restriction n))$$

is finite, hence $A(f)$ and $A(g)$ belong to $\text{Anti}(\mathbb{B})$ and are sharply almost disjoint. Let

$$\mathcal{A}\{\bar{J}\} = \{A(f) : f \in \{0,1\}^{\mathbb{N}}\}.$$

Any family of the form $\mathcal{A}\{\bar{J}\}$ is called a *perfect tree-like sharply almost disjoint family of antichains* in in $\mathbb{B}$.

See (9.1) for $\mathbb{B}_f$.

**Lemma 9.2.3.** *Every family $\mathcal{A}$ of sharply almost disjoint antichains in* $\text{Anti}(\mathbb{B})$ *is included in $\mathbb{B}_f$ for some $f \in \mathbb{N}^{\mathbb{N}}$.*

*If $\mathcal{A}$ is a perfect tree-like sharply almost disjoint family of antichains in $\mathbb{B}$, then there is a perfect tree-like sharply almost disjoint family of antichains $\mathcal{B}$ such that every element of $\mathcal{B}$ includes $2^{\aleph_0}$ elements of $\mathcal{A}$.*

Proof. If $\mathcal{A}$ is sharply almost disjoint, then for all $A, A'$ in $\mathcal{A}$ and all $n$ such that both $A \cap \mathbb{B}_n$ and $A' \cap \mathbb{B}_n$ are nonempty we have that $A \cap \mathbb{B}_n = A' \cap \mathbb{B}_n$. Being finite, both of these intersections are included in $\mathbb{B}_{n,(n)}$ for a sufficiently large $f(n)$, and so is $A'' \cap \mathbb{B}_n$ for all other $A'' \in \mathcal{A}$. Set $f(n) = 0$ if $A \cap \mathbb{B}_n = \emptyset$ for all $A \in \mathcal{A}$. Clearly $f$ is as required.

Since $\mathbb{B}_f$ is isomorphic to $\mathcal{P}(\mathbb{N})$, the second part follows from the special case of the first part when the families are perfect tree-like and from Lemma 3.3.7. $\square$

**9.2.3. Ideals $\mathcal{J}_{\text{cont}}$ and $\mathcal{J}_\sigma$.** Definition 9.2.5 below and the idea behind it is based on Definition 6.2.2, with a few twists, the smallest one of them being the following.

**Lemma 9.2.4.** *If $\mathbb{B}$ is an algebra of clopen subsets of a locally compact, non-compact, Polish, zero-dimensional space in standard form and $A$ is an antichain of $\mathbb{B}$ such that $A \cap \mathbb{B}_n$ is finite for all $n$, then*

$$\mathcal{P}(A) \ni B \mapsto \bigvee B \in \mathbb{B}$$

*is an isomorphic embedding of $\mathcal{P}(A)$ into $\mathbb{B} \restriction \bigvee A$.*

*In particular, if $\mathcal{A}$ is a perfect tree-like almost disjoint family of antichains of $\mathbb{B}$ then $\mathcal{P}(A)$ can be identified with a subalgebra of $\mathbb{B} \restriction \bigvee A$ for every $A \in \mathcal{A}$.* $\square$

For every $A \in \mathrm{Anti}(\mathbb{B})$ identify $\mathcal{P}(A)$ with the subalgebra of $\mathbb{B}$ 'completely generated' by $A$ as in Lemma 9.2.4.

**Definition 9.2.5.** Suppose that $\mathbb{B}$ and $\mathbb{B}'$ are algebras of clopen subsets of locally compact, non-compact, Polish, zero-dimensional spaces in standard form and $\Phi\colon \mathbb{B} \to \mathbb{B}'/\operatorname{Fin}(\mathbb{B}')$ is a homomorphism. Let

$$\mathcal{J}_{\mathrm{cont}}(\Phi) = \{A \in \mathrm{Anti}^+(\mathbb{B}) : \Phi \text{ has a continuous lifting on } \mathcal{P}(A)\},$$

$$\mathcal{J}_{\sigma}(\Phi) = \{A \in \mathrm{Anti}^+(\mathbb{B}) : \Phi \text{ has a lifting on } \mathcal{P}(A) \text{ whose graph can be covered by graphs of countably many Borel functions}\}.$$

As before, we will omit the parameter $\Phi$ whenever $\Phi$ is clear from the context.

Each one of $\mathcal{J}_{\mathrm{cont}}$ and $\mathcal{J}_\sigma$ is an ideal in $\mathbb{B}$, as in §6.2.2.

If $\mathbb{B}$ is a Boolean algebra and $a \in \mathbb{B}$ then we write

$$\mathbb{B} \restriction a = \{b \in \mathbb{B} : b \leq a\}. \tag{9.3}$$

**Lemma 9.2.6.** *If $\mathbb{C}$ and $\mathbb{D}$ are Boolean algebras, $\mathbb{C}$ is countable, $\mathcal{I}$ is an ideal in $\mathbb{D}$, and $\Phi\colon \mathbb{C} \to \mathbb{D}/\mathcal{I}$, then $\Phi$ has an additive lifting.*

Proof. Since the domain is countable, a lifting can be chosen by straightforward recursive construction. Since no infinite antichain in the domain has a supremum, every lifting is completely additive. □

**Lemma 9.2.7.** *If $A \in \mathcal{J}_{\mathrm{cont}}$, $B \in \mathrm{Anti}^+(\mathbb{B})$, $A \subseteq B$ and $B \setminus A$ is finite, then $B$ belongs to $\mathcal{J}_{\mathrm{cont}}$.*

Proof. Clearly $\mathcal{J}_{\mathrm{cont}}$ is closed under finite unions and taking subsets. It therefore suffices to prove that if $A \in \mathcal{J}_{\mathrm{cont}}$, $B \in \mathrm{Anti}^+(\mathbb{B})$, $A \subseteq B$, and $B \setminus A$ is a finite subset of $\operatorname{Fin}(\mathbb{N})$ then $B \in \mathcal{J}_{\mathrm{cont}}$. Since $\mathbb{B} \restriction a$ is countable, this follows from Lemma 9.2.6. □

The following is both an analog of the Radon–Nikodym property of Fin (Theorem 4.1.2) and its consequence.

**Lemma 9.2.8.** *Suppose that $\mathbb{B}$ and $\mathbb{B}'$ are algebras of clopen subsets of locally compact, non-compact, Polish, zero-dimensional spaces in standard form and $\Phi\colon \mathbb{B} \to \mathbb{B}'/\operatorname{Fin}(\mathbb{B}')$ is a homomorphism. For every $A \in \mathcal{J}_{\mathrm{cont}}$ there is a completely additive lifting of $\Phi$ on $\mathcal{P}(A)$.*

Proof. Let $F\colon \mathcal{P}(A) \to \mathbb{B}'$ be a continuous lifting of $\Phi$ on $\mathcal{P}(A)$. Its range is compact and therefore included in $\mathbb{B}'_f$ for some $f \in \mathbb{N}^{\mathbb{N}}$. Therefore, we have a homomorphism from $\mathcal{P}(\mathbb{N})$ to $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ with a continuous lifting, and the conclusion follows by Theorem 4.1.2. □

### 9.2.4. Ccc over $\operatorname{Fin}(\mathbb{B})$.

See Definition 9.2.1 for almost disjointness and sharply almost disjointness of antichains in $\mathrm{Anti}^+(\mathbb{B})$.

**Definition 9.2.9.** Suppose that $\mathbb{B}$ is an algebra of clopen subsets of a locally compact, non-compact, Polish, zero-dimensional space in standard form. An ideal $\mathcal{I}$ in $\mathbb{B}$ is *ccc over* $\operatorname{Fin}(\mathbb{B})$ if there is no uncountable family of $\mathcal{I}$-positive, pairwise disjoint modulo $\operatorname{Fin}(\mathbb{B})$, sets.

As in Definition 9.1.5, if $X$ is a locally compact space, a subset $Z$ of $X^*$ is *relatively ccc* if there is no uncountable family of disjoint open subsets of $X^*$ each of which intersects $Z$ nontrivially.

**Lemma 9.2.10.** *Suppose that $X$ is a locally compact, non-compact, Polish space and $Z \subseteq X^*$.*

*(1) If the ideal $\mathcal{I} = \{A \in \mathrm{Clop}(X) : A^* \cap Z = \emptyset\}$ intersects every perfect tree-like sharply almost disjoint family in $\mathrm{Clop}(X)$ then $Z$ is nowhere dense.*
*(2) If $Z$ is relatively ccc then it is nowhere dense.*

Proof. (1) Assume $Z$ is not nowhere dense and let $U \in \mathrm{Clop}(X)$ be such that $U^* \subseteq Z$. Write $X = \bigcup_n K_n$ as an increasing union of compacg clopen subsets. Then $U \cap (K_{n+1} \setminus K_n)$ is nonempty for infinitely many $n$. Enumerate these nonempty sets as $V_s$, for $s \in \{0,1\}^{<\mathbb{N}}$. Then $\{\bigcup V_{s\restriction n} : n \in \mathbb{N}\}$ is a perfect tree-like almost disjoint family in $\mathrm{Clop}(X)$ disjoint from $\mathcal{I}$.

(2) is a consequence of (1). □

## 9.3. Proof of Theorem 9.1.4

This proof is spread over five subsections with suggestive titles.

### 9.3.1. Reduction to a Boolean-algebraic statement and the strategy for its proof.

**Proposition 9.3.1.** *The first of the following statements implies the second and the third.*

*(1) If $\mathbb{B}$ and $\mathbb{B}'$ are algebras of clopen subsets of locally compact, non-compact, Polish, zero-dimensional spaces in standard form then every homomorphism $\Phi\colon \mathbb{B} \to \mathbb{B}'/\mathrm{Fin}(\mathbb{B}')$ has a completely additive on antichains lifting on an ideal which is ccc over $\mathrm{Fin}(\mathbb{B})$.*
*(2) For all $m, n$, and locally compact, non-compact, Polish, zero-dimensional spaces $X_i$, for $i < m$ and $Y_j$, for $j < n$, every continuous*

$$f\colon \prod_{i<m} X_i^* \to \prod_{j<n} Y_j^*$$

*is almost trivial.*
*(3) For all $m, n$, and locally compact, non-compact, Polish, zero-dimensional spaces $X_i$, for $i < m$, every continuous $f\colon \prod_{i<m} X_i^* \to (\beta\mathbb{N})^n$ is almost trivial.*

Proof. Proposition 9.1.7 implies that (2) is equivalent to its special case when $m = n = 1$ and that (3) is equivalent to its special case when $n = 1$. The implication from (1) to (2) and (3) now follow by using the Stone duality. To wit, the category of locally compact, non-compact, Polish, zero-dimensional spaces is equivalent to the category of Boolean algebras, as in §9.2.1. If $X$ is locally compact, non-compact, Polish, zero-dimensional, then $\beta X$ is the Stone space of $\mathrm{Clop}(X)$, which is a product of countable (possibly finite) Boolean algebras. Also, $\beta X \setminus X$ is the Stone space of $\mathrm{Clop}(X)/\mathrm{Cpct}(X)$, and $\mathrm{Cpct}(X) = \mathrm{Fin}(\mathrm{Clop}(X))$. Triviality of Boolean algebra homomorphisms corresponds to triviality of continuous maps, and the conclusion follows by Lemma 9.2.10. □

The following will be proved in §9.3.2.

**Proposition 9.3.2.** *Suppose that $\Phi\colon \mathbb{B} \to \mathbb{B}'/\mathrm{Fin}(\mathbb{B}')$, $B = \bigsqcup_n B_n$ is in $\mathscr{J}_\sigma$ and the $B_n$'s are elements of $\mathrm{Anti}^+(\mathbb{B})$ with disjoint supports. Then all but finitely many of the $B_n$'s belong to $\mathscr{J}_{\mathrm{cont}}$.*

The following will be proved in §9.3.4.

**Lemma 9.3.3.** *Assume that* $\mathrm{OCA_T}$ *holds,* $\mathbb{B}$ *and* $\mathbb{B}'$ *are algebras of clopen subsets of locally compact, non-compact, Polish, zero-dimensional spaces in standard form, and* $\Phi\colon \mathbb{B}\to\mathbb{B}'/\operatorname{Fin}(\mathbb{B}')$ *is a homomorphism. Then the following statements hold.*

(1) *The ideal* $\mathcal{J}_\sigma$ *intersects every uncountable tree-like sharply almost disjoint family of antichains nontrivially. In particular,* $\mathcal{J}_{\mathrm{cont}}\cap\mathbb{B}_f$ *is nonmeagre for all* $f\in\mathbb{N}^{\mathbb{N}}$.
(2) *The ideal* $\mathcal{J}_{\mathrm{cont}}$ *intersects every perfect tree-like sharply almost disjoint family of antichains nontrivially.*
(3) *If in addition* $\mathrm{MA}(\sigma\text{-linked})$ *holds, then* $\mathcal{J}_{\mathrm{cont}}$ *intersects every uncountable sharply almost disjoint family of antichains nontrivially.*

The following will be proved in §9.3.6.

**Theorem 9.3.4.** *Assume* $\mathrm{OCA_T}$. *Suppose that* $\mathbb{B}$ *and* $\mathbb{B}'$ *are algebras of clopen subsets of locally compact, non-compact, Polish, zero-dimensional spaces in standard form and* $\Phi\colon \mathbb{B}\to\mathbb{B}'/\operatorname{Fin}(\mathbb{B}')$ *is a homomorphism. Then there is a completely additive on antichains lifting of* $\Phi$ *on* $\mathcal{J}_{\mathrm{cont}}$.

Once proved, together with Proposition 9.3.1 these three facts will complete the proof of Theorem 9.1.4.

**9.3.2. From $\sigma$-Borel to continuous.** A bit of notation will be useful later on. As in §9.2.1, let $\mathbb{B}_{n,j}$, for $j\in\mathbb{N}$, be an increasing sequence of finite Boolean subalgebras of $\mathbb{B}_n$ such that $\mathbb{B}_n=\bigcup_j\mathbb{B}_{n,j}$ and let $\mathbb{B}'_{n,j}$, for $j\in\mathbb{N}$, be an increasing sequence of finite Boolean subalgebras of $\mathbb{B}'_n$ such that $\mathbb{B}'_n=\bigcup_j\mathbb{B}'_{n,j}$. Use these objects to define $\mathbb{A}_{n,j}$, $\mathcal{A}'_{n,j}$, $\mathbb{B}_f$, $\mathbb{A}_f$, $\mathbb{B}'_f$, and $\mathcal{A}'_f$ as in (9.1) and the paragraph preceding it:

$$\begin{aligned}
\mathbb{A}_{n,j}&=\{a: a\text{ is an atom in }\mathbb{B}_{n,j}\}, &\qquad \mathbb{A}_f&=\textstyle\bigcup_n\mathbb{A}_{n,f(n)}.\\
\mathbb{A}'_{n,j}&=\{a: a\text{ is an atom in }\mathbb{B}'_{n,j}\}, & \mathbb{A}'_f&=\textstyle\bigcup_n\mathbb{A}'_{n,f(n)},\\
\mathbb{B}_f&=\textstyle\prod_n\mathbb{B}_{n,f(n)}, & \mathbb{B}'_f&=\textstyle\prod_n\mathbb{B}'_{n,f(n)}.
\end{aligned}$$

We now prove an analog (and a consequence) of Proposition 6.4.1.

Proof of Proposition 9.3.2. Fix $\Phi\colon\mathbb{B}\to\mathbb{B}'/\operatorname{Fin}(\mathbb{B}')$, $B\in\mathcal{J}_\sigma$, and $B=\bigsqcup_n B_n$ where the $B_n$'s are elements of $\mathrm{Anti}^+(\mathbb{B})$ with disjoint supports. Fix C-measurable functions $\Theta_n$ whose graphs cover the graph of a lifting of $\Phi$. Let $\mathcal{Y}\subseteq\mathcal{P}(B)$ be a comeagre set such that the restriction of $\Theta_n$ to $\mathcal{Y}$ is continuous for all $n$ (Lemma A.1.1). By Corollary 3.1.4 there are a partition $B=A_0\sqcup A_1$, $C_0\subseteq A_0$, and $C_1\subseteq A_1$ such that for every $X\subseteq B$ both $(X\cap A_0)\cup C_1$ and $(X\cap A_1)\cup C_0$ belong to $\mathcal{Y}$. Therefore the function

$$\Theta_{mn}(X)=\Theta_m((X\wedge A_0)\vee C_1)\wedge\Phi_*(A_0)\vee\Theta_n((X\wedge A_1)\vee C_0)\wedge\Phi_*(A_1)$$

is continuous for all $m,n$ and the graphs of these functions cover a lifting of $\Phi$. Since each $\Theta_{mn}$ is continuous, the $\Phi_{mn}$-image of the compact set $\mathcal{P}(B)$ is a compact subset of $\mathbb{B}'$ and therefore included in $\mathbb{B}'_{g(m,n)}$ for some $g(m,n)\in\mathbb{N}^{\mathbb{N}}$. This countable family in $\mathbb{N}^{\mathbb{N}}$ can be dominated modulo finite by a single function $g$. Replace $\Phi_{mn}(\cdot)$ with $\Phi_{mn}(\cdot)\wedge[k,\infty)$ with $k$ large enough to have $g(m,n)\leq^k g$. Graphs of the new functions still cover the graph of a lifting of $\Phi$ on $\mathcal{P}(B)$.

The range of the restriction of $\Phi$ to $\mathcal{P}(B)$ is included in $\mathbb{B}'_g/(\operatorname{Fin}(\mathbb{B}')\cap\mathbb{B}'_g)$, which is an isomorphic copy of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$. We can therefore apply Proposition 6.4.1 to this restriction and conclude that all but finitely many of the $B_n$ belong to $\mathcal{J}_{\mathrm{cont}}$. $\square$

**9.3.3. Local liftings.** The following is the analog of Definition 6.3.3.

**Definition 9.3.5** (The set $\mathcal{X}_\mathcal{A}$ and the partition $K^{\Phi_*,\mathcal{A},n}$)**.** Suppose that $\mathbb{B}$ is an algebra of clopen subsets of a locally compact, non-compact, Polish, zero-dimensional space in standard form and $\mathcal{A}$ is a tree-like sharply almost disjoint family of antichains in $\mathbb{B}$. Let

$$\hat{\mathcal{A}} = \{\bigvee F : (\exists A \in \mathcal{A}) F \subseteq A \text{ is infinite}\}.$$

For every $b \in \hat{\mathcal{A}}$ there is a unique $A = A(b)$ in $\mathcal{A}$ such that $b \leq \bigvee A$. Let

$$\mathcal{X}_\mathcal{A} = \{(c, b) : b, c \in \hat{\mathcal{A}}, c \leq B\}.$$

For $x = (c, b)$ in $\mathcal{X}_\mathcal{A}$ we write $c = c(x)$, $b = b(x)$, and $A(b(x)) = A(x)$.

If in addition $\mathbb{B}'$ is an algebra of clopen subsets of a locally compact, non-compact, Polish, zero-dimensional space in standard form, $\Phi_* \colon \mathbb{B} \to \mathbb{B}'$, and $n \in \mathbb{N}$, then we define

$$[\mathcal{X}_\mathcal{A}]^2 = K_0^{\Phi_*,\mathcal{A},n} \cup K_1^{\Phi_*,\mathcal{A},n}$$

by setting $\{x, y\} \in K_0^{\Phi_*,\mathcal{A},n}$ if the following three conditions hold

$K_0^n$(i) $A(x) \neq A(y)$.
$K_0^n$(ii) $b(x) \wedge c(y) = c(x) \wedge b(y)$.
$K_0^n$(iii) $\operatorname{supp}(\Phi_*(b(x)) \wedge \Phi_*(c(y)))\Delta(\Phi_*(c(x)) \wedge \Phi_*(b(y)))) \not\subseteq n$.

Note that only ($K_0$(iii)) depends on $n$ and that $K_0^{\Phi_*,\mathcal{A},n} \supseteq K_0^{\Phi_*,\mathcal{A},n+1}$ for all $n$.

Endow $\mathcal{X}_\mathcal{A}$ with a separable metric topology $\tau^{\Phi_*,\mathcal{A}}$ by identifying it with a subspace of the Polish space $\mathbb{B}^2 \times \mathcal{A}(\mathbb{B}) \times (\mathbb{B}')^2$, $x \in \mathcal{X}_\mathcal{A}$ by associating the following to each $x \in \mathcal{X}_\mathcal{A}$

$$(c(x), b(x), A(x), \Phi_*(C(x)), \Phi_*(b(x))).$$

**Lemma 9.3.6.** *The set $K_0^{\Phi_*,\mathcal{A},\mathcal{K}}$ is a $\tau^{\Phi_*,\mathcal{A}}$-open subset of $[\mathcal{X}_\mathcal{A}]^2$.*

PROOF. Conditions ($K_0^n$(i)) and ($K_0^n$(iii)) are clearly open. The symmetric difference of $b(x) \cap c(y)$ and $b(y) \cap c(x)$ is included in $A(x) \cap A(y)$, but since the family $\mathcal{A}_0$ is tree-like, this is a finite set determined by the witnesses for ($K_0^n$(i)). Thus the condition ($K_0^n$(ii)) is open relative to ($K_0^n$(i)), and the conjunction of all three conditions defines an open partition. ☐

**9.3.4. A local version of Theorem 9.1.4.** The proof of Lemma 9.3.3 is similar to Lemma 6.3.2, and initial segments of their proofs are analogous. Subtle differences justify writing it out in full. The following is an analog of Definition 6.3.6.

**Definition 9.3.7.** For $a, b$ in $\mathbb{B}$ or $\mathbb{B}'$ it will be convenient to write

$$a =^n b \text{ if } \operatorname{supp}(a \Delta b) \subseteq n \text{ and}$$

$$a =^{\restriction n} b \text{ if } \operatorname{supp}(a \Delta b) \cap n = \emptyset.$$

This notation is extended to tuples of the same length in natural fashion.

PROOF OF LEMMA 9.3.3. Fix a homomorphism $\Phi \colon \mathbb{B} \to \mathbb{B}'/\operatorname{Fin}(\mathbb{B}')$ and a lifting $\Phi_*$ of $\Phi$.

(1) Fix an uncountable tree-like sharply almost disjoint family of antichains $\mathcal{A}$. We need to prove that $\mathcal{J}_\sigma \cap \mathcal{A}$ is nonempty. Write $K_0^n$ for $K_0^{\Phi_*,\mathcal{A},n}$. This is an open partition of $[\mathcal{X}_\mathcal{A}]^2$ in an appropriate separable metric topology by Lemma 9.3.6.

**Claim 9.3.8.** *There are no uncountable $Z \subseteq \{0,1\}^\mathbb{N}$ and function $f \colon Z \to \mathcal{X}_\mathcal{A}$ such that $\{f(z), f(z')\} \in K_0^{\Delta(z,z')}$ for all distinct $z, z'$ in $Z$.*

Proof. Assume otherwise and fix $Z$ and $f$. By ($K_0^n$(i)) and ($K_0^n$(ii)), for all distinct $z$ and $z'$ in $Z$ we have $A(f(z)) \neq A(f(z'))$ and

$$b(f(z)) \wedge c(f(z')) = c(f(z)) \wedge b(f(z')).$$

Since $c(f(z))$ and $c(f(z'))$ agree on the intersection of their supports for all $z$ and $z'$ in $Z$ and these supports are finite and disjoint, $c = \bigvee_{z\in Z} c(f(z))$ is well-defined. Then $c \wedge b(f(z)) = c(f(z))$ for all $z \in Z$. Since $\Phi_*$ is a lifting of $\Phi$ we have

$$(\Phi_*(c) \wedge \Phi_*(b(f(z))))\Delta\Phi_*(c(f(z))) \in \mathrm{Fin}(\mathbb{B}').$$

Since $Z$ is uncountable, there is $n \in \mathbb{N}$ such that the set $Z'$ of all of $z \in Z$ satisfying

$$(\Phi_*(c) \wedge \Phi_*(c(f(z)))) =^n \Phi_*(b(f(z)))$$

is uncountable. Thus for all $z$ and $z'$ in $Z'$ we have

$$\begin{aligned}\Phi_* b(f(z)) \wedge \Phi_*(c(f(z'))) &=^n \Phi_*(b(f(z))) \wedge \Phi_*(b(f(z') \wedge c)) \\ &=^n \Phi_*(c(f(z))) \wedge \Phi_*(b(f(z'))).\end{aligned}$$

Therefore $\Phi_*(b(f(z))) \wedge \Phi_*(c(f(z'))) =^n \Phi_*(c(f(z))) \wedge \Phi_*(b(f(z')))$. Since $Z'$ is uncountable, there are $z$ and $z'$ in $Z'$ such that $\Delta(z, z') > n$. Therefore ($K_0^n$(iii)) fails for $z$ and $z'$ and $\{f(z), f(z')\} \notin K_0^{\Delta(z,z')}$; contradiction. □

Since $\mathrm{OCA}_\infty$ is equivalent to $\mathrm{OCA_T}$ (Theorem A.3.5), Claim 9.3.8 implies that there are sets $\mathcal{X}_n$, for $n \in \mathbb{N}$, such that $\mathcal{X}_{\mathcal{A}} = \bigcup_n \mathcal{X}_n$ and $[\mathcal{X}_n]^2 \subseteq K_1^n$ for all $n$. Let $\mathcal{D}_n \subseteq \mathcal{X}_n$ be a countable and $\tau^{\Phi_*,\mathcal{A}}$ dense set. Since $\mathcal{A}$ is uncountable, there is

$$\tilde{A} \in \mathcal{A} \setminus \{A(x) : x \in \textstyle\bigcup_n \mathcal{D}_n\}.$$

Since $\tilde{A} \cap \mathbb{B}_n$ is finite for all $n$, every $F \subseteq \tilde{A}$ has a supremum in $\mathbb{B}$ and we can identify the Boolean subalgebra $\{\bigvee F : F \subseteq \tilde{A}\}$ of $\mathbb{B}$ with $\mathcal{P}(\tilde{A})$.

For $m \in \mathbb{N}$ define

$$\begin{aligned} m^+ = \min\{l > m : &(\forall n \leq m)(\forall x \in \mathcal{X}_n) A(x) = \tilde{A} \\ &\Rightarrow \quad (\exists d \in \mathcal{D}_n) \operatorname{supp}(b(d) \wedge \textstyle\bigvee \tilde{A}) \subseteq l \text{ and } (c(d), b(d)) =^{\restriction m} (c(x), b(x))\}\}.\end{aligned}$$

Because $\mathcal{D}_n$ is dense in $\mathcal{X}_n$ for every $n$, $\tilde{A} \neq A(x)$ for all $x \in \bigcup_j \mathcal{D}_j$, and the set $\mathcal{P}(\tilde{A}) \cap \prod_{j<m} \mathbb{B}_j$ is finite, there are only finitely many possibilities for the pair $(c(x), b(x)) \restriction m$, and $m^+$ is finite for every $m$. Recursively define $m(j)$ for $j \in \mathbb{N}$ by

$$m(0) = 0 \text{ and } m(i+1) = m(i)^+ + 1 \text{ for all } i.$$

This is a strictly increasing sequence, and we let

$$\begin{aligned} B_0 &= \textstyle\bigcup_i \bigcup_{j=m(2i)}^{m(2i+1)-1} (\mathbb{B}_j \cap \tilde{A}), & B_1 &= \tilde{A} \setminus B_0, \\ b_0 &= \textstyle\bigvee B_0, & \text{and } b_1 &= \textstyle\bigvee B_1,\end{aligned}$$

so that $B_0 \sqcup B_1 = \tilde{A}$ is a partition of $\tilde{A}$ and $b_0 \vee b_1 = \bigvee \tilde{A}$ is a partition of $\bigvee \tilde{A}$. We will prove that $B_j \in \mathcal{J}_\sigma$ for $j = 0, 1$.

For each $n$ let

$$\begin{aligned}\mathcal{Z}(n) = \{(x, y) \in \mathcal{P}(B_0) \times \mathbb{B}' : &(\forall j \geq n)(\exists d \in \mathcal{D}_n) \operatorname{supp}(b(d) \wedge \textstyle\bigvee \tilde{A}) \subseteq m(2j+2) \\ &\text{and } (c(d), b(d)) =^{\restriction m(2j+1)} (x, b_0)\}.\end{aligned}$$

**Claim 9.3.9.** *Suppose that $x \in \mathcal{X}_n$ and $b(x) = B_0$. Then the following holds.*

*(1)* $(c(x), \Phi_*(c(x))) \in \mathcal{Z}(n)$.

*(2) If* $(c(x), y) \in \mathcal{Z}(n)$ *then* $(\Phi_*(b_0) \wedge y) =^n (\Phi_*(c(x)) \wedge \Phi_*(b_0))$.

Proof. If $x \in \mathcal{X}_n$, $B(x) = B_0$, and $m(2j+1) \geq n$ then since $\mathcal{D}_n$ is $\tau^{\mathcal{A},\Phi_*}$-dense in $\mathcal{X}_n$, some $d \in \mathcal{D}_n$ satisfies $\operatorname{supp}(b(d) \wedge \bigvee \tilde{A}) \subseteq m(2j+2)$ and

$$(c(d), b(d)) =^{\restriction m(2j+1)} (c(x), b_0). \tag{9.4}$$

Since $j \geq n$ was arbitrary, $(c(x), \Phi_*(c(x)) \in \mathcal{Z}(n)$ follows.

To prove the second part of the claim, towards contradiction suppose $x \in \mathcal{X}_n$, $b(x) = b_0$, $y \in \mathbb{B}'$, $(c(x), y) \in \mathcal{Z}(n)$, but

$$\operatorname{supp}((\Phi_*(b_0) \wedge y)\Delta(\Phi_*(c(x)) \wedge \Phi_*(b_0))) \nsubseteq n.$$

Fix $j \geq n$ large enough to have

$$\operatorname{supp}((\Phi_*(b_0) \wedge y)\Delta(\Phi_*(c(x)))) \wedge \Phi_*(b_0))) \cap [n, m(2j+1)) \neq \emptyset. \tag{9.5}$$

Since $(c(x), y) \in \mathcal{Z}_n$, some $d \in \mathcal{D}_n$ satisfies $\operatorname{supp}(b(d) \wedge \bigvee \tilde{A}) \subseteq m(2j+2)$ and

$$(c(d), b(d)) =^{\restriction m(2j+1)} (c(x), b_0). \tag{9.6}$$

As $\operatorname{supp}(b_0)$ is disjoint from $[m(2j+1), m(2j+2))$ and $b(x) = b_0$, we have that $b_0 \wedge c(d) = c(x) \wedge b(d)$. As $\{x, d\} \in K_1^n$ and $A(d) \neq A(x)$, we have

$$\operatorname{supp}((\Phi_*(b_0) \wedge \Phi_*(c(d)))\Delta(\Phi_*(c(x)) \wedge \Phi_*(b(d)))) \subseteq n.$$

Together with (9.6) and (9.4) this implies

$$\operatorname{supp}((\Phi_*(b_0) \wedge y)\Delta((\Phi_*(c(x)) \cap \Phi_*(b_0))) \cap [n, m(2j+1)) = \emptyset,$$

contradicting (9.5). □

We claim that each $\mathcal{Z}(n)$ is a Borel subset of $\mathcal{P}(\tilde{A}) \times \mathbb{B}'$ (recall that $\mathbb{B}'$ carries a natural Polish topology). For a fixed $d \in \mathcal{D}_n$ and $j \in \mathbb{N}$ the set

$$\begin{aligned} \mathcal{Z}(n, d, j) = \{(x, y) \in \mathcal{P}(B_0) \times \mathbb{B}' : & \operatorname{supp}(b(d) \cap \textstyle\bigvee \tilde{A}) \subseteq m(2j+2) \text{ and} \\ & (c(d), b(d)) =^{\restriction m(2j+1)} (x, b_0)\} \end{aligned}$$

is closed. Thus $\mathcal{Z}(n) = \bigcap_j \bigcup_{d \in \mathcal{D}_n} \mathcal{Z}(n, d, j)$ is an $F_{\sigma\delta}$ set. By the Jankov, von Neumann theorem (Theorem A.1.2) there is a C-measurable function $\Theta_n$ whose domain includes $\{c(x) : x \in \mathcal{X}_n, b(x) = b_0\}$ such that $(x, \Theta_n(x)) \in \mathcal{Z}(n)$ for all $x$ in the domain of $\Theta_n$. Let $\tilde{\Theta}_n(\cdot) = \Theta_n(\cdot) \cap \Phi_*(b_0)$. Since $\mathcal{X}_{\mathcal{A}} = \bigcup_n \mathcal{X}_n$, Claim 9.3.9 implies that (as before, identifying $\mathcal{P}(B_0)$ with a subalgebra of $\mathbb{B}$) for every $x \subseteq B_0$ there is $n$ such that $\operatorname{supp}(\Phi_*(\bigvee x)\Delta\tilde{\Theta}_n(x)) \subseteq n$. Therefore, $B_0$ belongs to $\mathcal{J}_\sigma$, as required.

Analogous argument shows that $B_1 \in \mathcal{J}_\sigma$ and therefore $\tilde{A} \in \mathcal{J}_\sigma$, as promised. Since $\tilde{A} \in \mathcal{A} \setminus \{A(d) : d \in \bigcup_n \mathcal{D}_n\}$ was arbitrary, for every uncountable tree-like sharply almost disjoint family $\mathcal{A}$ all but countably many elements of $\mathcal{A}$ belong to $\mathcal{J}_\sigma$. This proves the first part of Lemma 9.3.3 (1).

It remains to prove that $\mathcal{J}_{\text{cont}} \cap \mathbb{B}_f$ is relatively nonmeagre in $\mathbb{B}_f$. Otherwise, by Corollary 3.2.3 there are disjoint finite subsets $S_n$, for $n \in \mathbb{N}$ of $\mathbb{A}_f$ such that for every infinite $X \subseteq \mathbb{N}$, $\bigcup_{n \in X} S_n$ does not belong to $\mathcal{J}_{\text{cont}}$. One can recursively find finite subsets $J_n$ of $\mathbb{A}_f$ such that each $J_n$ includes some $S_m$, and it is a union of maximal antichains in $\mathbb{A}_j$, for $j$ in some finite set $R_n$. Re-enumerate $J_n$ as $J_s$, for $s \in \{0,1\}^{<\mathbb{N}}$. Then $\mathcal{A}(J)$ is a perfect tree-like sharply almost disjoint family disjoint from $\mathcal{J}_{\text{cont}}$; contradiction.

(2) Fix a perfect tree-like sharply almost disjoint family $\mathcal{B}$. We need to prove that $\mathscr{J}_{\rm cont} \cap \mathcal{B}$ is nonempty. By the second part of Lemma 9.2.3, there is a perfect tree-like almost disjoint family $\mathcal{A}$ such that every $A \in \mathcal{A}$ includes infinitely many elements of $\mathcal{B}$. By (1), all but countably many elements of $\mathcal{A}$ belong to $\mathscr{J}_\sigma$, and this is witnessed by continuous functions into $\mathbb{B}'_g$ for some $g \in \mathbb{N}^{\mathbb{N}}$. Because we have $\mathbb{B}'_g \cong \mathcal{P}(\mathbb{N})$ and $\mathbb{B}'/(\mathrm{Fin}(\mathbb{B}') \cap \mathbb{B}'_g) \cong \mathcal{P}(\mathbb{N})/\mathrm{Fin}$, Proposition 9.3.2 implies that all but finitely many $A$ belong to $\mathscr{J}_{\rm cont}$, as required.

(3) Assume MA($\sigma$-linked) holds and fix an uncountable sharply almost disjoint family $\mathcal{A}$ of antichains. We need to prove that $\mathscr{J}_{\rm cont} \cap \mathcal{A}$ is nonempty. Lemma 9.2.3 implies that $\mathcal{A}$ is included in $\mathbb{B}_f$ for some $f$. Since $\mathbb{B}_f$ is isomorphic to $\mathcal{P}(\mathbb{N})$ and elements of $\mathcal{A}$ are almost disjoint when considered as subsets of $\mathbb{A}_f$, by MA($\sigma$-linked) and Lemma A.5.4 there are an uncountable almost disjoint family $\mathcal{B}$ such that for every $B \in \mathcal{B}$ the set $\{A \in \mathcal{A} : A \subseteq^* B\}$ is infinite, and there is a partition $B = B_0 \cup B_1$ such that each one of $\mathcal{A}_0 = \{A_0 : A \in \mathcal{A}\}$ and $\mathcal{A}_1 = \{A_1 : A \in \mathcal{A}\}$ is a tree-like almost disjoint family. By applying (1) to $\mathcal{A}_0$ and to $\mathcal{A}_1$, all but countably many $B \in \mathcal{B}$ belong to $\mathscr{J}_\sigma$, and Proposition 9.3.2 implies that $\mathscr{J}_{\rm cont}$ intersects $\mathcal{A}$ nontrivially. $\square$

**9.3.5. A smallness property.** The following simpleminded and not obviously useful idea related to the second part of Lemma 9.3.3 (1), will be the punchline in the proof of Theorem 9.1.4.

**Lemma 9.3.10.** *Suppose $\mathbb{B}$ is an algebra of clopen subsets of a locally compact, non-compact, Polish, zero-dimensional space in standard form. For $\mathcal{X} \subseteq \mathrm{Anti}^+(\mathbb{B})$ let*

$$\mathsf{NM}(\mathcal{X}) = \{f : \mathcal{X} \cap \mathrm{Anti}^+(\mathbb{B}_f) \textit{ is relatively nonmeagre in } \mathrm{Anti}^+(\mathbb{B}_f)\}.$$

*Then the set*

$$\mathcal{I}(\mathrm{Anti}^+(\mathbb{B})) = \{\mathcal{X} \subseteq \mathrm{Anti}^+(\mathbb{B}) : \mathsf{NM}(\mathcal{X}) \textit{ is not } \leq^*\textit{-cofinal in } \mathbb{N}^{\mathbb{N}}\}$$

*is a $\sigma$-ideal of subsets of $\mathrm{Anti}^+(\mathbb{B})$.*

Proof. It suffices to prove that $\mathcal{I} = \mathcal{I}(\mathrm{Anti}^+(\mathbb{B}))$ is closed under countable unions. Fix $\mathcal{X}_n$ in $\mathcal{I}$ for $n \in \mathbb{N}$. Since $\mathrm{Anti}^+(\mathbb{B}_f)$ is a Polish space for every $f$, we have that $\mathsf{NM}(\bigcup_n \mathcal{X}_n) = \bigcup_n \mathsf{NM}(\mathcal{X}_n)$. For each $n$ there is $f_n \in \mathbb{N}^{\mathbb{N}}$ such that $f_n \not\leq^* g$ for all $g \in \mathsf{NM}(\mathcal{X}_n)$. Let $f$ be such that $f \geq^* f_n$ for all $n$. If $g \geq^* f$ then $g \notin \bigcup_n \mathsf{NM}(\mathcal{X}_n) = \mathsf{NM}(\bigcup_n \mathcal{X}_n)$, showing that $\bigcup_n \mathcal{X}_n \in \mathcal{I}(\mathrm{Anti}^+(\mathbb{B}))$. $\square$

**9.3.6. Biba's trick and uniformisation.** According to the strategy laid out in §9.3.1, in order to complete the proof of Theorem 9.1.4 it remains to prove Theorem 9.3.4. Fix locally compact, non-compact, Polish, zero-dimensional spaces $X$ and $Y$, let $\mathbb{B} = \mathrm{Clop}(X)$ and $\mathbb{B}' = \mathrm{Clop}(Y)$ and fix a homomorphism $\Phi\colon \mathbb{B} \to \mathbb{B}'/\mathrm{Fin}(\mathbb{B}')$. By $\mathrm{OCA_T}$, Lemma 9.3.3 implies that $\mathscr{J}_{\rm cont}$ intersects every perfect sharply almost disjoint family of antichains nontrivially. We will find a lifting of $\Phi$ on $\mathscr{J}_{\rm cont}$ which is completely additive on antichains.

For each $A \in \mathscr{J}_{\rm cont}$ the restriction of $\Phi$ to $\mathcal{P}(A)$ has a continuous lifting, and by Lemma 9.2.8 it has a lifting of the form

$$\Psi_A\left(\bigvee X\right) = \bigvee g_A[X], \text{ for } X \subseteq A \tag{9.7}$$

for some $g_A\colon A \to \mathbb{B}'$ such that $g_A[A]$ is an antichain of $\mathbb{B}'$. This antichain is not required to be in $\mathrm{Anti}(\mathbb{B}')$ (and we cannot even expect that $g_A[A] \subseteq \bigcup_n \mathbb{B}_n$), but because each one of its subsets has a supremum in $\mathbb{B}'$, it automatically has the property that for every $m$ only finitely many $a \in A$ satisfy $g_A(a) \wedge 1_{\mathbb{B}'_m} \neq 0_{\mathbb{B}'}$.

The first step will be to verify that the family $g_A$, for $A \in \mathcal{J}_{\text{cont}}$, has an appropriate coherence property. This is Claim 9.3.13, a (more involved) analog of Claim 6.5.1 appropriate in the present context. The following lemma is of the 'surely known, but easier to prove than find a reference' brand.

**Lemma 9.3.11.** *Suppose that $\mathbb{B}$ and $\mathbb{B}'$ are finite Boolean algebras and that $\mathbb{C}$ and $\mathbb{D}$ are subalgebras of $\mathbb{B}$ that generate $\mathbb{B}$ such that $1_{\mathbb{B}} \in \mathbb{C} \cap \mathbb{D}$. Suppose moreover that $\varphi\colon \mathbb{C} \to \mathbb{B}'$ and $\psi\colon \mathbb{D} \to \mathbb{B}'$ are homomorphisms. Then the following are equivalent.*

*(1) There is a homomorphism $\theta\colon \mathbb{B} \to \mathbb{B}'$ that extends both $\varphi$ and $\psi$.*

*(2) $\varphi(1_{\mathbb{B}}) = \psi(1_{\mathbb{B}})$ and for all $c \in \mathbb{C}$ and $d \in \mathbb{D}$ we have that $c \wedge d = 0_{\mathbb{B}}$ implies $\varphi(c) \wedge \psi(d) = 0_{\mathbb{B}'}$,*

Proof. Only the converse implication requires a proof. Assume (2). Let $\mathbb{C}_0$ and $\mathbb{D}_0$ be the sets of atoms of $\mathbb{C}$ and $\mathbb{D}$, respectively. For $b \in \mathbb{B}$ define

$$\theta(b) = \bigvee\{\varphi(c) \wedge \psi(d) : c \in \mathbb{C}_0, d \in \mathbb{D}_0, c \wedge d \leq b\}.$$

Then $\theta(1_{\mathbb{B}}) = \bigvee_{c \in \mathbb{C}_0, d \in \mathbb{D}_0} \varphi(c) \wedge \psi(d) = \bigvee_{c \in \mathbb{C}_0} \varphi(c) \wedge \bigvee_{d \in \mathbb{D}_0} \psi(d) = 1_{\mathbb{B}'}$.

Since $\mathbb{C}$ and $\mathbb{D}$ generate $\mathbb{B}$, the atoms of $\mathbb{B}$ are nonzero elements of the form $c \wedge d$ where $c \in \mathbb{C}_0$ and $d \in \mathbb{D}_0$. Therefore, for $b, e \in \mathbb{B}$, $c \in \mathbb{C}_0$, and $d \in \mathbb{D}_0$ we have that $c \wedge d \leq b \vee e$ if and only $c \wedge d \leq b$ or $c \wedge d \leq e$ and therefore $\theta(b \vee e) = \theta(b) \vee \theta(e)$.

It remains to prove that $\theta(b^{\complement}) = \theta(b)^{\complement}$ for all $b \in \mathbb{B}$, and this is where we need (2). If $(c, d) \in \mathbb{C}_0 \times \mathbb{D}_0$ are such that both $c \wedge d \leq b$ and $c \wedge d \leq b^{\complement}$ hold (i.e., if $c \wedge d = 0_{\mathbb{B}}$) then $\varphi(c) \wedge \psi(d) = 0_{\mathbb{B}'}$. Therefore $\theta(b^{\complement})$ is equal to

$$\bigvee\{\varphi(c) \wedge \psi(d) : c \in \mathbb{C}_0, d \in \mathbb{D}_0, c \wedge d \not\leq b\}$$

which is $\theta(b)^{\complement}$. □

**Definition 9.3.12.** For $A$ and $B$ in $\mathcal{J}_{\text{cont}} \cap \text{Anti}^+(\mathbb{B})$ let $\text{Diff}(A, B)$ be the set of all $n$ such that both $A_n = A \cap \mathbb{B}_n$ and $B_n = B \cap \mathbb{B}_n$ are nonempty[2] and at least one of the following conditions holds.

(Diff1) $\bigvee_{a \in A_n} g_A(a) \neq \bigvee_{b \in B_n} g_B(b)$.

(Diff2) Some $a_n \in A_n$ and $b_n \in B_n$ satisfy $a_n \wedge b_n = 0_{\mathbb{B}}$ and $g_A(a_n) \wedge g_B(b_n) \neq 0_{\mathbb{B}'}$.

By Lemma 9.3.11, $n \in \text{Diff}(A, B)$ if and only if the restrictions of $g_A$ to $A \cap \mathbb{B}_n$ and of $g_B$ to $B \cap \mathbb{B}_n$ do not have common extension to a homomorphism whose domain is the algebra generated by $A \cap \mathbb{B}_n$ and $B \cap \mathbb{B}_n$.

**Claim 9.3.13.** *For all $A$ and $B$ in $\mathcal{J}_{\text{cont}} \cap \text{Anti}^+(\mathbb{B})$, the set $\text{Diff}(A, B)$ is finite.*

Proof. Assume otherwise and fix $A$ and $B$ such that $\text{Diff}(A, B)$ is infinite. Let $a_n = \bigvee_{a \in A_n} g_A(a)$ and $b_n = \bigvee_{b \in B_n} g_B(b)$. Suppose for a moment that the set

$$X = \{n \in \text{Diff}(A, B) : \text{(Diff1) holds for } n \text{, hence } a_n \neq b_n\},$$

is infinite. By using Ramsey's theorem as in the proof of Claim 6.5.1 and replacing $X$ with an infinite subset we may assume that one of the following applies.

(1) $a_m \wedge b_n = 0_{\mathbb{B}'}$ and $b_m \wedge a_n = 0_{\mathbb{B}'}$ for all $m < n$ in $X$.

(2) $a_m \wedge b_n \neq 0_{\mathbb{B}'}$ for all $m < n$ in $X$.

[2] Equivalently, both $A_n$ and $B_n$ are finite maximal antichains in $\mathbb{B}_n$, see Definition 9.2.1.

(3) $b_m \wedge a_n \neq 0_{\mathbb{B}'}$ for all $m < n$ in $X$.

Since each one of $g_A[A]$ and $g_B[B]$ is an antichain whose intersection with $\mathbb{B}'_k$ is finite for every $k$, if (1) applies then $\bigvee g_A[\bigcup_{n\in X} A_n]\Delta \bigvee g_B[\bigcup_{n\in X} B_n]$ does not belong to $\mathrm{Fin}(\mathbb{B}')$, contradicting $\bigvee_{n\in X}\bigvee A_n = \bigvee_{n\in X}\bigvee B_n$.

If (2) applies then let $X = \{n(j) : j \in \mathbb{N}\}$ be the increasing enumeration of $X$, let $X' = \{n(2j) : j \in \mathbb{N}\}$ and note that $\bigvee_{n\in X'}\bigvee A_n = \bigvee_{n\in X'}\bigvee B_n$ but $\bigvee g_A[\bigcup_{n\in X'} A_n]\Delta \bigvee g_B[\bigcup_{n\in X'} B_n]$ does not belong to $\mathrm{Fin}(\mathbb{B}')$; contradiction. Analogous argument shows that (3) also leads to contradiction.

We may therefore assume that the set $Y$ of $n$ such that (Diff2) applies is infinite. Then $a = \bigvee_{n\in Y} a_n$ and $b = \bigvee_{n\in Y} b_n$ satisfy $a \wedge b = 0_{\mathbb{B}}$ but $\Psi_A(a)\Delta\Psi_B(b)$ is not in $\mathrm{Fin}(\mathbb{B}')$; contradiction. □

Extend each $g_A$ to a function

$$g_A^+ \colon \mathrm{Fin}(\mathbb{B}) \to \mathbb{B}'$$

by setting $g_A^+(c) = 0_{\mathbb{B}'}$ for all $c \notin A$. Identify $A \in \mathcal{J}_{\mathrm{cont}}$ with the pair $(A, g_A^+)$ in $\mathrm{Anti}^+(\mathbb{B}) \times \mathbb{B}'^{\mathrm{Fin}(\mathbb{B})}$. The right-hand side is a Polish space and we use this identification to equip $\mathcal{J}_{\mathrm{cont}}$ with a separable metric topology $\tau$.

For $t \Subset \mathbb{N}$ and $A, B$ in $\mathcal{J}_{\mathrm{cont}}$ we say that $A$ and $B$ *conflict on* $t$ if $\mathrm{Diff}(A,B) \supseteq t$. The following is required in order to use $\mathrm{OCA}^\#$.

**Lemma 9.3.14.** *For every $t \Subset \mathbb{N}$ there are open sets $U^0_{t,i} \times U^1_{t,i}$, for $i \in \mathbb{N}$, included in $\mathcal{J}^2_{\mathrm{cont}}$ such that*

$$\{(A,B) \in \mathcal{J}^2_{\mathrm{cont}} : A, B \text{ conflict on } t\} = \textstyle\bigcup_i U^0_{t,i} \times U^1_{t,i}.$$

*Moreover, for every $i$ some large enough $k(i)$ satisfies $(A \cup B) \cap \mathbb{B}_n \subseteq \mathbb{B}_{n,k(i)}$ for all $(A,B) \in U^0_{t,i} \times U^1_{t,i}$.*

Proof. For the first claim, it suffices to prove that the set of all $(A,B)$ that conflict on $t$ is an open subset of $[\mathcal{J}_{\mathrm{cont}}]^2$ with respect to the separable metric topology $\tau$. Assume for a moment that $t$ is a singleton, $\{n\}$. Fix $A$ and $B$ that conflict on $\{n\}$. Then one of the conditions from Definition 9.3.12 holds for $g_A$ and $g_B$. Let $m$ be sufficiently large so that the functions $g^m_A(a) = g_A(a) \wedge \bigvee_{j<m} 1_j$ and $g^m_B(b) = g_B(b) \wedge \bigvee_{j<m} 1_j$ satisfy this condition. The set of pairs $(A', B')$ such that $A' \cap \mathbb{B}_n = A \cap \mathbb{B}_n$, $B' \cap \mathbb{B}_n = B \cap \mathbb{B}_n$, $g^m_{A'} = g^m_A$, and $g^m_{B'} = g^m_B$ is clearly open, and all such $A'$, $B'$ conflict on $t$.

This completes the proof in case when $t$ is a singleton, and the case when $t$ is an arbitrary finite subset of $\mathbb{N}$ follows by taking intersections. The second claim is automatic. □

For $m \geq 1$ define, using the notation from Lemma 9.3.14,

$$\mathcal{V}_m = \{U^0_{t,i} \times U^1_{t,i} : |t| \geq m + (4^{m+1} - 1)/3, i \in \mathbb{N}\}. \tag{9.8}$$

Clearly $\mathcal{V}_m \supseteq \mathcal{V}_{m+1}$ for all $m$.

The proof of the following is virtually identical to the proof of its analog, Claim 6.5.2.

**Claim 9.3.15.** *There is no $(Z, f, \rho)$ with $Z \subseteq \{0,1\}^{\mathbb{N}}$ uncountable, $f \colon Z \to \mathcal{J}_{\mathrm{cont}}$, and $\rho \colon \Delta(Z) \to \bigcup_m \mathcal{V}_m$ such that $\rho(s) \in \mathcal{V}_{|s|}$ for all $s$ and $\{f(x), f(y)\} \in \rho(x \wedge y)$ for all distinct $x, y$ in $Z$.*

PROOF. Assume otherwise and fix $Z$, $f$, and $\rho$. We may assume that $Z$ has no isolated points, in which case $(\Delta(Z), \sqsubseteq)$ is a perfect tree. (It is not necessarily downwards closed in $\{0,1\}^{<\mathbb{N}}$.) If $s \in \Delta(Z)$ then $\rho(s) = U^0_{t,i} \times U^1_{t,i}$ for some $t$ of cardinality $|s| + (4^{|s|+1} - 1)/3$ and some $i$. We write $A(s) = t$ and $i(s) = i$. By Lemma A.6.1 there are pairwise disjoint sets $B(s) \subseteq A(s)$, for $s \in \{0,1\}^{<\mathbb{N}}$, such that $|B(s)| = 2^{|s|}$ for all $m$ and all $s, t \in \Delta(Z)$. Let $\mathsf{S}_m$ denote the $m$-th level of $\Delta(Z)$ and note that $s \in \mathsf{S}_m$ implies $|B(s)| \geq 2^m$. There are therefore disjoint sets $J_t$, for $t \in \bigcup_m \{0,1\}^m$, such that $J_t \cap B(s) \neq \emptyset$ for all $s \in \mathsf{S}_m$ and $\bigcup\{B(s) : s \in \mathsf{S}_m\} = \bigcup\{J_t : t \in \{0,1\}^m\}$. For $n \in \mathbb{N}$ define, using $k(i)$ as in Lemma 9.3.14,

$$k(n) = \max\{k(i(s)) : s \in \{0,1\}^n\}.$$

For $h \in \{0,1\}^{\mathbb{N}}$ let

$$D(h) = \bigcup_n \bigcup_{j \in g \restriction n} \mathbb{A}_{j,k(n)}.$$

Since the finite sets $J_t$, for $t \in \{0,1\}^{<\mathbb{N}}$, are nonempty and disjoint, $\{D(h)\}$ is a perfect tree-like almost disjoint family. Therefore $\mathrm{OCA_T}$ and Proposition 6.3.1 together imply that $D(h) \in \mathscr{J}_{\mathrm{cont}}$ for some $h$. The salient property of $D(h)$ is that $D(h) \cap A(s) \neq \emptyset$ for all $s \in \Delta(Z)$. By Claim 6.5.1, for every $x \in Z$ there is $n(x)$ such that all $j \in (f(x) \cap D(h)) \setminus n(x)$ satisfy $h_{D(h)}(j) = h_{f(x)}(j)$.

Fix $n$ such that $Z' = \{x \in Z : n(x) = n\}$ is uncountable. As the sets $A(s)$ are nonempty and disjoint, the set $\{s \in \Delta(Z) : A(s) \cap n \neq \emptyset\}$ is finite. Hence, we can choose distinct $x$ and $y$ in $Z'$ such that $A(x \wedge y) \cap n = \emptyset$. Therefore, the set

$$\Delta = \bigcup_n J_{h \restriction n} \cap A(x \wedge y)$$

is nonempty and $f(x)$ and $f(y)$ conflict on this set. Fix $j \in \Delta$ and let $n$ be such that $j \in J_{h\restriction n}$. Then $(f(x) \cup f(y)) \cap \mathbb{B}_j \subseteq \mathbb{B}_{j,k(n)}$, and the restrictions of $g_{f(x)}$ to $A(f(x)) \cap \mathbb{B}_j$ and $g_{f(y)}$ to $A(f(y)) \cap \mathbb{B}_j$ have a common extension, namely the homomorphism determined by $g_{D(h)} \restriction \mathbb{B}_j$. This violates $j \in A(x \wedge y)$ and Lemma 9.3.11; contradiction. □

Since $\mathrm{OCA}^{\#}$ is a consequence of $\mathrm{OCA_T}$ (Theorem A.3.5), by Claim 6.5.2 and $\mathrm{OCA_T}$ there are $\mathcal{X}_n$, for $n \in \mathbb{N}$, such that $\mathscr{J}_{\mathrm{cont}} = \bigcup_n \mathcal{X}_n$ and $[\mathcal{X}_n]^2 \cap \mathcal{V}_n = \emptyset$ for all $n$. Since $\mathscr{J}_{\mathrm{cont}} \cap \mathbb{B}_f$ is nonmeagre (Lemma 9.3.3 (1)) for all $f$, by Lemma 9.3.10 there is $n$ such that the set of $f$ such that $\mathcal{X}_n \cap \mathbb{B}_f$ is relatively nonmeagre in $\mathbb{B}_f$ is $\leq^*$-cofinal in $\mathbb{N}^{\mathbb{N}}$. In terms of this lemma, $\mathcal{X}_n$ is $\mathcal{I}(\mathrm{Anti}^+(\mathbb{B}))$-positive.

Next, we attempt to recursively choose an increasing sequence $n(i)$, $k(i)$, for $i \in \mathbb{N}$, such that the following holds for all $m$.[3]

(1) The set $\mathcal{F}_{0,m} = \{A \in \mathcal{X}_n\colon A \in U^0_{n(i),k(i)} : \text{for all } i < m\}$ is in $\mathcal{I}(\mathrm{Anti}^+(\mathbb{B}))$.
(2) The set $\mathcal{F}_{1,m} = \{B \in \mathcal{X}_n\colon B \in U^1_{n(i),k(i)} : \text{for all } i < m\}$ is in $\mathcal{I}(\mathrm{Anti}^+(\mathbb{B}))$.

Since $[\mathcal{X}_n]^2 \cap \bigcup \mathcal{V}_n = \emptyset$, a recursive construction of such sequences has to stop at a finite stage. We therefore have $m$ (possibly $m = 0$, with $n(-1) = k(-1) = 0$) as well as $n(i)$ and $k(i)$ for $i < m$ such that for all $n > n(m-1)$ and all $k$ at least one of the sets

$$\{A \in \mathcal{F}_{0,m} : A \in U^0_{n(m),k}\} \qquad \text{or} \qquad \{B \in \mathcal{F}_{1,m} : B \in U^1_{n(m),k}\}$$

is meagre.

[3]The remaining part of the proof is Biba's trick.

Let $D$ be the set of all $n > n(m-1)$ such that for all $l \in \mathbb{N}$ each of the following sets is $\mathcal{I}(\mathrm{Anti}^+(\mathbb{B}))$-positive.

$$\mathcal{F}_0 = \{A \in \mathcal{F}_{0,m} : A \cap \mathbb{B}_{n,l} = \mathbb{A}_{n,l}\},$$
$$\mathcal{F}_1 = \{B \in \mathcal{F}_{1,m} : B \cap \mathbb{B}_{n,l} = \mathbb{A}_{n,l}\}.$$

For every choice of $n \in D$ and $l \in \mathbb{N}$ fix $g_0, g_1 \colon \mathbb{A}_{n,l} \to \mathbb{B}'$ such that the sets $\{A \in \mathcal{F}_0 : g_A \restriction \mathbb{A}_{n,l} = g_0\}$ and $\{B \in \mathcal{F}_1 : g_B \restriction \mathbb{A}_{n,l} = g_1\}$ are $\mathcal{I}(\mathrm{Anti}^+(\mathbb{B}))$-positive. By the definition of $\mathcal{V}_n$ ((9.8)) we have that $g_0$ and $g_1$ satisfy the assumptions of Lemma 9.3.11. Since each one of their domains is equal to $\mathbb{A}_{n,l}$, this is equivalent to $g_0 = g_1$. We can therefore define $g \colon \bigcup_{n \in \mathbb{D}} \mathbb{B}_n \to \mathbb{B}'$ by letting

$$(9.9) \quad g(a) = g_0(a), \text{ for } g_0 \text{ such that}$$
$$\text{the set } \{A \in \mathcal{F}_0 : g_A \restriction \mathbb{A}_{n,k} = g_0\} \text{ is in } \mathcal{I}(\mathrm{Anti}^+(\mathbb{B}))_+.$$

Then each one of the sets

$$\tilde{\mathcal{F}}_0 = \{A \in \mathcal{F}_{0,m} : (\forall n \in D) g_A \text{ and } g \text{ agree on } \mathbb{B}_n \cap A\}$$
$$\tilde{\mathcal{F}}_1 = \{B \in \mathcal{F}_{1,m} : (\forall n \in D) g_B \text{ and } g \text{ agree on } \mathbb{B}_n \cap B\}$$

is obtained by removing the union of countably many sets in $\mathcal{I}(\mathrm{Anti}^+(\mathbb{B}))$ from an $\mathcal{I}(\mathrm{Anti}^+(\mathbb{B}))$-positive set, and is therefore in $\mathcal{I}(\mathrm{Anti}^+(\mathbb{B}))$ by Lemma 9.3.10.

**Claim 9.3.16.** *The set $D$ is cofinite.*

Proof. Assume otherwise. Since $\tilde{\mathcal{F}}_0$ is $\mathcal{I}(\mathrm{Anti}^+(\mathbb{B}))$-positive, there is $f$ such that $\tilde{\mathcal{F}}_0 \cap \mathbb{B}_f$ is nonmeagre. By Corollary 3.2.3 there is $A \in \tilde{\mathcal{F}}_0$ such that the set $\{n : A \cap \mathbb{B}_n \neq \emptyset\}$ has infinite intersection with $\mathbb{N} \setminus D$; contradiction. □

**Claim 9.3.17.** *For all $A$ in $\mathcal{J}_{\mathrm{cont}}$ the set*

$$\mathrm{Diff}(A, g) = \{a \in A \cap \mathrm{dom}(g) : g_A(a) \neq g(a)\}$$

*is finite.*

Proof. Assume otherwise and fix $A$ such that $C = \mathrm{Diff}(A, g)$ is infinite. For $n \in C$ let $f(n)$ be large enough to have $A \cap \mathbb{B}_n \subseteq \mathbb{B}_{n,f(n)}$. Since $\tilde{\mathcal{F}}_{1,m}$ is $\mathcal{I}(\mathrm{Anti}^+(\mathbb{B}))$-positive, by increasing $f$ if needed we may assume that $\tilde{\mathcal{F}}_{1,m} \cap \mathrm{Anti}^+(\mathbb{B}_f)$ is relatively nonmeagre in $\mathrm{Anti}^+(\mathbb{B}_f)$. Therefore, there is $B \in \tilde{\mathcal{F}}_{1,m}$ such that $B \cap \mathbb{B}_n$ refines $\mathbb{A}_{n,f(n)}$ for infinitely many $n \in C$. Moreover $g_B$ agrees with $g$ on $\mathbb{B}_n \cap B$ for all $n$ in $C$. Since $C$ is infinite, $A$ and $B$ contradict Claim 9.3.13. □

It remains to define a lifting of $\Phi$ on $\mathcal{J}_{\mathrm{cont}}$ which is a completely additive on antichains. Since $D$ is cofinite (Claim 9.3.16), we can choose $n$ large enough to have $n \cup D = \mathbb{N}$. Lemma 9.2.6 implies that the restriction of $\Phi$ to $\prod_{i<n} \mathbb{B}_i$ has an additive lifting. We may therefore assume $n = 0$.

By Claim 9.3.16 the domain of the function $g$ defined in (9.9) is equal to $\bigcup_j \mathbb{B}_j$. For $x \in \prod_j \mathbb{B}_j$ let

$$\Psi(x) = \bigvee_j g(x \wedge 1_j).$$

This is a function from $\mathbb{B}$ into $\mathbb{B}'$ whose restriction to each $A \in \mathrm{Anti}(\mathbb{B})$ is completely additive. By Claim 9.3.17, $\Psi$ is a lifting of $\Phi$ on $\mathcal{J}_{\mathrm{cont}}$. Since Lemma 9.3.3 implies that $\mathcal{J}_{\mathrm{cont}}$ intersects every perfect tree-like almost disjoint family nontrivially, this completes the proof of Theorem 9.3.4.

**9.3.7. Proof of Theorem 9.1.6.** Suppose that $\mathrm{OCA_T}$ and MA($\sigma$-linked) hold, that $\mathbb{B}$ and $\mathbb{B}'$ are algebras of clopen subsets of locally compact, non-compact, Polish, zero-dimensional spaces in standard form, and that $\Phi\colon \mathbb{B} \to \mathbb{B}'/\operatorname{Fin}(\mathbb{B}')$ is a homomorphism. By Theorem 9.1.4. $\Phi$ has a completely additive on antichains lifting on $\mathscr{J}_{\mathrm{cont}}$.By Lemma 9.3.3 (3), $\mathscr{J}_{\mathrm{cont}}$ intersects every uncountable sharply almost disjoint family of antichains nontrivially, and the conclusion follows.

## 9.4. Concluding remarks

At the hindsight, the weak Extension Principle is a commutative version of a statement about *-homomorphisms between C*-algebras under forcing axioms. By the Gelfand–Naimark duality (e.g., [54, §1.3], the category of compact Hausdorff spaces is equivalent to the category of commutative C*-algebras. This gives another reformulation of Rudin/Shelah results, in terms of triviality of automorphisms of the C*-algebra $\ell_\infty(\mathbb{N})/c_0(\mathbb{N})$, equivalently $C(\mathbb{N}^*)$. $\mathrm{OCA_T}$ implies that all automorphisms of the Calkin algebra $\mathcal{Q} = \mathcal{B}(H)/\mathcal{K}(H)$ (the noncommutative analog of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$, see [169]) are inner ([53], also [54, §17]). The fact that under CH $\mathcal{Q}$ has outer automorphisms was also difficult to prove, see [135]. Forcing axioms imply rigidity results apply for coronas of other separable C*-algebras ([125], [165]), and even all endomorphisms of the Calkin algebra are inner ([160]). Rigidity has also been proved for uniform Roe coronas, C*-algebras associated with coarse metric spaces ([13]) and their centres, called Higson coronas ([166]). In this noncommutative context, obtaining non-rigidity results using CH proves to be considerably more difficult than in the context of analytic quotients. The only nontrivial examples of C*-algebras for which the question whether their coronas are isomorphic is independent from ZFC are obtained in [72] and [73] (for direct sums of matrix algebras) and [13, Theorem 1.5] and [15, Theorem 6.3] (for uniform Roe coronas). In both cases, the algebras in question have very large centres, hence arguably no truly noncommutative examples are known. By using the approach via C*-algebras, Vignati and Yilmaz proved that $\mathrm{OCA_T}$ and MA impy weak Extension Principle for all locally compact Polish spaces ([167]).

The Gelfand–Naimark duality also gives (in my humble opinion) an enlightening take on the results of Chapter 8. The main result of this chapter, Theorem 8.3.1, provides structure theory for continuous functions from a product of compact Hausdorff spaces into a $\beta\mathbb{N}$-space. Equivalently, it gives structure theory for *-homomorphisms from a C*-algebra of the form $C(X)$, for a compact $\beta\mathbb{N}$-space $X$, into any tensor product of unital, abelian, C*-algebras;[4] see [54, Theorem 15.6.34] and the discussion surrounding it. A noncommutative variant of this result was proved by Ghasemi ([74]) and improved in [168].

[4]Tensor products in C*-algebras are notoriously elusive, but not in the case of abelian C*-algebras.

CHAPTER 10

# Applications of the weak Extension Principle

It is the corollary time, once again, using results of Chapter 8 and Chapter 9. Assuming $\mathrm{OCA_T}$ (and occasionally MA), we prove that a homeomorphism between products of Čech–Stone remainders of locally compact, non-compact, Polish spaces has a continuous extension to the product of the corresponding Čech–Stone compactifications (Proposition 10.1.1). Thus, under these assumptions two Čech–Stone remainders of locally compact, non-compact, Polish spaces are homeomorphic if and only if they have co-compact homeomorphic subspaces. If the locally compact, non-compact, Polish spaces are minimal (Definition 10.1.4) then the latter conclusion extends to products (Theorem 10.1.6). Theorem 10.2.2 and Corollary 10.2.3 are analogs of Proposition 10.1.1 and Theorem 10.1.6 for surjective maps. Theorem 10.2.6 is the Dow–Hart result on remainders of compact, $\sigma$-compact, Hausdorff spaces that are continuous images of $\mathbb{N}^*$. Theorem 10.3.1 is a mild extension of a result by the same team, asserting that under $\mathrm{OCA_T}$ no Boolean algebra that carries a strictly positive Maharam submeasure (this includes the Lebesgue measure algebra and more exotic examples, [**151**]) embeds into $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$.

## 10.1. Homeomorphisms between remainders and their powers

For the weak Extension Principle wEP see Definition 9.1.1. For locally compact spaces $X$ and $Y$ we write

$$X \approx_{\mathrm{cpct}} Y$$

if there are compact $K \subseteq X$ and $L \subseteq Y$ such that $X\setminus K$ and $Y \setminus L$ are homeomorphic.

**Proposition 10.1.1.** *Assume $X_\xi$, for $\xi < \kappa$, and $Y_\eta$, for $\eta < \lambda$, are locally compact, non-compact, Polish spaces and $\mathrm{wEP}(X_\xi, Y_\eta)$ holds for all $\xi, \eta$. Then every homeomorphism $g\colon \prod_{\xi<\kappa} X^*_\xi \to \prod_{\eta<\lambda} Y^*_\eta$ has a continuous extension to $f\colon \prod_{\xi<\kappa} \beta X_\xi \to \prod_{\eta<\lambda} \beta Y_\eta$.*

*In particular, this follows from $\mathrm{OCA_T}$ if all spaces are Polish and zero-dimensional and from $\mathrm{OCA_T} + \mathrm{MA}$ if they are Polish.*

Proof. $\mathrm{OCA_T}$ implies wEP(Polish, 0-dim) (Theorem 9.1.4) and $\mathrm{OCA_T} + \mathrm{MA}$ implies wEP(Polish) ([**167**]).

It suffices to prove the assertion for $\pi_\eta \circ g$, for every $\eta < \lambda$. Since all $Y^*_\eta$ are $\beta\mathbb{N}$-spaces by Lemma 8.1.2, Theorem 8.1.3 implies that there is a clopen partition $\prod_{\xi<\kappa} X^*_\xi = \bigsqcup_{i<m} U^*_i$ such that $\pi_\eta \circ g$ depends on at most one coordinate, $\xi(\eta, i)$, on each $U^*_i$. Fix $i$. By wEP, there is a clopen partition $U^*_i = V^*_i \sqcup W^*_i$ such that $\pi_\eta \circ g[V^*_i]$ is nowhere dense, and the restriction of $g$ to $V^*_i$ has a continuous extension to $\prod_{\xi<\kappa} \beta X_\xi$ and therefore $g[W^*_i]$ is nowhere dense. Since $g$ is a homeomorphism, $W^*_i = \emptyset$ and the restriction of $g$ to $U^*_i$ has a continuous extension to $\prod_{\xi<\kappa} \beta X_\xi$. By applying this to all $i < m$, the conclusion follows. □

**Corollary 10.1.2.** *Assume* $\mathrm{OCA_T}$. *If $X$ and $Y$ are locally compact, non-compact, Polish, zero-dimensional then $X^*$ is homeomorphic to $Y^*$ if and only if $X \approx_{cpct} Y$.*

PROOF. Only the direct implication requires a proof. Suppose $f\colon X^* \to Y^*$ is a homeomorphism. By Theorem 9.1.4, there is a partition $X^* = U^* \sqcup V^*$ into clopen sets such that the $f$-image of $V^*$ is nowhere dense and the restriction of $f$ to the other is of the form $\beta g \restriction U^*$ for a continuous $g\colon X \to \beta Y$. Since $f$ is a homeomorphism, we have $V^* = \emptyset$. Hence $\beta f \restriction X^*$ is a homeomorphism and $X \approx_{\mathrm{cpct}} Y$ follows. □

Can wEP be used to give an elegant description of when products of Čech–Stone remainders of spaces to which wEP applies are homeomorphic and of their autohomeomorphism groups? It really depends on how elegant is this description expected to be.

**Example 10.1.3.** The spaces $[0,\infty)^* \times \mathbb{N}^*$ and $\mathbb{R}^* \times \mathbb{N}^*$ are homeomorhic. More generally, if $X, Y, Z$ are locally compact, non-compact, completely regular, $X \approx_{\mathrm{cpct}} Z \oplus Z$ and $Y \approx_{\mathrm{cpct}} Y \oplus Y$, then $X^* \times Y^* \approx_{\mathrm{cpct}} Z^* \times Y^*$. This is because each of the spaces is homeomorphic to the direct sum of two copies of $Z^* \times Y^*$.

There is still a hope that there is a reasonable characterisation of the homeomorphism relation between products of v Cech–Stone remainders of spaces to which wEP applies. Since I don't know what might be, we move on to a narrower (yet still fairly rich) class of spaces. The following definition is nonstandard.

**Definition 10.1.4.** A locally compact, non-compact, Polish space $X$ is *minimal* if for every partition $X = U \sqcup V$ into clopen, non-compact sets we have that $X \approx_{\mathrm{cpct}} U$ and $X \approx_{\mathrm{cpct}} V$.

If $X$ is minimal then for every clopen partition of $X^* = U^* \sqcup V^*$ we have that $X^* \approx_{\mathrm{cpct}} U^*$ and $X^* \approx_{\mathrm{cpct}} V^*$. Proof of the following is straightforward.

**Proposition 10.1.5.** *Each of the following locally compact, non-compact, Polish spaces is minimal.*

(1) $\mathbb{N}$.
(2) *Every indecomposable ordinal, with respect to the ordinal topology.*
(3) *Every connected locally compact, non-compact, Polish space.*
(4) *The product of any connected locally compact, non-compact, Polish space and a minimal locally compact, non-compact, Polish space.* □

It was proved by van Douwen ([25], §8.1 is largely about this) that if $X_\xi$, for $\xi < \kappa$, and $Y_\eta$, for $\eta < \lambda$, are families of $\beta\mathbb{N}$-spaces and $\prod_{\xi<\kappa} X^*_\xi \approx_{\mathrm{cpct}} \prod_{\eta<\lambda} Y^*_\eta$ then $\kappa = \lambda$.

**Theorem 10.1.6.** *Assume $X_\xi$, for $\xi < \kappa$, and $Y_\xi$, for $\xi < \kappa$, are families of minimal locally compact, non-compact, Polish spaces for which wEP holds, $X_\xi$ are pairwise non-$\approx_{cpct}$ and $Y_\eta$ are pairwise non-$\approx_{cpct}$. Then the following are equivalent.*

(1) $\prod_{\xi<\kappa} X^*_\xi \approx_{cpct} \prod_{\xi<\kappa} Y^*_\xi$.
(2) *There is a permutation $\sigma$ of $\lambda$ such that $X_{\sigma(\eta)} \approx_{cpct} Y_\eta$ for all $\eta < \lambda$.*

*In particular, this is true if the spaces are $locally compact, non-compact, Polish, zero-$ dimensional and* $\mathrm{OCA_T}$ *holds.*

Proof. The 'in particular' part will follow from Theorem 9.1.4, once the first part is proved.

Only the direct implication requires a proof. Fix $\eta < \kappa$ and consider $\pi_\eta \circ f$, where $f$ is the homeomorphism and $\pi_\eta$ is the projection to $Y^*_\eta$. By Theorem 8.1.3, there are a partition $\prod_{\xi<\kappa} X^*_\xi = \bigsqcup_{i<m} U^*_i$ into clopen rectangles and $\xi(i) < \kappa$, for $i < m$, such that the restriction of $\pi_\eta \circ f$ to $U^*_i$ depends only on the $\xi(i)$-th coordinate. To be precise, there are a clopen $V^*_i \subseteq X^*_{\xi(i)}$ and a continuous $g_i\colon V^*_i \to Y^*_\eta$ such that $g_i = \pi_\eta \circ f \circ \iota_{\xi(i)}$ on $V^*_i$. Since $f$ is a homeomorphism, its restriction to $U^*_i$ is a homeomorphism onto a clopen subset of $\prod_{\xi<\kappa} Y_|xi^*$. Therefore $X_{\xi(i)} \approx_{\mathrm{cpct}} Y_\xi$, and by our assumption $\xi(i)$ is the unique $X_\zeta$ with this property. By applying this argument to every $\eta < \lambda$, there is therefore an injection $\sigma\kappa \to \kappa$ such that $Y^*_\eta$ and $X^*_{\sigma(\eta)}$ are homeomorphic for all $\eta$. By applying the analogous argument to $f^{-1}$ and using the assumption that $Y_\eta$ are non-$\approx_{\mathrm{cpct}}$ we conclude that $\sigma$ is a permutation. □

Assuming wEP, one can describe autohomeomorphisms of finite powers of locally compact, non-compact, Polish spaces, but even in the case of $(\mathbb{N}^*)^2$ the description is somewhat messy.

## 10.2. Continuous images of remainders under wEP

By $\alpha X$ we denote the one-point compactification of a topological space $X$. We write $\infty$ for the unique point in $(\alpha X) \setminus X$; there will be no danger of confusion If $X$ is locally compact and Hausdorff then $\alpha X$ is compact and Hausdorff.

**Definition 10.2.1.** Recall that a function between topological spaces is *perfect* if it is continuous, closed, surjective, and the preimage of every point is compact. For locally compact, non-compact spaces $X$ and $Y$ we write

$$X \twoheadrightarrow_{\mathrm{cpct}} Y$$

if there is a perfect map from a co-compact subspace of $X$ onto a co-compact subspace of $Y$.

The first part of the following is well-known.

**Theorem 10.2.2.** *For all locally compact, non-compact, Polish spaces $X$ and $Y$ the following are equivalent.*

*(1) $X \twoheadrightarrow_{cpct} Y$.*
*(2) There are compact $K \subseteq X$ and $L \subseteq Y$ and a continuous surjection $f\colon \alpha X \setminus K \to \alpha Y \setminus L$ such that $f(\infty) = \infty$.*

*If* wEP$(X, Y)$ *holds, then* (1) *and* (2) *are equivalent to*

*(3) There is a continuous surjection from $X^*$ onto $Y^*$.*

Proof. (1) implies (2): If $f\colon X \setminus K \to Y \setminus L$ is perfect for some compact $K$ and $L$, then we can extend it to a continuous function from $\alpha X \setminus K$ to $\alpha Y \setminus L$ by setting $f(\infty) = \infty$. This function is continuous and as required.

(2) implies (1): If $f\colon \alpha X \setminus K \to \alpha Y \setminus L$ is continuous, $K, L$ compact, and $f(\infty) = \infty$, then the restriction of $f$ to $X \setminus K$ is as required.

(1) implies (3): If $f\colon X \setminus K \to Y \setminus L$ is perfect for some compact $K$ and $L$, then $f$ continuously extends to $\beta f\colon \beta(X \setminus K) \to \beta(Y \setminus L)$ and the restriction of $f$ to $(X \setminus K)^* = X^*$ is as required.

(3) implies (1): Fix a continuous surjection $g\colon X^* \to Y^*$. By wEP there is a clopen partition $X = U \sqcup V$ such that its restriction to $U^*$ is of the form $\beta f \restriction X^*$ for a continuous $f\colon U \to Y$ and $g[V^*]$ is nowhere dense. Since $Y^* \setminus \beta g[U^*]$ is open and $g$ is a surjection, $g[U^*] = Y^*$, and $f$ is as required. □

**Corollary 10.2.3.** *Suppose that* wEP *holds,* $m, n \geq 1$ *and* $X_i$*, for* $i < m$*, and* $Y_j$*, for* $j < n$*, are minimal locally compact, non-compact, Polish, zero-dimensional spaces. Then the following are equivalent.*

*(1) There is a continuous surjection from* $\prod_{i<m} X_i^*$ *onto* $\prod_{j<n} Y_j^*$*.*
*(2)* $m \geq n$ *and there is an injection* $\sigma\colon n \to m$ *such that* $X_{\sigma(j)} \twoheadrightarrow_{cpct} Y_j$ *for all* $j < n$*.* □

An immediate consequence is the following strengthening of [40, Theorem 4.6.3].

**Corollary 10.2.4.** *Suppose that* wEP(*Polish, 0-dim*) *holds,* $m, n \geq 1$ *and* $\alpha_i$*, for* $i < m$*, and* $\gamma_j$*, for* $j < n$*, are indecomposable countable ordinals. Then the following are equivalent.*

*(1) There is a continuous surjection from* $\prod_{i<m} \alpha_i^*$ *onto* $\prod_{j<n} \gamma_j^*$*.*
*(2)* $m \geq n$ *and some injection* $\sigma\colon n \to m$ *satisfies* $\alpha_{\sigma(j)} \geq \gamma_j$ *for all* $j < n$*.* □

This implies that, under wEP(Polish, 0-dim), there is no surjectively universal space among the Čech–Stone remainders of countable ordinals ([40, Theorem 4.6.3]). It is not difficult to see that there is a surjectively universal space among the Čech–Stone remainders of locally compact, non-compact, Polish spaces.

**Example 10.2.5.** Let $X$ be $\mathbb{N} \times \{0,1\}^{\mathbb{N}}$ (equivalently, the space obtained by removing a single point from the Cantor space $\{0,1\}^{\mathbb{N}}$). Then $X^* \twoheadrightarrow Y^*$ for every locally compact, non-compact, Polish space $Y$. Write $Y = \bigcup_n Y_k$, where each $Y_n$ is compact and $Y_n \subseteq Y_{n+1}$. Since every compact second-countable space is a continuous image of $\{0,1\}^{\mathbb{N}}$, there is a continuous surjection from the $n$-th copy of $\{0,1\}^{\mathbb{N}}$ in $X$ onto $\overline{Y_{n+1} \setminus Y_n}$. This defines a surjection from $X$ onto $Y$, implying $X^* \twoheadrightarrow Y^*$.

The following is the main result of [30].

**Theorem 10.2.6.** *Assume* $\mathrm{OCA_T}$*. If* $X$ *is locally compact,* $\sigma$*-compact, Hausdorff, and not compact then* $\mathbb{N}^* \twoheadrightarrow X^*$ *if and only if* $X$ *is homeomorphic to the direct sum of* $\mathbb{N}$ *and a compact space.*

Proof. A simple topological argument ([30, §1.1])shows that if $\mathbb{N}^* \twoheadrightarrow X^*$ then there is a surjection $f\colon X^* \to [0,\infty)^*$.[1] Furthermore, a recursive construction produces a clopen subset $U$ of $\mathbb{N}^*$ such that $f[U]$ contains infinitely many intervals of the form $[n, n+1]$ and $[0,\infty)^* \setminus f[U]$ is non-compact. This implies $\mathbb{N}^* \twoheadrightarrow (\mathbb{N} \times [0,1])^*$. Another argument using only $\mathrm{OCA_T}$ implies that ($\omega^2$ is the indecomposable ordinal, often denoted $\mathbb{D}$) $\mathbb{N}^* \twoheadrightarrow (\omega^2)^*$, contradicting wEP(Polish, 0-dim). □

## 10.3. Lebesgue measure algebra does not always embed

Theorem 10.3.1 below is a mild strengthening of the main result of [31]. A complete Boolean algebra $\mathbb{B}$ is called *weakly* $(\aleph_0, \aleph_0)$*-distributive* if for every sequence $\mathcal{D}_n$, for $n \in \mathbb{N}$, of countable maximal antichains in $\mathbb{B}$ there is a maximal

[1] This already contradicts wEP, and hence by the main result of [167] it contradicts $\mathrm{OCA_T} + \mathrm{MA}$.

antichain $\mathcal{D}$ such that for every $a \in \mathcal{D}$ and every $n$ the set $\{b \in \mathcal{D} : a \wedge b \neq 0_{\mathbb{B}}\}$ is finite. This property is shared by the Lebesgue measure algebra and, more generally, all complete Boolean algebras that carry a strictly positive Maharam submeasure ([**151**]; note that at the time when [**31**] was proven the question whether every such algebra carries a strictly positive measure was wide open).

**Theorem 10.3.1.** *Assume* $\mathrm{OCA_T}$*. Then no atomless weakly* $(\aleph_0, \aleph_0)$*-distributive complete Boolean algebra* $\mathbb{D}$ *embeds into* $\mathrm{Clop}(X)/\mathrm{Cpct}(X)$*, for any locally compact, non-compact, Polish, zero-dimensional space* $X$*.*

*In particular, neither the Lebesgue measure algebra nor any atomless complete Boolean algebra that carries a Maharam submeasure embed into* $\mathrm{Clop}(X)/\mathrm{Cpct}(X)$ *for any locally compact, non-compact, Polish, zero-dimensional space* $X$*.*

Proof. Assume otherwise and fix $\mathbb{D}$ and $X$. Choose a sequence $\mathcal{D}_n$, for $n \in \mathbb{N}$, of maximal countably infinite antichains such that $\mathcal{D}_{n+1}$ refines $\mathcal{D}_n$ for all $n$ and $\{a \in \mathcal{D}_{n+1} : a \leq b\}$ is infinite for all $b \in \mathcal{D}_n$. and $\bigcup_n \mathcal{D}_n$ generates $\mathbb{D}$.

Enumerate $\mathcal{D}_0$ as $\{a_n : n \in \mathbb{N}\}$. Let $\mathbb{D}_n$ be the subalgebra of $\mathbb{D}$ generated by $\{a \in \bigcup_j \mathcal{D}_j : a \leq a_n\}$ and write

$$\mathbb{D}' = \prod_n \mathbb{D}_n.$$

With $\mathbb{B} = \mathrm{Clop}(X)$, let

$$\Phi \colon \mathbb{D}' \to \mathcal{P}(\mathbb{N})/\mathrm{Fin}$$

be the restriction of the embedding to $\mathbb{D}'$. By Theorem 9.1.4 or Proposition 9.3.1, there is a completely additive

$$\Psi \colon \mathbb{D}' \to \mathcal{P}(\mathbb{N})$$

which is a completely additive almost lifting of $\Phi$ on a nonmeagre ideal. Fix a nonzero $A \in \mathbb{D}'$ such that the restriction of $\Psi$ has a completely additive lifting. We may assume $A = 1_{\mathbb{B}}$. Since $\Psi$ is completely additive, $\mathcal{U}_j = \Psi^{-1}(\{A \in \mathcal{P}(\mathbb{N}) : j \in A\})$ is an ultrafilter in $\mathbb{D}'$ for every $j$. Since $\mathbb{D}'$ is atomless, each $\mathcal{U}_j$ is nonprincipal and it contains $1_{\mathbb{D}_{n(j)}}$ for some $n(j)$. Therefore for each $m$, the set

$$\tilde{\mathcal{E}}_m = \{a \in \mathbb{D}' : (\forall j \leq m) a \notin \mathcal{U}_j\}$$

is dense open in $\mathbb{D}$. For each $m$ let $\mathcal{E}_m \subseteq \tilde{\mathcal{E}}_m$ be a maximal antichain. By the weak $(\aleph_0, \aleph_0)$-distributivity of $\mathbb{B}$, there is a nonzero $a \in \mathbb{D}'$ such that for every $m$ the set $\{b \in \mathcal{E}_m : b \wedge a \neq 0_{\mathbb{B}}\}$ is finite. Therefore, $a \notin \mathcal{U}_j$ for all $j$. This implies that $\Psi(a) = \emptyset$, contradicting the assumption that $\Phi$ is an embedding. $\square$

## 10.4. Concluding remarks

There is no a priori reason for wEP to be restricted only to a class of locally compact, non-compact Polish spaces. It is conceivable that forcing axioms imply wEP for all locally compact, $\sigma$-compact, Hausdorff, noncompact spaces (see Theorem 10.2.6). See [**40**, Theorem 4.10.2] for one example.

Regarding Theorem 10.3.1, by Theorem 5.3.1 the Lebesgue measure algebra, and even the Haar measure algebra on $\{0,1\}^{\mathfrak{c}}$, embeds into $\mathcal{P}(\mathbb{N})/\mathcal{Z}_0$. Notably, this embedding has a completely additive lifting. I do not know whether forcing axioms imply that the Lebesgue measure algebra embeds into $\mathcal{P}(\mathbb{N})/\mathcal{I}$ for an $F_\sigma$, or whether every embedding of the Lebesgue measure algebra into the quotient over a countably $\aleph_0$-determined ideal with the Fubini property has a completely additive lifting.

Another related (and still open) question was asked in [**31**], whether forcing axioms imply that every complete Boolean algebra that embeds into $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ is $\sigma$-centered. Encouraged by Theorem 10.3.1, one could ask whether forcing axioms imply that every complete Boolean algebra that embeds into $\operatorname{Clop}(X)/\operatorname{Cpct}(X)$ for some locally compact, non-compact, Polish, zero-dimensional space $X$ is $\sigma$-centered It is not difficult to see that every $\sigma$-centered Boolean algebra embeds into $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$.

**Question 10.4.1** (Bell)**.** *Is it consistent that for every Boolean algebra $\mathbb{B}$, if $\mathcal{P}(\mathbb{N})$ embeds into $\mathbb{B}/\mathcal{I}$ for some countably generated ideal $\mathcal{I}$, then $\mathcal{P}(\mathbb{N})$ embeds into $\mathbb{B}$ itself?*

For more information and some partial results regarding this question (in particular, a negative answer using the Continuum Hypothesis) see [**28**].

CHAPTER 11

# The dark side: Rigidity lost

The Continuum Hypothesis, CH, implies that analytic quotients $\mathcal{P}(\mathbb{N})/\mathcal{I}$ exhibit maximal possible non-rigidity. Erdös and Monk (see [26]) constructed an isomorphism between $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ and the summable ideal $\mathcal{P}(\mathbb{N})/\mathcal{I}_{1/n}$ (see §1.5.1) using the Continuum Hypothesis. In [95], Just and Krawczyk provided an explanation for this fact: under CH, $\aleph_1$-saturation of these quotients implies their full saturation and therefore isomorphism follows from elementary equivalence, by the well-known theorem of Keisler that two saturated models of the same cardinality are isomorphic if and only if they are elementarily equivalent (see e.g., [16] or [54, Corollary 16.6.7]). Quotients over all ordinal ideals $\mathcal{O}_\alpha$ and ideals in the class of layered ideals are also $\aleph_1$-saturated (Definition 11.1.1), and therefore isomorphic if CH holds. Countable saturation of reduced products $\prod_{\mathcal{I}} \mathfrak{A}_n$ was investigated in [133] and [138].

More interesting is the result of Just and Krawczyk who proved, assuming CH, that all quotients over EU-ideals are isomorphic ([95]). From the modern point of view, quotients over analytic P-ideals are metric structures (Proposition 5.2.2), and therefore subject to analysis using continuous logic ([8], [78]). All EU-ideals are density ideals (Theorem 2.7.8) and every quotient over a density ideal is isomorphic to a reduced product $\prod_{\operatorname{Fin}}(\mathcal{P}(I_n), \mu_n)$ (Proposition 11.2.1). Such reduced products are $\aleph_1$-saturated as metric structures ([61], [54, Theorem 16.5.2]). Therefore once again isomorphism (and even isometric isomorphism) reduces to elementary equivalence in logic of metric structures.[1] Thus, the 'right' proof of the Just–Krawczyk result proceeds by showing elementary equivalence of quotients over EU-ideals in logic of metric structures (as atomless Boolean algebras, they are elementarily equivalent but this is inconsequential). We present such proof and also compute theories of quotients over other dense density ideals and dense LV-ideals and prove analogous isomorphism results (all of these results had been proved in a pre-logic of metric structures paper, [49], by adapting methods from [95]).

A deeper explanation for the reason why CH implies 'maximal non-rigidity' of analytic quotients is provided by Woodin's $\Sigma^2_1$-absoluteness theorem (see e.g., [117, §3.2] for the theorem and [59, §12] for the discussion of its relevance to the present context). No analogous explanation of the rigidity effect of forcing axioms is presently known. It is worth pointing out that CH is not necessary for non-rigidity results. In [142] it was proved that $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ has nontrivial automorphisms after adding $\aleph_2$ Cohen reals to a model of CH. Not much else is known along these lines.

[1] One has to be a little bit careful here, since some of the arguments involve renormalising submeasures used to define a given density ideal, and renormalisation process possibly changes the metric and the theory of the quotient.

## 11.1. Discrete saturation

We give a well-known characterisation of $\aleph_1$-saturated atomless Boolean algebras (Proposition 11.1.3), condition for a quotient $\mathcal{P}(\mathbb{N})/\mathcal{I}$ to be $\aleph_1$ saturated (Theorem 11.1.7), and prove that quotients over all $F_\sigma$ ideals, ideals $\mathcal{O}_\alpha(P)$, and $\mathrm{CB}_\alpha(X)$ are $\aleph_1$-saturared (Corollary 11.1.8).

Definitions and results of this section are taken from [49]; it turns out that they were known to Galvin in 1967 ([71]).

**Definition 11.1.1.** An ideal $\mathcal{I}$ on $\mathbb{N}$ is *layered* if there is $f\colon \mathcal{P}(\mathbb{N}) \to [0,\infty]$ such that

(L1) $A \subseteq B$ implies $f(A) \leq f(B)$,
(L2) $\mathcal{I} = \{A : f(A) < \infty\}$,
(L3) $f(A) = \infty$ implies $f(A) = \sup_{B \subseteq A} f(B)$.

A class of ideals introduced by Galvin in [71] is easily verified to coincide with layered ideals. Galvin also proved that the reduced product of any family of metric structures in a countable language associated with a layered ideal is $\aleph_1$-saturated. This result was rediscovered in [20]. In [96] layered ideals were called G-ideals. Another class of ideals called $\nabla$-ideals was considered in [96], and it was shown that if $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is atomless, then it is $\aleph_1$-saturated (as a discrete structure) if and only if $\mathcal{I}$ is a $\nabla$-ideal.

**Proposition 11.1.2.** *(1) Every $F_\sigma$-ideal is layered.*

*(2) If $\alpha$ is an indecomposable countable ordinal, then $\mathcal{O}_\alpha$ is layered. If $L$ is a countable linear well-order and then $\mathcal{O}_\alpha(L)$ is layered (see Definition 1.9.1).*

*(3) If $\alpha$ is a multiplicatively indecomposable countable ordinal then $\mathcal{W}_\alpha$ is layered. If $X$ is a countable topological space whose Cantor–Bendixson rank is at least $\alpha$, then $\mathrm{CB}_\alpha(X)$ is layered (see Definition 1.9.4).*

*(4) If $\mathcal{J}$ is a layered ideal and $\mathcal{I}$ is an arbitrary ideal on $\mathbb{N}$, then $\mathcal{J} \times \mathcal{I}$ is layered.*

Proof. (1) If $\mathcal{I}$ is $F_\sigma$ then by Theorem 1.4.6, $\mathcal{I} = \mathrm{Fin}(\varphi)$ for some lower semicontinuous submeasure $\varphi$. Then $f = \varphi$ satisfies conditions (L1)–(L3) from Definition 11.1.1.

(2) Take an increasing sequence $\alpha_n$, for $n \in \mathbb{N}$, of ordinals converging to $\alpha$ and let $f(A) = \min\{n : \alpha_n \text{ does not embed into } A\}$. Since $P$ is well-ordered, the conditions (L1)–(L3) are easily verified.

(3) Let $\alpha_n$, for $n \in \mathbb{N}$, be an increasing sequence of ordinals converging to $\alpha$ and let $f(A) = \min\{n : \text{the Cantor–Bendixson rank of } A \text{ is smaller than } \alpha_n\}$. Conditions (L1)–(L3) are easily verified.

(4) Let $f_{\mathcal{J}}$ be a function satisfying (L1)–(L3) for $\mathcal{J}$, and let (for $A \subseteq \mathbb{N}^2$ we write $A_n = \{m : (n,m) \in A\}$) $f(A) = f_{\mathcal{J}}\{n : A_n \notin \mathcal{I}\}$. Then (L1) and (L2) are clearly satisfied. To prove (L3), fix $A$ such that $f(A) = \infty$. If $B = \{n : A_n \notin \mathcal{I}\}$, for each $n$ find $B_n \subseteq B$ such that $f_{\mathcal{J}}(B_n) \geq n$. Then $f(A \cap (B_n \times \mathbb{N})) = f_{\mathcal{J}}(B_n) \geq n$ for each $n$, therefore (L3) is satisfied. □

The following is essentially [97, Corollary 2.4], we include a proof for reader's convenience (for $\mathrm{P}^+$-ideals see Definition 1.3.2 and Lemma 1.3.4 ).

**Proposition 11.1.3.** *For an ideal $\mathcal{I}$ on $\mathbb{N}$ such that $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is atomless the following are equivalent:*

*(1) The quotient $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is $\aleph_1$-saturated.*
*(2) $\mathcal{I}$ is a $P^+$-ideal.*
*(3) $\mathcal{P}(\mathbb{N})/\mathcal{I}$ has no $\aleph_0$-limits.*

*If $\mathcal{I}$ is an analytic P-ideal, then the above conditions are equivalent to*

*(4) $\mathcal{I}$ is $F_\sigma$.*

This result dates back to the overlooked [71]. We include a proof for reader's convenience.

Clearly (1) implies (2), and (2) is equivalent to (3) by Lemma 1.3.4. Implication from (3) to (1) and (4) are proved after three lemmas.

**Lemma 11.1.4.** *If $\mathbb{B}$ is an atomless Boolean algebra and $\mathbf{t}(x)$ is a satisfiable quantifier-free 1-type over some $X \subseteq \mathbb{B}$, then there are $X' \subseteq \mathbb{B}$ of cardinality $|X| + \aleph_0$ and a satisfiable 1-type such that each one of its conditions is of one of the following forms:*

$$a \wedge x = 0, \qquad b \setminus x = 0, \qquad c \wedge x \neq 0, \qquad d \setminus x \neq 0 \tag{11.1}$$

*for some $a, b, c,$ or $d$ in $X'$.*

Proof. Every condition in $\mathbf{t}(x)$ is of the form $w(x) = 0$ or $w(x) \neq 0$ for some Boolean expression $w$ over $X$. Let us consider a condition of the form $w(x) = 0$. We may present $w$ in disjunctive normal form $\bigvee_{j<m}(a_j \wedge x) \vee \bigvee_{j<n}(b_j \setminus x)$, for some $a_j, b_j$ in $\mathcal{P}(\mathbb{N})/\mathcal{I}$. This condition can be replaced with $m+n$ conditions $a_j \wedge x = 0$ for $j < m$ and $b_j \setminus x = 0$ for $j < n$.

Now consider a condition of the form $w_k(x) \neq 0$. Again present $w$ in disjunctive normal form (now we have to be a bit more careful to keep track on how parameters depend on the condition) $\bigvee_{j<m(k)}(a_{j,k} \wedge x) \vee \bigvee_{j<n(k)}(b_{j,k} \setminus x)$, for some $a_{j,k}, b_{j,k}$ in $X$. Then $w_k(x) \neq 0$ is equivalent to (here $\bigvee$ stands for disjunction)

$$\bigvee_{j<m(k)}(a_{j,k} \wedge x \neq 0) \vee \bigvee_{j<n(k)}(b_{j,k} \setminus x \neq 0).$$

Since $\mathbf{t}(x)$ is consistent, for every finite set of conditions $F$ in $\mathbf{t}(x)$ there is $c_F$ in $X$ such that for every condition $w_k(x) \neq 0$ in $F$ there is $i < m(k)$ such that $a_{i,k} \wedge c_F \neq 0$ or $j < n(k)$ such that $b_{j,k} \setminus c + F \neq 0$. Let $g_F$ be the function on $F$ defined by Define function $f$ on $\mathbb{N}$ by setting

$$g_F(k) = \begin{cases} (a, i), & \text{if } i < m(k) \text{ is the least such that } a_{i,k} \wedge c_F \neq 0, \text{ or} \\ (b, j), & \text{if } j < n(k) \text{ is the least such that } b_{j,k} \setminus c + F \neq 0. \end{cases}$$

If $\mathbf{t}(x)$ has only finitely many conditions then consistency implies satisfiability, hence we may assume it has infinitely many conditions. Let $\mathcal{U}$ be a nonprincipal ultrafilter on the set of all finite sets of conditions in $\mathbf{t}(x)$. Since there are only finitely many possibilities for the value of $g_F(k)$ for every fixed $k$, we can set $g(k)$ to be such that

$$\{F : g(k) = g_F(k)\} \in \mathcal{U}.$$

Let $\mathbf{t}'(x)$ be obtained by replacing condition $w_k(x)$ with $a_{i,k} \wedge x \neq 0$ if $g(k) = (a, i)$ and with $b_{j,k} \setminus x \neq 0$ if $g(k) = (b, j)$. This type is consistent, each of its conditions is of the form as in (11.1), every realisation of $\mathbf{t}''$ is a realisation of $\mathbf{t}$.

The set $X'$ of parameters occurring in the conditions of $\mathbf{t}'$ clearly has cardinality not greater than $|X| + \aleph_0$. $\square$

**Lemma 11.1.5.** *Suppose that $\mathbb{B}$ is an atomless Boolean algebra with no $\aleph_0$-limits, that $\mathcal{J}$ is an ideal in $\mathbb{B}$, that $d \in \mathbb{B}$ is nonzero and that $c_n \in \mathbb{B}$, for $n \in \mathbb{N}$, are $\mathcal{J}$-positive. Then there is nonzero $\tilde{d} \leq d$ such that $c_n \setminus \tilde{d}$ is $\mathcal{J}$-positive for all $n$.*

Proof. Recursively choose a decreasing sequence $d = d_0 \geq d_1 \geq \dots$ in $\mathbb{B} \setminus \{0\}$ such that $c_n \setminus d_n$ is $\mathcal{J}$-positive for all $n$. This is possible because $\mathbb{B}$ is atomless, hence we can split $d_n$ in two disjoint positive sets, and at least one of them will not be greater than $c_n$. Since $\mathbb{B}$ has no $\aleph_0$-limit, we can choose a positive $\tilde{d}$ such that $\tilde{d} \leq d_n$ for all $n$, and such $\tilde{d}$ is as required. □

Conclusion of the following lemma asserts that $\mathcal{P}(\mathbb{N})/\mathcal{I}$ has no $\aleph_0, \aleph_0$-gaps (also called $\omega, \omega$-gaps, but this author believes that the symbol $\omega$ is used to excess).

**Lemma 11.1.6.** *Suppose that $\mathcal{I}$ is an ideal on $\mathbb{N}$ such that $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is atomless and has no $\aleph_0$-limits. If $\mathcal{A}$ and $\mathcal{B}$ are two countable families in $\mathcal{P}(\mathbb{N})$ such that $A \cap B \in \mathcal{I}$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$, then there is $C \in \mathcal{P}(\mathbb{N})$ such that $B \setminus C \in \mathcal{I}$ and $A \cap C \in \mathcal{I}$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$.*

Proof. Fix enumerations $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$ and $\mathcal{B} = \{B_n : n \in \mathbb{N}\}$. Let

$$C = \bigcup_n B_n \setminus \bigcup_{j \leq n} A_j.$$

For every $n$ the set $B_n \setminus C$ is included in $\bigcup_{j \leq n} A_j$, hence belongs to $\mathcal{I}$, and $A_n \cap C$ is included in $\bigcup_{j \leq n} B_j$, hence belongs to $\mathcal{I}$. □

Proof of Proposition 11.1.3. Assume (2), hence $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is atomless and has no $\aleph_0$-limits and its theory admits elimination of quantifiers (because this theory is $\aleph_0$-categorical, [16, Proposition 1.4.5], see [16, Exercise 1.5.3]).

It therefore suffices to prove that every countable, satisfiable, quantifier-free type $\mathbf{t}(x)$ over $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is realised in $\mathcal{P}(\mathbb{N})/\mathcal{I}$. By a standard proof by induction it suffices to consider 1-types. By Lemma 11.1.4, it suffices to consider 1-types all of whose conditions are as in (11.1). By lifting these elements of $\mathcal{P}(\mathbb{N})/\mathcal{I}$ to $\mathcal{P}(\mathbb{N})$, we obtain four countable (possibly finite) subsets of $\mathcal{P}(\mathbb{N})$, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ such that for $E \subseteq \mathbb{N}$, $[E]_\mathcal{I}$ realises $\mathbf{t}'(x)$ if $A \cap E \in \mathcal{I}$, $B \setminus E \in \mathcal{I}$, $C \cap E \notin \mathcal{I}$, and $D \setminus E \notin \mathcal{I}$ for all $A, B, C, D$, in the corresponding sets.

By Lemma 11.1.5 applied to $\mathbb{B} = \mathcal{P}(\mathbb{N})/\mathcal{I}$ and $\mathcal{J}$ the ideal of $\mathbb{B}$ generated by $\mathcal{B}$ for each $C \in \mathcal{C}$ there is $\tilde{C}$ such that $[D \setminus \tilde{C}]_\mathcal{I} \notin \mathcal{J}$ for all $D \in \mathcal{D}$. (The assumptions of Lemma 11.1.5 are satisfied because the type $\mathbf{t}'(x)$ is satisfiable.) Similarly, for each $D \in \mathcal{D}$ there is $\tilde{D} \subseteq D$ such that $[C \setminus \tilde{D}]_\mathcal{I} \notin \mathcal{J}$ for all $C \in \mathcal{C}$.

Let $\mathcal{A}' = \mathcal{A} \cup \{\tilde{C} : C \in \mathcal{C}\}$ and $\mathcal{B}' = \mathcal{B} \cup \{\tilde{D} : D \in \mathcal{D}\}$. These families satisfy the assumptions of Lemma 11.1.6, and therefore there exists $E \subseteq \mathbb{N}$ such that $B \setminus E \in \mathcal{I}$, $D \cap E \notin \mathcal{I}$, $A \cap E \in \mathcal{I}$, and $C \setminus E \notin \mathcal{I}$ for $A, B, C, D$ in the appropriate sets. Therefore $[E]_\mathcal{I}$ satisfies $\mathbf{t}'(x)$, and therefore $\mathbf{t}(x)$. Since the latter was an arbitrary countable quantifier-free 1-type, $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is $\aleph_1$-saturated and (1) follows.

It remains to prove that if $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is $\aleph_1$-saturated and $\mathcal{I}$ is an analytic P-ideal, then $\mathcal{I}$ is $F_\sigma$. Assume $\mathcal{I}$ is an analytic P-ideal and that it is not $F_\sigma$. By [146, Theorem 3.1], $\mathcal{I} = \mathrm{Exh}(\varphi)$ for a lower semicontinuous submeasure $\varphi$ and since $\mathcal{I}$ is not $F_\sigma$, Corollary 1.4.8 implies that there is a partition of $\mathbb{N}$ into $\mathcal{I}$-positive sets $A_n$, for $n \in \mathbb{N}$, such that $\varphi(A_n) \leq 2^{-n}$ for all $n$. Therefore, the sets $Y_n = \bigcup_{i=n}^\infty X_i$ form an $\aleph_0$-limit in $\mathcal{P}(\mathbb{N})/\mathcal{I}$. □

**Theorem 11.1.7.** *If $\mathcal{I}$ is a layered ideal then $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is $\aleph_1$-saturated.*

Proof. It suffices to check that (2) of Proposition 11.1.3 fails. Let $f$ be a witness that $\mathcal{I}$ is layered. Let $A_i$ ($i \in \mathbb{N}$) be a decreasing sequence of $\mathcal{I}$-positive sets. For each $i$ pick $B_i \subseteq A_i$ such that $f(B_i) \geq i$. Then $A = \bigcup_n B_n$ satisfies $f(A) \geq i$ for all $i$, hence it is $\mathcal{I}$-positive. Also, $A \setminus A_i \subseteq \bigcup_{j=1}^{i-1} B_j \in \mathcal{I}$, and $A$ is as required. □

**Corollary 11.1.8.** *The quotient over each of the following ideals is $\aleph_1$-saturated.*

1. *Every $F_\sigma$-ideal.*
2. *$\mathcal{O}_\alpha$ and $\mathcal{O}_\alpha(P)$ if $P$ is a well-ordered linear order and $\alpha$ is an additively indecomposable countable ordinal.*
3. *$\mathcal{W}_\alpha$ and $\mathrm{CB}_\alpha(X)$, if $X$ is a countable topological space whose Cantor-Bendixson rank is at least $\alpha$ and $\alpha$ is multiplicatively indecomposable countable ordinals.*
4. *$\mathcal{J} \times \mathcal{I}$, if $\mathcal{J}$ is a layered ideal and $\mathcal{I}$ is an arbitrary ideal on $\mathbb{N}$.* □

**Proposition 11.1.9.** CH *implies that every quotient over an analytic ideal that includes* Fin *embeds into every quotient over an analytic ideal that includes* Fin.

Proof. It suffices to prove that if $\mathcal{J} \supseteq \mathrm{Fin}$ is an analytic ideal then $\mathcal{P}(\mathbb{N})/\mathcal{J}$ embeds into $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ and vice versa. By Corollary 3.2.3, $\mathrm{Fin} \leq_{\mathrm{RB}} \mathcal{J}$ and therefore $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ embeds into $\mathcal{P}(\mathbb{N})/\mathcal{J}$. Corollary 11.1.8 implies that the quotient $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ is $\aleph_1$-saturated, and therefore $\mathfrak{c}$-saturated if CH holds. Therefore, $\mathcal{P}(\mathbb{N})/\mathcal{J}$ embeds into $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$. □

Part (1) of the following theorem was proved in [95].

**Corollary 11.1.10.** *Assume CH. Then the quotients over all of the following ideals are isomorphic.*

1. *All $F_\sigma$ ideals that include* Fin.
2. *All $\mathcal{O}_\alpha(P)$ for an indecomposable countable ordinal $\alpha$ and well-ordered linear order $P$ (Definition 1.9.1).*
3. *All Cantor–Bendixson ideals $\mathrm{CB}_\alpha(X)$ (§1.9.2).*
4. *All ideals of the form $\mathcal{I} \times \mathcal{J}$ where $\mathcal{I}$ is as in the previous items and $\mathcal{J}$ is an arbitrary ideal.*

*In particular, all quotients over ordinal ideals $\mathcal{O}_\alpha$ for $\alpha$ countable and indecomposable, all Weiss ideals $\mathcal{W}_\alpha$ for $\alpha$ countable and multiplicatively indecomposable, and all $F_\sigma$ ideals that include* Fin *are pairwise isomorphic.*

Proof. By Proposition 11.1.2 all of these ideals are layered, hence Theorem 11.1.7 implies that their quotients are $\aleph_1$-saturated. Since the theory of atomless Boolean algebras is complete, all of these quotients are elementarily equivalent and (because of the CH) saturated structures of the same cardinality, and therefore isomorphic by [54, Corollary16.6.7]. □

Saturation of quotients over $F_\sigma$ ideals was proved in [95, §1] (although it is an immediate consequence of Galvin's result and the characterisation of $F_\sigma$ ideals, Theorem 1.4.6). The fact that $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ is $\aleph_1$-saturated is consequence of an earlier result of P. Olin (see [89]).

Since the restriction of an $F_\sigma$ ideal that includes Fin to a positive set is $F_\sigma$ and it includes Fin, Corollary 11.1.10 immediately implies the following.

**Corollary 11.1.11.** *Assume CH. If $\mathcal{I}$ is an $F_\sigma$ ideal that includes* Fin*, then $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is homogeneous.* □

By using CH one can construct $2^{2^{\aleph_0}}$ nontrivial automorphisms of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$, most of them not very interesting. Conjugacy of trivial automorphisms of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ was studied in [14] and [15], resulting in more interesting automorphisms. The main result of the former asserts that CH implies that the left and right shift on $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ are conjugate. I am not aware of any results on conjugacy of trivial automorphisms of quotients over other analytic ideals under CH. Of course this relation is trivialised under $\mathrm{OCA_T}$ and MA($\sigma$-linked).

It is not known whether every analytic ideal such that the quotient $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is $\aleph_1$-saturated is layered. In [96, Corollary 2.20] it was shown that there exists a $\nabla$-ideal $\mathcal{I}$ that is no layered, although $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is atomless. This ideal is not analytic.

## 11.2. Continuous saturation

We prove that CH implies all quotients over EU-ideals are isomorphic (Theorem 11.2.3 (1)), that all quotients over dense density ideals that are not isomorphic to EU-ideals are isomorphic (Theorem 11.2.3 (2)), that there are six isomorphism classes of quotients over dense ideals of the form $\operatorname{Exh}(\sup_n \mu_n)$ where $\mu_n$, for $n \in \mathbb{N}$, are measures concentrating on disjoint subsets of $\mathbb{N}$ (Theorem 11.2.5), and that all quotients over dense LV-ideals are isomorphic (Theorem 11.2.6). All of these results of this section are taken from [95] and [49], but we offer novel proofs. They use the fact that each one of these quotients is equipped with a complete metric (Proposition 5.2.2) and is $\aleph_1$-saturated (Proposition 11.2.2) as a metric structure (see [8], [78] for continuous logic). For some time it was clear that such a proof ought to exist (see [59, the text preceding Corollary 6.5], [54, Notes to Chapter 16]) but finding one took surprisingly (at the hindsight) long time.

No quotient over a dense generalised density ideal is $\aleph_1$-saturated *when considered as a discrete structure* by Proposition 4 (5). However, the quotient over every analytic P-ideal $\operatorname{Exh}(\varphi)$ is equipped with a complete metric $d_\varphi$ (Proposition 5.2.2[2]).

The following is taken from [59, Proposition 5.11].

**Proposition 11.2.1.** *Suppose that $\mathcal{Z}_\varphi$ is a generalised ideal defined by parameters $I_n, \varphi_n$. Then $\mathcal{P}(\mathbb{N})/\mathcal{Z}_\varphi$ is isomorphic to the reduced product*

$$\textstyle\prod_{\operatorname{Fin}}(\mathcal{P}(I_n), \mu_n).$$

Proof. To define an isomorphism, identify $X \subseteq \mathbb{N}$ with the sequence $(X \cap I_n)_n$. This defines a bijection between $\mathcal{P}(\mathbb{N})$ and $\prod_n \mathcal{P}(I_n)$ that also satisfies (with $\varphi = \sup_n \varphi_n$, so that $\mathcal{Z}_\varphi = \operatorname{Exh}(\varphi)$) $\liminf_k \varphi(X \setminus k) = \limsup_n \varphi_n(X \cap I_n)$, and hence defines an isometric isomorphism between these structures. □

**Proposition 11.2.2.** *Suppose that $\mathcal{Z} = \operatorname{Exh}(\varphi)$ is a generalised density ideal. Then the metric sructure $(\mathcal{P}(\mathbb{N})/\mathcal{Z}, d_\varphi)$ is $\aleph_1$-saturated.*

Proof. Let $I_n$, $\varphi_n$ be disjoint sets and a strictly positive submeasures on them, such that $\mathcal{Z} = \operatorname{Exh}(\sup_n \varphi_n)$. Consider $(\mathcal{P}(I_n), \varphi_n)$ as a metric structure with respect to the metric

$$d_\varphi(s,t) = \varphi(s \Delta t).$$

[2] This metric is discrete if and only if the ideal is $F_\sigma$, which begs the question whether every quotient over an analytic P-ideal is $\aleph_1$-saturated when considered as a metric structure?

Since replacing $\varphi$ with the function $\varphi'(A) = \varphi(A) + \sum_{n\in A} 2^{-n}$ does not affect $\mathrm{Exh}(\varphi)$ or $\mathrm{Fin}(\varphi)$, we may assume that $\varphi$ is strictly positive and $d$ is a metric. This metric structure is denoted $(\mathcal{P}(I_n), \mu_n)$, with $\varphi_n$ suppressed since it is definable from $\mu_n$.[3] The desired conclusion now follows by Proposition 11.2.1. □

The EU-ideal case of Theorem 11.2.3 was proved in [95] and the remaining part was proved in [49], which also contains a different proof of the first part. The original proofs of these results were quite long and, I dare say, fairly opaque. we finally have the right model-theoretic statement and a natural proof.

The following stands in stark contrast with the fact that, assuming $\mathrm{OCA_T}$ and MA($\sigma$-linked), by Theorem 7.1.1 quotients over density ideals are isomorphic if and only if the ideals are RK-isomorphic, and that in particular quotients over $\mathcal{Z}_0$ and $\mathcal{Z}_{\log}$ are not isomorphic (Corollary 7.1.2).

**Theorem 11.2.3.** *Suppose that $\mathcal{Z}_\mu$ and $\mathcal{Z}_\nu$ are dense density ideals.*

*(1) If they are both EU-ideals, then their quotients are isomorphic and homogeneous.*

*(2) If neither of them is an EU-ideal, then their quotients are isomorphic.*

*If* CH *holds, then the quotients over all EU-ideals are isomorphic and the quotients over all density ideals that are not EU-ideals are not isomorphic.*

By Proposition 5.1.7, if $\mathcal{Z}_\mu$ is an EU-ideal and $\mathcal{Z}_\nu$ is a dense density ideal that is not an EU-ideal, then the corresponding quotients are not isomorphic.

Proof. (1) All structures, reduced products, and ultraproducts in this proof are metric. Fix parameters $I_n, \mu_n, J_n, \nu_n$ determining $\mathcal{Z}_\mu$ and $\mathcal{Z}_\nu$. Thus, $I_n$ are disjoint finite subsets of $\mathbb{N}$ and $\mu_n$ are probability measures on these sets such that $\lim_n \max_{j\in J_n} \mu_n(\{j\}) = 0$ and $\mu_n(I_n) = 1$ for all $n$. Also, $J_n, \nu_n$ share these properties. Consider $(\mathcal{P}(I_n), \mu_n)$ as a metric structure. It is a finite Boolean algebra with a probability measure and the metric $d_n(s,t) = \mu_n(s\Delta t)$. (We may assume that $\mu_n$ is strictly positive, and since $d_n$ is definable from $\mu_n$, we suppress it.) Then $\mathcal{P}(\mathbb{N})/\mathcal{Z}_0$ is isomorphic to the reduced product $\prod_{\mathrm{Fin}}(\mathcal{P}(I_n), \mu_n)$ in the logic of metric structures (Proposition 11.2.1). We will prove that the theories of $(\mathcal{P}(I_n), \mu_n)$ converge as $n \to \infty$.

Let $\mathcal{U}$ be a nonprincipal ultrafilter on $\mathbb{N}$ and consider the ultraproduct $(\mathbb{B}_\mathcal{U}, \mu_\mathcal{U}) = \prod_\mathcal{U}(\mathcal{P}(I_n), \mu_n)$. We only need to prove that the theory of this structure does not depend on the choice of $\mathcal{U}$, and observe that the analogous statement holds for ultraproducts of $(\mathcal{P}(J_n), \nu_n)$. This can be done in at least two ways. The hard way is to prove that $(\mathbb{B}_\mathcal{U}, \mu_\mathcal{U})$ is a Maharam-homogenepus probability measure algebra of Maharam character $2^{\aleph_0}$ and apply Maharam's theorem (see e.g., [66, §331]). We will instead take a separable elementary sumbodel and show that it is isomorphic to the Lebesgue measure algebra (this uses the easy, separable, case of Maharam's theorem).

Before this, note that (by the continuous Łoś's Theorem) $\mu_\mathcal{U}$ is a finitely additive, strictly positive, probability measure on $\mathbb{B}_\mathcal{U}$. The continuity of $\mu_\mathcal{U}$ implies

[3] One minor detail needs to be addressed. We are not assuming that the sequence $\varphi_n(I_n)$ is bounded, which technically presents a problem because in continuous logic the metric structures are supposed to be of uniformly bounded diameter. However, this problem is routinely resolved by introducing domains of quantification, see e.g., [78]. This change affects the syntax, but not the semantics, and is therefore innocuous. Alternatively, one could replace $\varphi_n$ by $\min(1, \varphi_n)$; this would of course affect the metric, but not the isomorphism type of the quotient.

that it is even $\sigma$-additive. Therefore $(\mathbb{B}_\mathcal{U}, \mu_\mathcal{U})$ is a probability measure algebra. Since $\lim_n \max_{j\in J_n} \mu_n(\{j\}) = 0$, it is atomless. A separable elementary submodel is therefore an atomless probability measure algebra and therefore isomorphic to the Lebesgue measure algebra $(\mathbb{B}, \lambda)$ on $[0, 1]$.

Thus the theory of $(\mathbb{B}_\mathcal{U}, \mu_\mathcal{U})$ does not depend on the choice of $\mathcal{U}$. By the continuous Feferman–Vaught theorem ([75], see [54, Theorem 16.5.2]), $\prod_{\mathrm{Fin}}(\mathcal{P}(I_n), \mu_n)$ is elementarily equivalent to $\prod_{\mathrm{Fin}}(\mathbb{B}, \lambda)$. This applies to $\prod_{\mathrm{Fin}}(\mathcal{P}(J_n), \nu_n)$.

Since the restriction of a density ideal to a positive set is a density ideal, homogeneity of the quotient follows.

(2) The proof is very similar to the proof of (1), modulo the issue of having to deal with metric structures of arbitrarily large diameter, but see [78]. Theorem 2.7.8 implies that $\sup_n \mu_n(I_n) = \infty$ and $\sup_n \nu_n(J_n) = \infty$. Again $\mathcal{P}(\mathbb{N})/\mathcal{Z}_\mu$ is isomorphic to $\prod_{\mathrm{Fin}}(\mathcal{P}(I_n), \mu_n)$, and it suffices to prove that the theories of $(\mathcal{P}(I_n), \mu_n)$ and $(\mathcal{P}(J_n), \nu_n)$ converge to the same limit. This is not quite true, and we will have to consider two cases.

Assume for a moment that $\lim_n \mu_n(I_n) = \infty$ and $\lim_n \nu_n(J_n) = \infty$. We will show that the theory of the ultraproduct $(\mathbb{B}_\mathcal{U}, \mu_\mathcal{U}) = \prod_\mathcal{U}(\mathcal{P}(I_n), \mu_n)$ does not depend on $\mathcal{U}$. Again we will show that this theory is the theory of a familiar structure. A separable elementary submodel of $(\mathbb{B}_\mathcal{U}, \mu_\mathcal{U})$ is a measure algebra with a strictly positive, $\sigma$-additive, unbounded measure. By separability, this measure is $\sigma$-finite and therefore this ultraproduct is isomorphic to the Lebesgue measure algebra on $\mathbb{R}$. The conclusion follows as in (1).

In the remaining part of the proof we show that if $\sup_n \mu_n(I_n) = \infty$ then $\prod_{\mathrm{Fin}}(\mathcal{P}(I_n), \mu_n)$ is elementarily equivalent to $\prod_{\mathrm{Fin}}(\mathcal{P}(J_n), \nu_n)$ if $\lim_n \nu_n(J_n) = \infty$. Fix a sequence $k(n)$ in $\mathbb{N}$ such that $\lim_n \mu_{k(n)}(I_{k(n)}) = \infty$ and let $l(n)$, for $n \in \mathbb{N}$, enumerate the complement of $\{k(n) : n \in \mathbb{N}\}$ (if this complement is finite, then $\lim_n \mu_n(I_n) = \infty$). Let $\tilde{I}_n = I_{k(n)} \cup I_{l(n)}$ and $\tilde{\mu}_n = \mu_{k(n)} + \mu_{l(n)}$. Then $A \subseteq \mathbb{N}$ is $\mathcal{Z}_{\tilde{\mu}}$-positive if and only if it is $\mathcal{Z}_\mu$-positive (as in the proof of Lemma 2.7.4). Thus the ideal $\mathcal{Z}_\mu$ is presented by parameters $\tilde{I}_n, \tilde{\mu}_n$ that satisfy the assumptions of first part, and this concludes the proof.

Metric reduced products over Fin are $\aleph_1$-saturated. This follows [61, Theorem 1.5], see [54, Theorem 16.5.1] for a simpler proof. Therefore CH implies that they are saturated hence (isometrically)[4] isomorphic if and only if they are elementarily equivalent. Therefore (1) and (2) imply the desired conclusion. □

This proof gives additional information that may be worth recording. Let $(\mathbb{B}, \lambda)$ denote the Lebesgue measure algebra on $[0, 1]$, with the metric $d_\lambda(A, B) = \lambda(A \Delta B)$ associated with $\lambda$, let $\{0, 1\}^\mathbb{N}$ denote the Cantor space, and let $\mathbb{B}_\infty$ denote the Lebesgue measure algebra on $\mathbb{R}$, with $\lambda$ and $d_\lambda$ as in $\mathbb{B}$.

**Theorem 11.2.4.** *(1) If $\mathcal{Z}_\mu$ is an EU-ideal, then $(\mathcal{P}(\mathbb{N})/\mathcal{Z}_\mu, d_\mu)$ is elementarily equivalent to the structure $\mathbb{B}_\infty$ whose elements are $d_\lambda$-continuous functions from $\{0, 1\}^\mathbb{N}$ into $\mathbb{B}$, equipped with the uniform metric*

$$d(f, g) = \max_{x\in\{0,1\}^\mathbb{N}} d_\lambda(f(x), g(x)).$$

[4] Note that, in order to obtain isometry between quotients, we had to modify the submeasures as in the proof of Lemma 2.7.4 in order to assure that all $\mu_n$, $\nu_n$, are probability measures. In the general case we obtain only isomorphism of discrete structures.

*(2) If $\mathcal{Z}_\mu$ is a density ideal such that $\lim_n \mu_n(I_n) = \infty$, then $(\mathcal{P}(\mathbb{N})/\mathcal{Z}_\mu, d_\mu)$ is elementarily equivalent to the structure whose elements are $d_\lambda$-continuous functions from $\{0,1\}^{\mathbb{N}}$ into $\mathbb{B}_\infty$, equipped with the uniform metric*

$$d(f,g) = \max_{x \in \{0,1\}^{\mathbb{N}}} d_\lambda(f(x), g(x)).$$

Proof. We prove only the first part since the proof of the second part is analogous. The proof of Theorem 11.2.3 (1) shows that $\mathcal{P}(\mathbb{N})/\mathcal{Z}_\mu$ is elementarily equivalent to $\prod_{\mathrm{Fin}}(\mathbb{B}, \lambda)$. By [55, Proposition 3.5 (1)],[5] this reduced product is elementarily equivalent to $\mathbb{B}_\infty$ (in the notation of [55], $\mathbb{B}_\infty$ is $K(\mathbb{B}, \lambda)$). □

The following is [49, Theorem 7.3].

**Theorem 11.2.5.** *Assume* CH *and let $\mathcal{I} = \mathrm{Exh}(\sup_n \mu_n)$ for a sequence $(\mu_n)$ of measures concentrating on disjoint, possibly infinite, subsets of $\mathbb{N}$. if $\mathcal{I}$ is dense, then $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is isomorphic to the quotient of exactly one of the following.*

*(1) $\mathcal{Z}_0$.*
*(2) $\mathcal{Z}_\infty$, a dense density ideal that is not an EU-ideal.*
*(3) The summable ideal $\mathcal{I}_{1/n}$.*
*(4) $\mathcal{I}_{1/n} \oplus \mathcal{Z}_0$.*
*(5) $\mathcal{I}_{1/n} \oplus \mathcal{Z}_\infty$.*
*(6) $\mathcal{I}_\infty$ (see Lemma 2.8.1).*

Proof. Suppose that $\mathcal{I} = \mathcal{Z}_\mu$ and measures $\mu_n$ concentrate on disjoint sets $I_n$. If all $I_n$ are finite, then $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is isomorphic to the quotient over $\mathcal{Z}_0$ or of $\mathcal{Z}_\infty$ by Theorem 11.2.3. If there is $A \in \mathcal{I}$ such that $I_n \setminus A$ is finite for all $n$, then the same conclusion applies.

If there is no such $A$ then for some $n$ the restriction of $\mathcal{Z}_\mu$ to $I_n$ is a summable ideal. If there are only finitely many such $n$, then the restriction of $\mathcal{I}$ to the union of those $I_n$ is summable and (since all quotients over summable ideals are isomorphic by Corollary 11.1.10) $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is isomorphic to the quotient over $\mathcal{I}_{1/n} \oplus \mathcal{Z}_0$, to that over $\mathcal{I}_{1/n} \oplus \mathcal{Z}_\infty$, or to that over $\mathcal{I}_{1/n}$ (the last case applies if $\lim_n \mu_n(I_n) = 0$).

Therefore, there are infinitely many $n$ such that the restriction of $\mathcal{I}$ to $I_n$ is a summable ideal. By Lemma 2.8.2, we may assume that this is the case for all $n$, hence by Proposition 11.2.1 we have (the following is a metric reduced product)

$$\mathcal{P}(\mathbb{N})/\mathcal{I} \cong \prod\nolimits_{\mathrm{Fin}} \mathcal{P}(I_n)/\mathcal{I}_n$$

where $\mathcal{I}_n$ is the summable ideal on $I_n$ associated with $\mu_n$. The submeasure associated to $\mu_n$ induces a discrete metric on the atomless Boolean algebra $\mathcal{P}(I_n)/\mathcal{I}_n$, hence all of these quotients are elementarily equivalent (in continuous logic). By the Feferman–Vaught theorem, the theory of $\mathcal{P}(\mathbb{N})/\mathcal{I}$ does not depend on the choice of summable ideals $\mathcal{I}_n$. Therefore this quotient is elementarily equivalent to that over $\mathcal{I}_\infty$, and the desired conclusion follows. □

The second part of the following is [49, Theorem 5.5] (see Definition 1.7.7 for LV-ideals).

**Theorem 11.2.6.** *All quotients over dense LV-ideals such that $\varphi_n(I_n) = 1$ for all $n$ are elementarily equivalent as metric structures. Therefore,* CH *implies that all quotients over dense LV-ideals are isomorphic and homogeneous.*

[5] This proposition gives more information than necessary, it describes an elementary embedding between these structures.

Proof. Fix finite sets $I_n$ and submeasures $\varphi_n$ on $I_n$ that satisfy conditions (LV3)–(LV2) of Definition 1.7.7, thus $\lim_n \max_{j\in I_n} \varphi_n(\{j\}) = 0$, $\varphi_i(I_i) \geq 1$ for all $i$, and for all $k$, $\varepsilon > 0$, and all large enough $n$ we have

$$(\forall a_0, \dots, a_k \subseteq I_n)|\varphi_n(a_0 \Delta a_k) - \max_{i<k} \varphi_n(a_i \Delta a_{i+1})| < \varepsilon, \tag{11.2}$$

As in the proof of Theorem 11.2.3, it suffices to prove that the theory of the metric ultraproduct $(\mathbb{B}_{\mathcal{U}}, \varphi_{\mathcal{U}}) = \prod_{\mathcal{U}}(\mathcal{P}(I_n), \varphi_n)$ does not depend on the choice of a nonprincipal ultrafilter $\mathcal{U}$.

Assume $\varphi_i(I_i) = 1$ for all $i$. By (11.2), every finite family of elements $a_0, \dots, a_k$ of $\mathbb{B}_{\mathcal{U}}$ satisfies $\varphi_{\mathcal{U}}(\bigcup_{j\leq k} a_j) = \max_{j\leq k} \varphi_{\mathcal{U}}(a_j)$. For $0 < r \leq 1$ let

$$\mathcal{X}_r = \{a \in \mathbb{B}_{\mathcal{U}} : \varphi_{\mathcal{U}}(a) = r\}.$$

Then for every $0 < t \leq 1$ the set $\bigcup_{r<t} \mathcal{X}_r$ is an ideal in $\mathbb{B}_{\mathcal{U}}$, and for all $r < t$ and $a \in \mathcal{X}_t$ the set $\{b \in \mathcal{X}_r : b \leq a\}$ is infinite.

Perhaps the easiest way to prove elementary equivalence of Boolean algebras with submeasures of this form is via taking a countable dense set. To every $(\mathcal{B}_{\mathcal{U}}, \varphi_{\mathcal{U}})$ of this form associate the discrete two-sorted structure with sorts $\mathcal{B}_{\mathcal{U}}$ and $[0, 1]$ and function $\varphi_{\mathcal{U}}$. Let $(\mathcal{B}_0, \mathcal{D}_0)$ be a countable elementary submodel. It is rather straightforward to show that a back-and-forth argument gives an isomorphism between any two such models $(\mathcal{B}_0, \mathcal{D}_0)$ and $(\mathcal{B}_1, \mathcal{D}_1)$ such that $\mathcal{D}_0 = \mathcal{D}_1$ (this condition can be easily arranged). This isomorphism is isometric, hence extends to an isomorphism between completions. These completions are separable metric elementary submodels, and this completes the proof.

Now assume CH. By the first part, all quotients over dense LV-ideals whose submeasures satisfy $\varphi_n(I_n) = 1$ for all $n$ are isometrically isomorphic. In general case, replace $\varphi_n$ with $\min(1, \varphi_n)$ and note that this affects nether the ideal nor conditions (LV3)–(LV1) and apply the first part.

Since the restriction of a dense LV-ideal to a positive set is a dense LV-ideal, homogeneity of its quotient follows. □

Since the restriction of an LV-ideal to a positive set is an LV-ideal (Lemma 1.7.9), Theorem 11.2.6 implies the following.

**Corollary 11.2.7.** *Assume CH. If $\mathcal{I}$ is an LV-ideal, then $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is homogeneous.* □

CHAPTER 12

# Other directions

While the main emphasis in this text is given to rigidity of quotients of the form $\mathcal{P}(\mathbb{N})/\mathcal{I}$, other directions of ideal-related research ought to be mentioned. Rigidity for coronas of C*-algebras (noncommutative version of the results from Chapter 9), was discussed briefly in §9.4, see also [59] and the extensive discussion of other threads of research in [59, §11].

**Rigidity of reduced products in other categories.** For every ideal $\mathcal{I}$, the quotient $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is naturally isomorphic to the reduced product of the two-element Boolean algebra with respect to the ideal $\mathcal{I}$. What categories, other than the category of two-element Boolean algebras, admit rigidity results for their reduced product associated with Fin (or other analytic ideals)? This question was first considered by Just for lattices ([93], [94]) and in [42] the author connected rigidity results with Ulam-stability. Not much else happened for over two decades. In [21] (see [58, Theorem 1] for a deeper take on this condition) a condition of 'recognising coordinates' was introduced and proved that if a category $\mathcal{C}$ recognises coordinates then $\mathrm{OCA_T}$ and MA($\sigma$-linked) imply strong rigidity results for reduced products $\prod_{\mathrm{Fin}} \mathfrak{M}_n$, for countable $\mathfrak{M}_n \in \mathcal{C}$, for $n \in \mathbb{N}$. This was used to prove that forcing axioms imply rigidity for reduced products of countable linear orders, random graphs ([21]) and some (but not all) classes of groups ([58]) over Fin. On the other hand, if the theory of $\prod_{\mathrm{Fin}} \mathfrak{M}_n$ is stable (in the model-theoretic sense) then this reduced product is fully saturated, hence its isomorphism class depends on its first-order theory only (see [20]). The question whether analogous rigidity results extend to other ideals hadn't been studied yet, but for the continuous version and application see [23] and [22]. For more see [59].

**Forcing with quotients.** Quotients $\mathcal{P}(\mathbb{N})/\mathcal{I}$ for a Borel ideal were considered as forcing notions in [85], see also [82]. By [85, Theorem 1.3], under certain definability assumptions on $I$, the $\mathcal{P}(\{0,1\}^{<\mathbb{N}})/\operatorname{tr}(I)$ is forcing equivalent to the iteration of Borel $/I$ and $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$, and in particular proper (see also §1.7.4.1).

**Tukey reductions.** Tukey order on directed sets, and ideals in particular, has attracted considerable attention ([65]). Its behaviour on LV-ideals is very similar to that on the orders considered here ([119]), but with respect to it the summable ideal $\mathcal{I}_{1/n}$ has a considerably more prominent place ([148]). As in the case of the Katětov order, category and measure play a role in understanding the order ([148]).

## 12.1. Convergence

Much effort has been made towards understanding convergence along ideals (e.g., [4]). For example, in [5] the notion of ideal limit points has been studied.

Given an ideal $\mathcal{I}$ on $\mathbb{N}$ and a sequence $\bar{x} = (x_n)$ in a first-countable topological space $X$, one considers the set of $\mathcal{I}$-limit points defined as

$$\Lambda_{\bar{x}}(\mathcal{I}) = \{\lim_{j\to\infty} x_{n(i)} : \{n(j) : j \in \mathbb{N}\} \in \mathcal{I}_+, \text{ limit exists}\}.$$

In [5] it was proved that if $\mathcal{I}$ is an analytic P-ideal and not all points in $X$ are isolated, then $\Lambda_x(\mathcal{I})$ is closed for all sequences $\bar{x}$ in $X$ if and only if $\mathcal{I}$ is $F_\sigma$.

**Proposition 12.1.1.** *Suppose that $X$ is a first-countable space not all of whose points are isolated and $\mathcal{I}$ is an ideal on $\mathbb{N}$ such that $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is atomless, and consider the following four statements.*

*(1) $\Lambda_{\mathcal{I}}(\bar{x})$ is closed for all sequences $\bar{x}$ in $X$.*
*(2) For every family $A_n$, for $n \in \mathbb{N}$ of disjoint $\mathcal{I}$-positive sets there is an $\mathcal{I}$-positive $B$ such that $B \cap A_n$ is finite for all $n$.*
*(3) The quotient $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is $\aleph_1$-saturated.*
*(4) The quotient $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is $\aleph_1$-saturated and $\mathcal{I}$ is a P-ideal.*

*Then* (1) $\Leftrightarrow$ (2), (1) $\Rightarrow$ (3), *and* (4) $\Rightarrow$ (1).

PROOF. Let $x$ be a non-isolated point in $X$ and fix a nontrivial sequence $(y_n)$ converging to $x$.

(2) $\Leftrightarrow$ (1): Since $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is atomless, we can fix a partition $\mathbb{N} = \bigsqcup_n A_n$ into $\mathcal{I}$-positive sets. Let $x_j = y_n$ if $j \in A_n$. Then $y_n \in \Lambda_{\bar{x}}(\mathcal{I})$ for all $n$, and $x \in \Lambda_{\bar{x}}(\mathcal{I})$ if and only if there is an $\mathcal{I}$-positive $B$ such that $B \cap A_n$ is finite for all $n$.

(2) $\Rightarrow$ (3): Assume $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is not $\aleph_1$-saturated. Since $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is atomless, $\mathcal{I} \supseteq \mathrm{Fin}$ and by Proposition 11.1.3 there is an $\aleph_0$-limit in $\mathcal{P}(\mathbb{N})/\mathcal{I}$ (see Lemma 1.3.4). Thus, there are $\mathcal{I}$-positive sets $B_n \subseteq \mathbb{N}$ such that $B_n \supseteq B_{n+1}$ for all $n$ and there is no $\mathcal{I}$-positive set $B$ such that $B \setminus B_n \in \mathcal{I}$ for all $n$. This implies that infinitely many of the differences $B_n \setminus B_{n+1}$ are $\mathcal{I}$-positive, and by passing to a subsequence we may assume that all of them are $\mathcal{I}$-positive. Hence, $A_n = B_n \setminus B_{n+1}$ are disjoint $\mathcal{I}$-positive sets, but there is no $\mathcal{I}$-positive set $B$ such that $B \cap A_n$ is finite for all $n$, and (2) fails.

(4) $\Rightarrow$ (2): Assume $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is $\aleph_1$-saturated and $\mathcal{I}$ is a P-ideal. By Proposition 11.1.3, there are no $\aleph_0$-limits in $\mathcal{P}(\mathbb{N})/\mathcal{I}$. If $A_n$, for $n \in \mathbb{N}$, are as in (2), then the sets $B_n = \bigcup_{j\geq n} A_j$ do not form an $\aleph_0$-limit in $\mathcal{P}(\mathbb{N})/\mathcal{I}$, hence there is an $\mathcal{I}$-positive $B \subseteq \mathbb{N}$ such that $B \setminus B_n \in \mathcal{I}$ for all $n$. Since $\mathcal{I}$ is a P-ideal, there is $C \in \mathcal{I}$ such that $(B \setminus B_n) \setminus C$ is finite for all $n$. Therefore, $B \setminus C$ witnesses that the conclusion of (2) holds for $(A_n)$. Since this sequence was arbitrary, (2) follows. □

The $\aleph_1$-saturation of $\mathcal{P}(\mathbb{N})/\mathcal{I}$ alone does not suffice to imply (1) of Proposition 12.1.1, since in [5, Example 4.2] it was proved that the ideal $\mathrm{Fin} \times \mathrm{Fin}$ (also known as $\mathcal{O}_{\omega^2}$, see Definition 1.9.1) does not satisfy this implication. Since its quotient is $\aleph_1$-saturated by Corollary 11.1.8, this implies that the properties of $\Lambda_{\bar{x}}(\mathcal{I})$ depend not only on the quotient, but on the ideal itself. I don't know whether (1) of Proposition 12.1.1 holds for any ideal such that $\mathcal{P}(\mathbb{N})/\mathcal{I}$ is atomoless that is not $F_\sigma$, but Proposition 12.1.2 below (a straightforward application of the ideas in [5, Example 4.2]) rules out all known examples of layered ideals, and therefore all known analytic ideals that include Fin whose quotient is $\aleph_1$-saturated; see §11.

**Proposition 12.1.2.** *Suppose that $\mathcal{I} \supseteq \mathrm{Fin}$ is an ideal on $\mathbb{N}$ and there is $h\colon \mathbb{N} \to \mathbb{N}$ that witnesses $\mathrm{Fin} \leq_{\mathrm{RK}} \mathcal{I}$ such that the following holds.*

*(1) $h^{-1}(\{n\})$ is infinite for all $n$.*

*(2) If $A \subseteq \mathbb{N}$ is such that $h \restriction A$ is finite-to one, then $A \in \mathcal{I}$.*

*Then in every first-countable space with a non-isolated point there is a sequence $\bar{x}$ such that $\Lambda_{\mathcal{I}}(\bar{x})$ is not closed.*

Proof. Let $\mathbb{N} = \bigsqcup_n C_n$ be a partition into infinite sets. Then $A_n = h^{-1}(C_n)$ is $\mathcal{I}$-positive for all $n$. If $B$ is such that $B \cap A_n$ is finite for all $n$, then $h \restriction B$ is finite-to-one and $B \in \mathcal{I}$, hence (2) of Proposition 12.1.1 fails. □

**Corollary 12.1.3.** *Suppose that $X$ is a first-countable space not all of whose points are isolated and $\mathcal{I}$ is an analytic ideal on $\mathbb{N}$ such that* $\mathrm{Fin} \subseteq \mathcal{I}$.

*(1) If $\mathcal{I}$ is an $F_\sigma$ ideal $\Lambda_{\bar{x}}(\mathcal{I})$ is closed for every choice of $\bar{x}$.*

*(2) If $\mathcal{I}$ is a P-ideal, then $\Lambda_{\bar{x}}(\mathcal{I})$ is closed for every choice of $\bar{x}$ if and only if $\mathcal{I}$ is $F_\sigma$.*

*(3) If $\alpha > \omega$ is an indecomposable ordinal, then $\Lambda_{\bar{x}}(\mathcal{O}_\alpha)$ (see Definition 1.9.1) is not closed for some choice of $\bar{x}$.*

*(4) If $\alpha > \omega$ is a multiplicatively indecomposable ordinal, then then $\Lambda_{\bar{x}}(\mathcal{W}_\alpha)$ (see Definition 1.9.3) is not closed for some choice of $\bar{x}$.*

Proof. The first two parts are immediate consequences of Proposition 12.1.1.

(3) Let $(\alpha_n)$ be an increasing sequence of limit ordinals whose supremum is $\alpha$ and let $h\colon \alpha \to \mathbb{N}$ be defined by $h(\xi) = n$ if $\alpha_n \leq \xi < \alpha_{n+1}$. Clearly $h$ satisfies the assumptions of Proposition 12.1.2.

(4) The proof is analogous to that of (3). □

APPENDIX A

# Appendix

## A.1. Descriptive set theory

We will interchangeably use the terms Borel-measurable and Borel for functions between Polish spaces.

**Lemma A.1.1.** *If $F$ is a Baire measurable function from a Polish space $X$ into a second-countable space, then there is a dense $G_\delta$ subset $X_0$ of $X$ such that the restriction of $F$ to $X_0$ is continuous.*

A function between Polish spaces is called *C-measurable* if it is measurable with respect to the $\sigma$-algebra generated by analytic sets of the domain. A proof of the following classical theorem can be found e.g., in [**105**, 18.A].

**Theorem A.1.2** (Jankov, von Neumann). *If $R \subseteq \mathcal{P}(\mathbb{N}) \times \mathcal{P}(\mathbb{N})$ is analytic and $\mathcal{X} = \{a : \langle a, b\rangle \in R$ for some $b\}$, then there is a function $f\colon \mathcal{X} \to \mathcal{P}(\mathbb{N})$ such that the graph of $f$ is included in $R$ and the $f$-preimage of every open subset of $\mathcal{P}(\mathbb{N})$ belongs to the $\sigma$-algebra generated by the analytic subsets of $\mathcal{P}(\mathbb{N})$.* □

The following is a consequence of Theorem A.1.2.

**Theorem A.1.3.** *Let $X$ and $Y$ be Polish spaces, and let $\mathcal{Z} \subseteq X \times Y$ be analytic. Then there is a C-measurable selection $\Theta$ for $\mathcal{Z}$; that is, a C-measurable function $\Theta$ whose domain is the projection of $\mathcal{Z}$ to $X$ and such that $(x, \Theta(x)) \in \mathcal{Z}$ for every $x \in \mathrm{dom}(\Theta)$.* □

The following two theorems use the common notation $A_x$ for vertical sections of $A \subseteq X \times Y$. The first one is [**105**, Theorem 29.3]

**Theorem A.1.4** (Novikov). *Suppose that $X$ and $Y$ are Polish spaces and a subset $A$ of $X \times Y$ is analytic, then each of the sets $\{x \in X : A_x$ is nonmeagre$\}$ and $\{x \in X : A_x$ is comeagre$\}$ is analytic.* □

The following is [**105**, §8.K].

**Theorem A.1.5** (Kuratowski–Ulam). *Suppose that $X$ and $Y$ are Polish spaces and $A \subseteq X \times Y$ has the property of Baire. Then $A$ is meagre if and only if $\{x : A_x$ is nonmeagre in $Y\}$ is meagre in $X$.* □

## A.2. Saturation

We will use model-theoretic definition of saturation. For definition of types and more details see any text on model theory such as [**16**].

**Definition A.2.1.** A structure $\mathfrak{M}$ in language $\mathcal{L}$ is $\aleph_1$-*saturated* if every consistent type $\mathbf{t}$ over a countable subset of $\mathfrak{M}$ is realised in $\mathfrak{M}$.

Since we will be considering only structures in a countable language, the requirement that **t** is a type over a countable set is equivalent to the requirement that it is countable. Since the theory of atomless Boolean algebras admits quantifier elimination and the Boolean algebras of the form $\mathcal{P}(\mathbb{N})/\mathcal{I}$ are rather specific, the types that we will need are of a very special form (see the proof of Proposition 11.1.3).

## A.3. Open Colouring Axioms

Suppose that $X$ is a separable metric space, and by $[X]^2$ denote the set of all unordered pairs of its elements,

$$[X]^2 = \{\{x, y\} : x \neq y \ \& \ x, y \in X\}.$$

Subsets of $[X]^2$ are naturally identified with the symmetric subsets of $X \times X$ minus the diagonal. A partition (or *colouring*) $[X]^2 = K_0 \cup K_1$ is *open* if $K_0$, when identified with a symmetric subset of $X \times X$, is open in the product topology. We say that a subset $Y$ of $X$ is $K_i$*-homogeneous* if $[Y]^2$ is included in $K_i$ $(i = 0, 1)$.

**Definition A.3.1.** $\mathrm{OCA_T}$ is the following statement. If $X$ is a separable metric space and $[X]^2 = K_0 \cup K_1$ is an open partition, then $X$ either has an uncountable $K_0$-homogeneous subset or it can be covered by a countable family of $K_1$-homogeneous sets.

A set covered by countably many $K_1$-homogeneous sets is called $\sigma$-$K_1$*-homogeneous*. We should first say a word to clarify our use of the phrase "open colouring." Spaces such as $\mathcal{P}(\mathbb{N})$, $\mathbb{N}^{\mathbb{N}}$, $\mathrm{Fin}^{\mathbb{N}}$, and the finite products of such spaces, are considered with their natural separable metric product topology. In order to be able to apply $\mathrm{OCA_T}$ to a partition $[X]^2 = K_0 \cup K_1$, it suffices to know that *there is* a separable metric topology $\tau$ on $X$ which makes $K_0$ open.

**Example A.3.2.** Given $X \subseteq \mathcal{P}(\mathbb{N})$ and $f_x \in \mathbb{N}^{\mathbb{N}}$ for each $x \in X$. Consider the partition $[X]^2 = K_0 \cup K_1$ defined by

$$\{x, y\} \in K_0 \text{ if and only if } f_x(n) \neq f_y(n) \text{ for some } n \in x \cap y.$$

Then $K_0$ is not necessarily open in the topology inherited from $\mathcal{P}(\mathbb{N})$. However, $K_0$ is open in the subspace topology on $X$ obtained by identifying it with a subset of $\mathcal{P}(\mathbb{N}) \times \mathbb{N}^{\mathbb{N}}$ via the embedding $x \mapsto \langle x, f_x \rangle$.

Definition A.3.4 below gives sharpening of the reformulation of $\mathrm{OCA_T}$ introduced in [**38**] (Definition A.3.3) that will considerably simplify some of the uniformisation arguments.

For distinct $a$ and $b$ in $\{0, 1\}^{\mathbb{N}}$ let

$$\Delta(a, b) = \min\{n : a(n) \neq b(n)\},$$
$$a \wedge b = a \restriction \Delta(a, b).$$

If $Z \subseteq \{0, 1\}^{\mathbb{N}}$ then let

$$\Delta(Z) = \{a \wedge b : a, b \text{are distinct elements of } Z\}.$$

Equivalently, $\Delta(Z) = \{s \in \{0, 1\}^{<\mathbb{N}} : [s^\frown j] \cap Z \neq \emptyset \text{ for } j = 0, 1\}$.

The following apparent strengthening of $\mathrm{OCA_T}$ is equivalent to it ([**129**], see also [**54**, Theorem 8.6.6])), but we state it as a warm-up for $\mathrm{OCA}^{\#}$.

**Definition A.3.3.** $\mathrm{OCA}_\infty$ is the following statement. If $X$ is a separable metric space and $[X]^2 = K_0^n \cup K_1^n$, for $n \in \mathbb{N}$ are open partitions such that $K_0^n \supseteq K_0^{n+1}$ for all $n$ then one of the following applies

(1) There are sets $X_n$, for $n \in \mathbb{N}$, such that $X = \bigcup_n X_n$ and $[X_n]^{\subseteq} K_1^n$. (We say that $X$ is $\sigma$-$K_1^*$-homogeneous.)
(2) There are an uncountable $Z \subseteq \{0,1\}^{\mathbb{N}}$ and a continuous $f\colon Z \to X$ such that $\{f(a), f(b)\} \in K_0^{\Delta(a,b)}$ for all $a, b$ in $Z$.

In [38] a weaker axiom was called $\mathrm{OCA}_\infty$, although a proof that of $\mathrm{OCA}_\infty$ as stated in Definition A.3.3 is a consequence of PFA was given. This proof contains a horrible (albeit obvious) typo; in [38, 4. of Lemma 3.1], 'whenever $s \subseteq t$' should be 'for some $t$ such that $s \subseteq t$.'

**Definition A.3.4.** $\mathrm{OCA}^\#$ is the following statement. Suppose that $X$ is a separable metric space and that $\mathcal{V}_j$ is a countable family of symmetric open subsets of $[X]^2$ such that $\bigcup \mathcal{V}_j \supseteq \bigcup \mathcal{V}_{j+1}$ for all $j \in \mathbb{N}$. Then one of the following alternatives holds.

(1) There are $X_n$, for $n \in \mathbb{N}$, such that $X = \bigcup_n X_n$ and $[X_n]^2 \cap \bigcup \mathcal{V}_n = \emptyset$ for all $n$.
(2) There are an uncountable $Z \subseteq \{0,1\}^{\mathbb{N}}$, an injective $f\colon Z \to X$, and $\rho\colon \Delta(Z) \to \bigcup_j \mathcal{V}_j$[1] such that $\rho(s) \in \mathcal{V}_{|s|}$ for all $s$ and all distinct $a$ and $b$ in $Z$ satisfy $\{f(a), f(b)\} \in \rho(a \wedge b)$.

With

$$K_0^n = \bigcup_{j \geq n} U_{m,0}^j \times U_{m,1}^j \tag{A.1}$$

the first alternatives of $\mathrm{OCA}_\infty$ and $\mathrm{OCA}^\#$ are equivalent, and the second alternative of the latter is a sharper variant of the second alternative of the former.

The following is [21, Theorem 3.3], whose proof is based on [129, §5].

**Theorem A.3.5.** $\mathrm{OCA}_{\mathrm{T}}$, $\mathrm{OCA}_\infty$, *and* $\mathrm{OCA}^\#$ *are equivalent.*

Proof. It suffices to prove that $\mathrm{OCA}_{\mathrm{T}}$ implies $\mathrm{OCA}^\#$. Define $Y = \{0,1\}^{\mathbb{N}} \times \prod_j \mathcal{V}_j \times X$. Since each $\mathcal{V}_j$ is countable, $\prod_j \mathcal{V}_j$ is naturally homeomorphic to the Baire space and $Y$ has a natural separable metrisable topology. Define a subset $K_0$ of $[Y]^2$ by letting $\{(a,\mu,x),(b,\nu,y)\} \in K_0$ if

$$(K_0) \qquad a \neq b,\ x \neq y,\ \mu(\Delta(a,b)) = \nu(\Delta(a,b)), \text{ and } \{x,y\} \in \mu(\Delta(a,b)).$$

Since each element of $\bigcup_j \mathcal{V}_j$ is symmetric, $K_0$ is a symmetric subset of $Y^2$, clearly disjoint from the diagonal. It is evidently an open subset of $[Y]^2$ in its natural topology.

Assume that $H \subseteq Y$ is uncountable and $K_0$-homogeneous. We will prove that the alternative (2) of $\mathrm{OCA}^\#$ holds. Since $H$ is $K_0$-homogeneous, if $(a,\mu,x)$ and $(b,\nu,y)$ are distinct elements of $H$ then $a \neq b$ and $x \neq y$. Therefore the set

$$Z = \{a : (a,\mu,x) \in H \text{ for some } \mu, x\}$$

is uncountable and $f(a) = x$ if $(a,\mu,x) \in H$ for some $\mu$ defines an injection from $Z$ into $X$. Define $\rho\colon \Delta(Z) \to \bigcup_j \mathcal{V}_j$ as follows. For $s \in \Delta(Z)$ choose $(a,\mu,x) \in H$ with $a(|s|) = 0$ such that there exists some $(b,\nu,y) \in H$ with $a \wedge b = s$ and let

[1] The reader will hopefully forgive us for pointing out the obvious, that $\bigcup \mathcal{V}_j$ and $\bigcup_j \mathcal{V}_j$ are two very different sets.

$\rho(s) = \mu(|s|)$. To see that $\rho(s)$ does not depend on the choice of $(a, \mu, x)$, pick $(a', \mu', x') \in H$ with $a'(|s|) = 0$ such that there exists some $(b', \nu', y') \in H$ with $a' \wedge b' = s$. Then necessarily $a' \wedge b = s$ and since both $\{(a, \mu, x), (b, \nu, y)\} \in K_0$ and $\{(a', \mu', x'), (b, \nu, y)\} \in K_0$ we have $\mu(|s|) = \nu(|s|) = \mu'(|s|)$, as required. This shows $\rho$ is well defined. Further, $\rho(s) = \mu(|s|) \in \mathcal{V}_{|s|}$ by construction. Lastly, if $a$ and $b$ are distinct members of $Z$ then there are unique $(a, \mu, x)$ and $(b, \nu, y)$ in $H$ such that $f(a) = x$, $f(b) = y$, and $\{x, y\} \in \rho(a \wedge b) = \mu(\Delta(a, b))$, and therefore we have the alternative (2) of $\mathrm{OCA}^{\#}$.

Now suppose that $Y$ has no uncountable $K_0$-homogeneous subsets. By $\mathrm{OCA_T}$ it can be covered by the union of sets $Y_k$, for $k \in \mathbb{N}$, such that $[Y_k]^2 \cap K_0 = \emptyset$ for all $k$. Because $K_0$ is an open subset of $[Y]^2$, the property $[Y_k]^2 \cap K_0 = \emptyset$ is preserved by replacing $Y_k$ by its closure and therefore we can and will assume that for each $k \in \mathbb{N}$, $Y_k$ is a closed subset of $Y$. We will infer that alternative (1) of $\mathrm{OCA}^{\#}$ holds. By the Baire Category Theorem, for each $x \in X$ there exists $k \in \mathbb{N}$ such that the closed set

$$Z_{k,x} = \{(a, \mu) \in \{0, 1\}^{\mathbb{N}} \times \prod_j \mathcal{V}_j : (a, \mu, x) \in Y_k\}$$

is nonmeagre in $\{0, 1\}^{\mathbb{N}} \times \prod_j \mathcal{V}_j$. We can therefore pick for each $x \in X$ some $k = k_x$, $m = m_x$, $s_x \in \{0, 1\}^m$, and $t_x \in \prod_{j<m} \mathcal{V}_j$ such that $[s_x] \times [t_x] \subseteq Z_{k,x}$.

Fix $m, k$ in $\mathbb{N}$, $s \in \{0, 1\}^m$, and $t \in \prod_{j<m} \mathcal{V}_j$. Let

$$X_{k,s,t} = \{x \in X : k_x = k, s_x = s, \text{ and } t_x = t\}.$$

We will prove that $[X_{k,s,t}]^2$ is disjoint from $\bigcup \mathcal{V}_{|s|}$. Towards this, let $m = |s|$ and fix distinct $x$ and $y$ in $X_{k,s,t}$. Let $V \in \mathcal{V}_m$. In order to prove that $\{x, y\} \notin V$, we additionally fix some $a, b \in [s]$ and $\mu, \nu \in [t]$ such that $\Delta(a, b) = m$ and $\mu(m) = \nu(m) = V$. Then $(a, \mu) \in Z_{k,x}$ and $(b, \nu) \in Z_{k,y}$, so since $[X_n]^2$ is disjoint from $K_0$, yet $\mu(\Delta(a, b)) = \nu(\Delta(a, b)) = V$, we necessarily have $\{x, y\} \notin V$. Since $V \in \mathcal{V}_m$ was arbitrary, this implies that $\{x, y\} \notin \bigcup \mathcal{V}_m$. Since $x$ and $y$ were arbitrary, this proves that $[X_{k,s,t}]^2$ is disjoint from $\bigcup \mathcal{V}_m$.

To complete the proof, it remains to re-enumerate the family $\{X_{k,s,t}\}$ as $\tilde{X}_n$, for $n \in \mathbb{N}$, so that $[\tilde{X}_n]^2 \cap \bigcup \mathcal{V}_n = \emptyset$ for all $n$. Since the sets $\bigcup \mathcal{V}_n$ form a decreasing sequence, we only need to make sure that $n$ such that $\tilde{X}_n = X_{k,s,t}$ is not smaller than $|s| = |t|$. Since we are allowed to have $\tilde{X}_j = \emptyset$ for infinitely many $j$, this is straightforward. □

### A.4. $\mathrm{OCA_T}$ and $\mathbb{N}^{\mathbb{N}}$

Let $g \leq^j$ if $g(i) \leq f(i)$ for all $i \geq j$ and let $g \leq^* f$ if $g \leq^j f$ for some $j \in \mathbb{N}$.

**Lemma A.4.1.** *Assume* $\mathrm{OCA_T}$. *If* $\mathcal{H}$ *is an uncountable subset of* $\mathbb{N}^{\mathbb{N}}$ *then there is* $g \in \mathbb{N}^{\mathbb{N}}$ *such that* $\{f \in \mathcal{H} : f \leq g\}$ *is uncountable.*

Proof. $\mathrm{OCA_T}$ implies that every subset of $\mathbb{N}$ of cardinality $\aleph_1$ is $\leq^*$-bounded ([**152**]). Fix $g'$ such that $\{f \in \mathcal{H} : f \leq^* g'\}$ is uncountable. Let $n$ be such that $\mathcal{H}' = \{f \in \mathcal{H} : f \leq^n g\}$ is uncountable. Finally, for some $s\colon n \to \mathbb{N}$ the set $\{f \in \mathcal{H}' : f \restriction n = s\}$ is uncountable. Let $g \geq g'$ be such that $g \restriction n = s$; it is as required. □

**Lemma A.4.2.** *If* $\mathbb{N}^{\mathbb{N}} = \bigcup_n \mathcal{F}_n$ *then for some* $j$ *and* $n$, *for every* $g \in \mathbb{N}^{\mathbb{N}}$ *some* $f \in \mathcal{F}_n$ *satisfies* $g \leq^j f$.

For $f \in \mathbb{N}^{\mathbb{N}}$ let

$$\Gamma_f = \{(m,n) : n \leq f(m)\}.$$

A *coherent family of partial functions indexed by* $\mathbb{N}^{\mathbb{N}}$ is a family $h_f$ ($f \in \mathbb{N}^{\mathbb{N}}$) such that the following holds for all $f$ and $g$ in $\mathbb{N}^{\mathbb{N}}$.

(1) $h_f \colon \Gamma_f \to \mathbb{N}$, and
(2) $h_f((m,n)) = h_g((m,n))$ for all but finitely many $(m,n) \in \Gamma_f \cap \Gamma_g$.

Such coherent family is *trivial* if there is $g \colon \mathbb{N}^2 \to \mathbb{N}$ such that every $f \in \mathbb{N}^{\mathbb{N}}$ satisfies $f((m,n)) = g((m,n))$ for all but finitely many $(m,n) \in \Gamma_f$.

The following is [152, Theorem 8.7].

**Corollary A.4.3.** *OCA implies that every coherent family $\mathcal{F}$ of partial functions indexed by $\mathbb{N}^{\mathbb{N}}$ is trivial.* □

## A.5. Martin's Axiom and almost disjoint families

The following is well-known but the proof had been omitted in both [162] and [40] and we included it here.

**Lemma A.5.1.** *Assume* MA($\sigma$-*centered*). *If $\mathcal{A}$ is an uncountable almost disjoint family, then there is an uncountable almost disjoint family $\mathbb{B}$ such that for every $B \in \mathbb{B}$ the set $\{A \in \mathcal{A} : A \subseteq^* B\}$ is infinite.*

Proof. Let $A_\alpha$, for $\alpha < \aleph_1$, enumerate a subset of $\mathcal{A}$. it will suffice to construct $\mathbb{B}$ in a forcing extension obtained by adding $\aleph_1$ dominating reals, $d_\xi$ for $\xi < \aleph_1$. (To be precise, we are considering finite-support iteration, so that $d_\xi$ dominates $\mathbb{N}^{V[\langle d_\eta : \eta < \xi \rangle]}$ for all $\xi$.) For each $\xi \leq \aleph_1$ let

$$B_\xi = \bigcup_n (A_{\xi \cdot \omega + n} \setminus d_\xi(n))$$

and let $\mathbb{B}_\xi = \{B_\eta : \eta < \xi\} \cup \{A_\alpha : \alpha \geq \xi \cdot \omega\}$. Note that $\mathbb{B}_0 = \mathcal{A}$. By induction one proves that each $\mathbb{B}_\xi$ is almost disjoint, as follows. If $\mathbb{B}_\xi$ is almost disjoint, then for every $B \in B_\xi$ we have the function $f_B \in \mathbb{N}^{\mathbb{N}}$ defined by $f_B(n) = \max(B \cap A_{\xi \cdot \omega \cap n})$ (with $\max(\emptyset) = 0$). Since $d_\xi \geq^* f_B$ for all $B \in \mathbb{B}_\xi \setminus \{A_{\xi \cdot \omega + n} : n \in \omega\}$, the family $\mathbb{B}_{\xi+1}$ is almost disjoint. If $\xi$ is a limit ordinal, then any two elements of $\mathbb{B}_\xi$ belong to $\mathbb{B}_\eta$ for some $\eta < \xi$, therefore $\mathbb{B}_\xi$ is almost disjoint. Thus $\mathbb{B}_{\aleph_1}$ is as required. □

Lemma A.5.2 below was proven in [162, Lemma 2.3] with MA($\sigma$-linked) replaced with MA. An inspection of the proof shows that the proof of Claim given on p. 8 shows that any two conditions $p$ and $q$ in the poset $\mathcal{P}$ with the same 'working part' (the tuple $(n, e^0, e^1)$) are compatible. Although the $\Delta$-system lemma is invoked in fourth line of the proof, the fact that the conditions form a $\Delta$-system is not used.

**Lemma A.5.2.** *Assume* MA($\sigma$-*linked*). *Then for every uncountable almost disjoint family $\mathcal{A}$ there are an uncountable subfamily $\mathcal{A}'$ of $\mathcal{A}$ and a partition $A = A_0 \cup A_1$ of each $A \in \mathcal{A}'$ such that each one of $\mathcal{A}_0 = \{A_0 : A \in \mathcal{A}\}$ and $\mathcal{A}_1 = \{A_1 : A \in \mathcal{A}\}$is a tree-like almost disjoint family.* □

The assumption of the following lemma follows both from $\mathrm{OCA_T}$ and from MA($\sigma$-linked) (and even MA($\sigma$-centered)).

**Lemma A.5.3.** *Assume $\mathfrak{b} > \aleph_1$. Then for every uncountable almost disjoint family $\mathcal{A}$ there is an uncountable almost disjoint family $\mathcal{B}$ such that for every $B \in \mathcal{B}$ the set $\{A \in \mathcal{A} : A \subseteq^* B\}$ is infinite.*

Proof. Since $\mathfrak{b} > \aleph_1$, for every uncountable almost disjoint family $\mathcal{A}$ and every countably infinite $\mathcal{A}_0 \subseteq \mathcal{A}$ there exists $B \subseteq \mathbb{N}$ such that $A \subseteq^* B$ for all $A \in \mathcal{A}_0$ and $A \cap B \in \text{Fin}$ for all $A \in \mathcal{A}_1$, where $\mathcal{A}_1 = \mathcal{A} \setminus \mathcal{A}_0$. Let us prove this (well-known) fact. Enumerate $\mathcal{A}_0$ as $A_n$, for $n \in \mathbb{N}$. For $A \in \mathcal{A}_1$ let $g_A(n) = \max\{m : A \cap A_n \subseteq m\}$ Since $\mathfrak{b} > \aleph_1$, there is $g \in \mathbb{N}^{\mathbb{N}}$ such that $g \geq^* g_A$ for all $A \in \mathcal{A}_1$. Let $B = \bigcup_n (A_n \setminus g(n))$. Then $A_n \subseteq^* B$ and $A \cap B \in \text{Fin}$ for all $A \in \mathcal{A}_1$. By recursively applying this argument $\aleph_1$ times we obtain $\mathcal{B}$ as required. □

The following is an immediate consequence of Lemma A.5.2 and Lemma A.5.3.

**Lemma A.5.4.** *Assume* MA($\sigma$-*linked*). *Then for every uncountable almost disjoint family $\mathcal{A}$ there is an uncountable almost disjoint family $\mathcal{B}$ such that for every $B \in \mathcal{B}$ the set $\{A \in \mathcal{A} : A \subseteq^* B\}$ is infinite, and there is a partition $B = B_0 \cup B_1$ such that each one of $\mathcal{A}_0 = \{A_0 : A \in \mathcal{A}\}$ and $\mathcal{A}_1 = \{A_1 : A \in \mathcal{A}\}$is a tree-like almost disjoint family.* □

An ideal $\mathcal{J}$ on $\mathcal{P}(\mathbb{N})$ is *ccc over* Fin if there is no uncountable family of $\mathcal{J}$-positive sets that are almost disjoint modulo Fin.

**Corollary A.5.5.** *Assume* MA($\sigma$-*linked*). *An ideal on* $\mathbb{N}$ *intersects every uncountable tree-like almost disjoint family if and only if it is ccc over* Fin. □

## A.6. Disjoint refinements of families of finite sets

This section includes two very simple lemmas that will be used in uniformisation proofs of our main lifting theorems in conjunction with Biba's trick, together with a version of one of these lemmas that uses MA($\sigma$-linked). The following is [**21**, Lemma 3.8].

**Lemma A.6.1.** *(1) Suppose that $S(n, i)$, for $n \in \mathbb{N}$ and $i < 2^n$, are finite sets such that $|S(n, i)| \geq n + (4^{n+1} - 1)/3$ for all $n$ and $i$. Then there are pairwise disjoint sets $F(n, i) \subseteq S(n, i) \setminus n$ such that $|F(n, i)| = 2^n$ for all $n$ and $i$.*

*(2) If $A(s) \in \text{Fin}$ for $s \in D \subseteq \{0, 1\}^{<\mathbb{N}}$ satisfy $|A(s)| \geq |s| + (4^{|s|+1} - 1)/3$ for all $s \in D$, then there are pairwise disjoint $B(s) \subseteq A(s) \setminus |s|$ for $s \in D$ such that $|B(s)| = 2^{|s|}$ for all $s \in D$.*

Proof. (1): If $S(n, i)$ are as described, a fairly dumb algorithm produces the sets $F(n, i)$. Let $F(0, 0)$ be any element of $S(0, 0)$. Assume that $F(k, i)$ as required had been chosen for all $k < n$ and $i < 2^k$. Then $I = \bigcup_{k<n, i<2^k} F(k, i) \cup n$ has cardinality not greater than $n + \sum_{k<n, i<2^k} 2^k = n + (4^n - 1)/3$. Therefore $S(n, i) \setminus I$ has cardinality at least $4^n$ for each $i < n$. We can now choose pairwise disjoint $F(n, i) \subseteq S(n, i) \setminus I$ of cardinality $2^n$ each recursively in $i$.

(2): Enumerate the $n$th level of $\{0, 1\}^{<\mathbb{N}}$ as $s(n, i)$, for $i < 2^n$. Define the sets $S(n, i)$, for $n \in \mathbb{N}$ and $i < 2^n$ by setting $S(n, i) = A(s(n, i))$ when $s(n, i) \in D$ and by choosing $S(n, i)$ to be an arbitrary set of cardinality $n + (4^{n+1} - 1)/3$ when $s(n, i) \notin D$. By part (1) of this lemma, there are pairwise disjoint $F(n, i) \subseteq S(n, i) \setminus n$ for $n \in \mathbb{N}$ and $i < 2^n$, of cardinality $2^n$. Then by defining for every $s \in D$ the set $B(s) = F(n, i)$, where $n$ and $i < 2^n$ are such that $s = s(n, i)$, we find that the sets $B(s) \subseteq A(s) \setminus |s|$ are as required. □

Lemma A.6.2 below is based on a lemma due to J.W. Roberts that was one of the (simpler) ideas involved in the solution to Maharam's problem ([**151**]). The main difference with Lemma A.6.1 is that we need to handle finite sets instead of singletons.

**Lemma A.6.2.** *For every $m \in \mathbb{N}$, if for all $s \subseteq m$ there are $l(s) \geq 2^{2m}$ and an $l(s)$-tuple $\vec{n}(s)$: $n_0(s) < n_1(s) < \cdots < n_{l(s)}(s)$ in $\mathbb{N}$, then there are $A(t) \subseteq \mathbb{N}$, for $t \subseteq m$, such that for all $s$ and $t$ some $i = i(s,t)$ satisfies $[n_i(s), n_{i+1}(s)) \subseteq A(t)$.*

Proof. Enumerate $2^m$ as $s(i)$, for $i < 2^m$, by using the following algorithm. Choose $s(0)$ so that $n_{2^m}(0)$ is minimal possible. If $s(i)$ for $i < k$ had been chosen, then let $s(k)$ be such that $n_{(k+1)2^m}(s)$ is minimal possible among $\{s < 2^m : s \neq s(i)$ for $i < k\}$. This describes the construction. For all $k < 2^m$ we have $n_{k2^m}(s(k)) \geq n_{k2^m}(s(k-1))$ hence the intervals $J_k = [n_{k2^m}(s(k)), n_{(k+1)2^m}(s(k)))$, for $k < 2^m$, are disjoint and the sequence $n_{(k+1)2^m}(s(k))$, for $k < 2^m$, is nondecreasing.

Fix a bijection. $g\colon 2^m \to \mathcal{P}(m)$ and for $t \subseteq m$ let

$$A(t) = \bigcup_{k<2^m} [n_{k2^m+g(t)}(s(k)), n_{k2^m+g(t)+1}(s(k))).$$

The sets $A(t)$ are disjoint, and for all $s = s(k)$ and $t$ we have

$$A(t) \supseteq [n_{k2^m+g(t)}(s(k)), n_{k2^m+g(t)+1}(s(k))),$$

as required. □

The fact that Roberts's lemma does not have a reasonable extension to infinite families of finite sequences is one of the reasons why the construction of a pathological Maharam submeasure given in [151] is deep and beautiful. In our situation, we can get away with a little use of MA($\sigma$-linked), see Lemma A.6.5 below. Since the existence of a pathological Maharam submeasure is obviously a $\Sigma^1_2$ statement,[2] MA($\sigma$-linked) would be of no help there.

The following is an elementary but important property of ccc posets.

**Lemma A.6.3.** *Suppose that $\mathbb{P}$ has countable chain condition. If $Z$ is an uncountable subset of $\mathbb{P}$, then some condition $q \in \mathbb{P}$ forces that the intersection of the generic filter with $Z$ is uncountable.*

Proof. Suppose otherwise. Let $p_\xi$, for $\xi < \aleph_1$, enumerate a subset of $Z$ of cardinality $\aleph_1$.[3] Then the set of conditions $q \in \mathbb{P}$ such that $p_\xi$ is incompatible with $q$ for all but countably many $\xi$ is then dense in $\mathbb{P}$. Let $q_m$, for $m \in \mathbb{N}$, be a maximal antichain included in this dense set and let $\alpha_m < \aleph_1$ be such that $p_\xi$ is incompatible with $q_m$ for all $\xi > \alpha_m$. Then $\alpha = \sup_m \alpha_m$ is countable, hence $p_{\alpha+1}$ is incompatible with all $q_m$; contradiction. □

**Lemma A.6.4.** *Suppose that $X$ is an uncountable set, $\mathbb{P}$ has countable chain condition, and $\dot{Z}$ is a $\mathbb{P}$-name for a subset of $X$ such that for every $x \in X$ some $p_x \in \mathbb{P}$ forces that $x \in \dot{Z}$. Then some condition $q \in \mathbb{P}$ forces that $\dot{Z}$ is uncountable.*

Proof. Apply Lemma A.6.3 to $\{p_x : x \in X\}$. □

For $Z \subseteq \{0,1\}^{\mathbb{N}}$ let $\Delta(Z) = \{x \wedge y : x \in Z, y \in Z, x \neq y\}$. This is a tree with respect to $\sqsubseteq$ and its $i$-th level is equal to $\{s \in \Delta(Z) : |\{t \in \Delta(Z) : t \sqsubset s\}| = i+1\}$.

[2]It is a bit less obvious that it is a $\Delta_0$ statement, but this follows from Talagrand's result, because every provable statement is provably $\Delta_0$.

[3]A reader interested in $\aleph_{42}$ and such may easily formulate and prove analogs of this lemma for higher cardinals.

**Lemma A.6.5.** *Assume* MA($\sigma$-*linked*). *If* $Z \subseteq \{0,1\}^{\mathbb{N}}$ *is uncountable and for every* $s \in \Delta(Z)$, $I(s)$ *is a finite interval in* $\mathbb{N}$ *such that* $|s| \leq \min(I(s))$. *Then there are an uncountable* $Z' \subseteq Z$, *an increasing sequence* $k_i$, *for* $i \in \mathbb{N}$, *in* $\mathbb{N}$, *and* $i\colon \Delta(Z') \to \mathbb{N}$ *such every* $s \in \Delta(Z')$ *satisfies* $I(s) \subseteq [k_{i(s)}, k_{i(s)+1})$. *Moreover* $\mathsf{S}_i = \{s \in \Delta(Z') : i(s) = i\}$ *is the* $i$*-th level of* $\Delta(Z')$.

Proof. We may assume that $Z$ has no isolated points (by removing any, if they exist). Let $\mathbb{P}$ be the poset of all $p = (F(p), \vec{k}(p))$ such that $F(p) \Subset Z$ and $\vec{k}(p)$ is a tuple $k_0(p) < k_1(p) < \dots < k_{l(p)}(p)$ for some $l(p) \in \mathbb{N}$ such that the following holds.

(1) For all $s \in \Delta(F(p))$, $I(s) \subseteq [k_i(p), k_{i+1}(p))$ for some $i < l(p)$.
(2) For all $i < l(p)$, the set $\{s \in \Delta(F(p)) : I(s) \subseteq [k_i(p), k_{i+1}(p))\}$ is equal to the $i$-th level of $\Delta(F(p))$.

For every $z \in Z$ the condition $p_z$ satisfying $F(p_z) = \{z\}$, $l(p_z) = 0$ and $k_0(p_z) = 0$ belongs to $\mathbb{P}$.

In order to prove that $\mathbb{P}$ is $\sigma$-linked, it suffices to prove that every two conditions $p$ and $q$ such that $\vec{k}(p) = \vec{k}(q)$ and (with $l = l(p) = l(q)$ and $k = k_l(p) = k_l(q)$) they satisfy $\{z \restriction k : z \in F(p)\} = \{z \restriction k : z \in F(q)\}$ are compatible.

Assume for a moment that $F(p) \cap F(q) = \emptyset$. Thus for every $x \in F(p)$ the unique $x' \in F(q)$ such that $x \restriction k_l = x' \restriction k_l$ is distinct from $x$. Then let $F(r) = F(p) \cup F(q)$, $l(r) = l+1$, $k_i(r) = k_i(p)$ for $i \leq l$, and $k_{l+1}(r) = \max(I(\Delta(x,x')) : x \in F(p)) + 1$. Since $I(x \wedge x') \geq |x \wedge x'| \geq k$, the condition $r = (F^r, \vec{k}(r))$ belongs to $\mathbb{P}$ and extends both $p$ and $q$.

Now consider the case when the set $F_0 = F(p) \cap F(q)$ is nonempty. Thus $F_0$ is the set of all $x \in F(p)$ such that the unique $x' \in F(q)$ which satisfies $x \restriction k_l = x' \restriction k_l$ is equal to $x$. Since $Z$ has no isolated points, for every $x \in F_0$ we can find $x' \in Z \setminus \{x\}$ such that $x \restriction k = x' \restriction k$. Let $F(r) = F(p) \cup F(q) \cup \{x' : x \in F_0\}$, $l(r) = l+1$, $k_i(r) = k_i(p)$ for $i \leq l$, and $k_{l+1}(r) = \max(I(\Delta(x,x')) : x \in F(p)) + 1$. As in the previous case, $r = (F(r), \vec{k}(r))$ is a condition that extends both $p$ and $q$.

Since $\mathbb{P}$ is $\sigma$-linked, some condition $p \in \mathbb{P}$ forces that $\{z \in Z : p_z \in G\}$ is uncountable. Therefore MA($\sigma$-linked) implies that for some filter $G \subseteq \mathbb{P}$ the set $Z' = \{z \in Z : p_z \in G\}$ is uncountable and that for every $l$ there is $p \in G$ satisfying $l(p) \geq l$. Since $G$ is a filter, for every $F \Subset Z$ there is $p \in G$ such that $F(p) \supseteq F$. For each $i$ let $k_i = k_i(p)$ for some $p \in G$ such that $l(p) \geq l$. Since $G$ is a filter, $k_i$ is well-defined. Clearly $Z'$ and the sequence $k_i$ are as required. □

## A.7. Ye olde uniformisation proof

The original proof of the OCA lifting theorem for Fin (Theorem 6.1.3) is included in this section. I have to admit that one of the reasons for including this proof is of a sentimental nature. It showcases a technique pioneered in [40], where MA is applied to an uncountable 0-homogeneous set provided by $\mathrm{OCA_T}$ to produce a large family of uncountable 0-homogeneous sets. In addition to sentimentality, keeping this proof here is justified by the possible applicability of this technique to other problems (and the fact that the presentation from [40] has been greatly improved here).

Assume $\mathrm{OCA_T}$ and MA and fix a homomorphism $\Phi\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})/\mathit{Fin}$. By Proposition 6.3.1, the ideal $\mathscr{J}_{\mathrm{cont}}$ (Definition 6.2.2) is ccc over Fin and by the

Radon–Nikodym property of Fin (Theorem 4.1.2), for every $A \in \mathcal{J}_{\text{cont}}$ the restriction $\Phi \restriction \mathcal{P}(A)$ has a completely additive lifting, $\Theta_A$.

The following is an addendum to Definition 6.2.2.

**Definition A.7.1.** If $\mathcal{I}$ is an ideal on $\mathbb{N}$ and is $\mathcal{K}$ is a closed approximation to $\mathcal{I}$, then for a homomorphism $\Phi\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})/\mathcal{I}$

$$\begin{aligned}\mathcal{J}_{\text{dec}} &= \{D : \Phi \restriction \mathcal{P}(D) \text{ has a completely additive lifting on } \mathcal{J}_{\text{cont}} \cap \mathcal{P}(D)\}\\ \mathcal{J}^*_{\text{dec}} &= \{D : \text{there is a completely additive } \Theta_D\colon \mathcal{P}(A) \to \mathcal{P}(\mathbb{N}) \text{ such that}\\ &\qquad \{A \in \mathcal{J}_{\text{cont}} : \Theta_D \text{ lifts } \Phi \text{ on } \mathcal{P}(A \cap D)\} \text{ is nonmeagre}\}\end{aligned}$$

We need to prove that $\mathcal{J}_{\text{dec}} = \mathcal{J}^*_{\text{dec}} = \mathcal{P}(\mathbb{N})$. Following Shelah's original proof, we first prove that $\mathcal{J}_{\text{dec}}$ is a P-ideal.

**Lemma A.7.2.** *For every finite $F \subseteq \mathcal{J}_{\text{cont}}$ the following are equivalent.*

*(1) There is completely additive $\Theta\colon \bigcup F \cap D \to \mathcal{P}(\mathbb{N})$ such that $\Theta$ and $\Theta_A$ agree on $\mathcal{P}(A \cap D)$.*

*(2) For al $A$ and $B$ in $F$ we have the following.*

*(a) All $s \Subset A \cap B \cap D$ satisfy $\Theta_A(s) = \Theta_B(s)$.*

*(b) For all $s \Subset A \cap D$ and $t \Subset B \cap D$, if $s \cap t = \emptyset$ then $\Theta_A(s) \cap \Theta_B(t) = \emptyset$.*

Proof. Only the converse implication requires a proof. For $X \subseteq \bigcup F \cap D$ let $\Theta(X) = \bigcup_{A \in F} \Theta_A(X \cap A)$. Then (2a) and the fact that $\Theta_A$ is completely additive for all $A \in F$ together imply that $\Theta(X \cup Y) = \Theta(X) \cup \Theta(Y)$ and even $\Theta(\bigcup_n X_n) = \bigcup_n \Theta(X_n)$ for all subsets of the domain of $\Theta$. On the other hand, (2b) implies that if $X \cap Y = \emptyset$ then $\Theta(X) \cap \Theta(Y) = \emptyset$. Therefore $\Theta$ is completely additive, as required. □

**Lemma A.7.3.** *For all $D$ and $E$ in $\mathcal{J}^*_{\text{dec}}$ there is $m$ such that some completely additive $\Theta\colon (D \cup E) \setminus m \to \mathcal{P}(\mathbb{N})$ agrees with $\Theta_D$ on $\mathcal{P}(D \setminus m)$ and with $\Theta_E$ on $\mathcal{P}(E \setminus m)$.*

Proof. The idea is similar to that of Claim 9.3.13. If no such $\Theta$ exists, then Lemma A.7.2 implies that one of the following two possibilities happens.

(1) There are pairwise disjoint $s_n \Subset D \cap E$, for $n \in \mathbb{N}$, such that $\Theta_D(s_n) \neq \Theta_E(s_n)$ for all $n$.

(2) There are disjoint $s_n \subseteq D$ and $t_n \subseteq E$ such that $\Theta_D(s_n) \cap \Theta_E(t_n) \neq \emptyset$.

If the first case applies then there is an infinite $X$ such that $A = \bigcup_{n \in X} s_n$ belongs to $\mathcal{J}_{\text{cont}}$ and (after some refining as in the proof of Claim 9.3.13) that $\Theta_D(A) \Delta \Theta_E(A)$ is infinite. Since each of these two sets is equal modulo finite to $\Theta_A(A)$, this is a contradiction.

Otherwise, there is an infinite $X$ such that $A = \bigcup_{n \in X} s_n$ and $B = \bigcup_{n \in X} t_n$ both belong to $\mathcal{J}_{\text{cont}}$ and (after some refining as in the proof of Claim 9.3.13) are disjoint, but that $\Theta_D(A) \cap \Theta_E(A)$ is infinite. Since the first of these two sets is equal modulo finite to $\Theta_{A \cup B}(A)$ and the second is equal modulo finite to $\Theta_{A \cup B}(B)$, this is a contradiction. □

Let

$$\begin{aligned}\mathcal{X} = \{(A, \Theta) : &A \in \mathcal{J}_{\text{dec}} \text{ and } \Theta\colon \mathcal{P}(A) \to \mathcal{P}(\mathbb{N})\\ &\text{is a completely additive lifting of } \Phi \text{ on } \mathcal{P}(A) \cap \mathcal{J}_{\text{cont}}\}.\end{aligned}$$

For all $(A, \Theta) \in \mathcal{X}$, $\Theta$ is uniquely determined by its restriction to $\mathrm{Fin}(A) = \mathcal{P}(A) \cap \mathrm{Fin}$. Therefore $\mathcal{X}$ is equipped with the following separable metric.
(A.2)
$$d((A,\Theta),(A',\Theta')) = \frac{1}{\min((A\Delta A') \cup \{n \in A \cap A' : \Theta(\{n\}) \neq \Theta'(\{n\})\}) + 1}.$$

**Definition A.7.4.** For $D \subseteq \mathbb{N}$ define a partition $[\mathcal{X}]^2 = L_0(D) \cup L_1(D)$ by
$$\{(A,\Theta),(B',\Theta')\} \in L_0(D)$$
if and only if no completely additive function from $(A \cup A') \cap D$ into $\mathcal{P}(\mathbb{N})$ extends both $\Theta$ and $\Theta$'

Lemma A.7.3 gives a finitary equivalent definition of this partition, and it is open in the metric defined in (A.2).

**Lemma A.7.5.** *For every $D \subseteq \mathbb{N}$ the following are equivalent.*
*(1)* $D \in \mathcal{J}_{\mathrm{dec}}$.
*(2)* $D \in \mathcal{J}^*_{\mathrm{dec}}$.
*(3)* $\mathcal{J}_{\mathrm{cont}}$ *has a nonmeagre $L_1(D)$-homogeneous subset.*

Proof. Clearly $\mathcal{J}_{\mathrm{dec}} \subseteq \mathcal{J}^*_{\mathrm{dec}}$. Suppose $D \in \mathcal{J}^*_{\mathrm{dec}}$. By Lemma A.7.3, there is $n$ such that the set
$$\{A \in \mathcal{J}_{\mathrm{cont}} : \Theta_A \text{ and } \Theta_D \text{ agree on } \mathcal{P}((A \cap D) \setminus n)\}$$
is nonmeagre. This set is clearly $L_1(D)$-homogeneous.

It remains to prove that if $\mathcal{J}_{\mathrm{cont}}$ has a nonmeagre $L_1(D)$-homogeneous subset $\mathcal{Y}$ then $D \in \mathcal{J}_{\mathrm{dec}}$. By Lemma A.7.2, some completely additive $\Theta\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ extends $\Theta_A$ for all $A \in \mathcal{Y}$. Fix $B \in \mathcal{J}_{\mathrm{cont}}$. For every $A \in \mathcal{Y}$, by Lemma A.7.3 there exists $m = m(A)$ such that $\Theta_A$ and $\Theta_B$ agree on $(D \cap A) \setminus m$. The set $\mathcal{Y}' = \{A \in \mathcal{Y} : m(A) = m\}$ is nonmeagre for some $m$. In particular, $\bigcup \mathcal{Y}' \supseteq (A \setminus k)$ for a large enough $k = k(A)$.

Choose $n$ large enough so that $\{A \in \mathcal{J}_{\mathrm{cont}} : n \geq \max(m(A), k(A))\}$ is nonmeagre. Then $\Theta_A$ and $\Theta$ agree on $\mathcal{P}(D \setminus n)$ for a nonmeagre set of $A \in \mathcal{J}_{\mathrm{cont}}$. We can modify the restriction of $\Theta$ to the finite Boolean algebra $\mathcal{P}(D \cap n)$ to obtain a lifting $\Theta'$ of $\Phi$ such that $\Theta'(s)$ disjoint from $\mathcal{P}(\Theta_D(D \setminus n))$ for all $s \subseteq D \cap n$. Then $\Theta_D(X) = \Theta'(X \cap D \cap m) \cup \Theta((X \cap D) \setminus m)$ is a completely additive lifting of $\Phi$ on $\mathcal{J}_{\mathrm{cont}} \cap \mathcal{P}(D)$, as required. □

**Lemma A.7.6.** *Suppose that* $\mathrm{OCA_T}$ *and* MA *hold and* $\Phi\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})/\mathrm{Fin}$ *is a homomorphism. Then the ideal* $\mathcal{J}_{\mathrm{dec}}$ *is a P-ideal.*

Proof. Assume $\mathcal{J}_{\mathrm{dec}}$ is not a P-ideal. Fix a sequence $A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots$ of sets in $\mathcal{J}_{\mathrm{dec}}$ such that there is no $A_\infty \in \mathcal{J}_{\mathrm{dec}}$ which almost includes all $A_n$'s. Therefore $A_{n+1} \setminus A_n$ is infinite infinitely many $n$, and by passing to a subsequence we may assume that this difference is infinite for all $n$. For convenience we identify $\mathbb{N}$ with $\mathbb{N} \times \mathbb{N}$ by identifying each $A_n$ with $(n+1) \times \mathbb{N}$ and assume that $\Phi\colon \mathcal{P}(\mathbb{N}^2) \to \mathcal{P}(\mathbb{N})/\mathrm{Fin}$. For $f \in \mathbb{N}^{\mathbb{N}}$ let
$$\Gamma_f = \{\langle k, m\rangle : m \geq f(k)\}.$$
By the choice of $\{A_k\}$, the set $\mathbb{N}^2 \setminus \Gamma_f$ is $\mathcal{J}_{\mathrm{dec}}$-positive for all $f$.

Assume for a moment that there is no $\bar{f}\colon \mathbb{N} \to \mathbb{N}$ such that for all $f \geq^* \bar{f}$ the set $\Gamma_f \setminus \Gamma_{\bar{f}}$ is in $\mathcal{J}_{\mathrm{dec}}$. Then one can recursively find functions $\{f_\xi\}_{\xi<\omega_1}$ such that

$\Gamma_{f_{\xi+1}} \setminus \Gamma_{f_\xi}$ does not belong to $\mathcal{J}_{\rm dec}$, contradicting the fact that $\mathcal{J}_{\rm cont}$ is ccc over Fin (Proposition 6.3.1).

Therefore, there is $\bar{f}\colon \mathbb{N} \to \mathbb{N}$ such that $\Gamma_f \setminus \Gamma_{\bar{f}}$ is in $\mathcal{J}_{\rm cont}$ for all $f \geq^* \bar{f}$. Since $\Gamma_{\bar{f}} \cap A_n$ is finite for all $n$, by restricting our attention to $\mathbb{N}^2 \setminus \Gamma_{\bar{f}}$ (and restricting $\Phi$ to the power set of this set modulo finite), we may assume that $\bar{f}$ is identically equal to zero.

We claim that there is a $\leq^*$-cofinal $\mathcal{F}$ in $\mathbb{N}^{\mathbb{N}}$ such that (writing $\Theta_f$ for $\Theta_{\Gamma_f}$) $\{(\Gamma_f, \Theta_f)\} : f \in \mathcal{F}\}$ is $L_1(\mathbb{N}^2)$-homogeneous. Otherwise, by $\mathrm{OCA_T}$ there is an uncountable $\mathcal{G} \subseteq \mathbb{N}^{\mathbb{N}}$ such that $\{(\Gamma_f, \Theta_f)\} : f \in \mathcal{G}\}$ is $L_0(\mathbb{N}^2)$-homogeneous. Since $\mathfrak{b} > \aleph_1$, we may assume that $\mathcal{G}$ is bounded by some $g \in \mathbb{N}^{\mathbb{N}}$. By Lemma A.7.3, this easily leads to contradiction.

Let $\mathcal{F}$ be $\leq^*$-cofinal and such that $\{(\Gamma_f, \Theta_f)\} : f \in \mathcal{F}\}$ is $L_1(\mathbb{N}^2)$-homogeneous. By Lemma A.7.2, some completely additive $\Theta\colon \mathcal{P}(\mathbb{N}^2) \to \mathcal{P}(\mathbb{N})$ extends $\Theta_f$ for all $f \in \mathcal{F}$. We claim that for every $A \in \mathcal{J}_{\rm cont}$ there is $m$ such that $\Theta_A$ and $\Theta$ agree on $A \cap ((m, \infty) \times \mathbb{N})$. Assume otherwise and fix $A \in \mathcal{J}_{\rm cont}$. Then there is $g \in \mathbb{N}^{\mathbb{N}}$ such that $\Theta_A$ and $\Theta$ do not agree on $\Gamma_g \cap A$. If $f \geq^g$ is in $\mathcal{F}$, then this implies that $\Theta_f$ and $\Theta_A$ violate Lemma A.7.3; contradiction. □

**A.7.1. Martin's Axiom and liftings, I. Poset $\mathcal{P}$.** The fact that $\mathcal{J}_{\rm dec}$ is a P-ideal will be used to define a ccc poset. The ideas from this proof, taken from [40], are related to the notion of Y-cc studied in [17].

**Lemma A.7.7.** *Suppose that $\Phi\colon \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})/\mathrm{Fin}$ is a homomorphism. If $\mathbb{N}$ is not in $\mathcal{J}_{\rm dec}$, then there are an uncountable $\subseteq^*$ increasing sequence $B_\alpha$, for $\alpha < \aleph_1$ and $\Theta_\alpha\colon \mathcal{P}(B_\alpha) \to \mathcal{P}(\mathbb{N})$ that is a lifting of $\Phi$ on $\mathcal{J}_{\rm cont} \cap \mathcal{P}(B_\alpha)$ such that the set $\{(B_\alpha, \Theta_\alpha) : \alpha < \aleph_1\}$ is $L_0(\mathbb{N})$-homogeneous.*

Proof. If $\mathbb{N} \notin \mathcal{J}_{\rm dec}$ then by Lemma A.7.5 $\mathcal{J}_{\rm cont}$ does not have a nonmeagre $L_1(\mathbb{N})$-homogeneous subset. Therefore, $\mathrm{OCA_T}$ implies that there is an uncountable $L_0(\mathbb{N})$-homogeneous subset of $\mathcal{J}_{\rm cont}$ that we can enumerate as $A_\alpha$, for $\alpha < \aleph_1$. Since $\mathcal{J}_{\rm dec}$ is a P-ideal, there are $B_\alpha \in \mathcal{J}_{\rm dec}$, for $\alpha < \aleph_1$, such that $B_\alpha \subseteq^* B_\beta$ and $A_\alpha \subseteq B_\alpha$ for all $\beta < \aleph_1$. Modify $\Theta_{B_\alpha}$ so that it agrees with $\Theta_{A_\alpha}$ on $A_\alpha$ and call the resulting function $\Theta_\alpha$. This is a completely additive lifting of $\Phi$ on $\mathcal{P}(B_\alpha) \cap \mathcal{J}_{\rm cont}$. Also, $\{(B_\alpha, \Theta_\alpha), (B_\beta, \Theta_\beta)\}$ belongs to $L_0(\mathbb{N})$ for all $\alpha \neq \beta$, hence the set $\{(B_\alpha, \Theta_\alpha) : \alpha < \aleph_1\}$ is as required. □

It will be convenient to shift our attention from completely additive almost liftings $\Theta_A$ to their restriction to $\mathrm{Fin}(A) = \mathcal{P}(A) \cap \mathrm{Fin}$. Let

$$g_A = \Theta_A \restriction \mathrm{Fin}(A)$$

and let

$$\mathcal{X} = \{B_\alpha : \alpha < \aleph_1\}. \tag{A.3}$$

Note that $g_\alpha$ determines $\Theta_\alpha$ uniquely.

Let $[\mathcal{X}]^2 = L_0 \cup L_1$ be defined by setting $\{B_\alpha, B_\beta\} \in L_0$ if

$$\{(B_\alpha, \Theta_\alpha), (B_\beta, \Theta_\beta)\} \in L_0(\mathbb{N}).$$

The following lemma will be used in the analysis of the forcing notion $\mathcal{P}$ defined in Definition A.7.9 below (an $L$-homogeneous rectangle is a set of the form $\mathcal{Y} \times \mathcal{Z}$ included in $L$).

**Lemma A.7.8.** *Suppose that $\Phi\colon \mathcal{P}(\mathbb{N})/\operatorname{Fin} \to \mathcal{P}(\mathbb{N})/\operatorname{Fin}$ is a homomorphism and $\mathcal{X}$ is as in (A.3). Then for every $n \in \mathbb{N}$ the following holds.*

(1) *$\mathcal{X}$ can be covered by countably many $L_0([n,\infty))$-homogeneous subsets.*
(2) *If $\mathcal{Y}$ and $\mathcal{Z}$ are uncountable subsets of $\mathcal{X}$, then there are uncountable $\mathcal{Y}' \subseteq \mathcal{Y}$ and $\mathcal{Z}' \subseteq \mathcal{Z}$ such that $\mathcal{Y} \times \mathcal{Z} \subseteq L_0([n,\infty))$.*

Proof. To prove the first part, for $g\colon n \to \operatorname{Fin}$ let

$$\mathcal{X}(g) = \{A \in \mathcal{X} : g_A(j) = g(j) \text{ for all } j \in A \cap n\}.$$

Fix $g$ and distinct $A$ and $B$ in $\mathcal{X}(g)$. Since $\mathcal{X}$ is $L_0$-homogeneous, $g_A(k) \neq g_B(k)$ for some $k \in A \cap B$ or $g_A(k) \cap g_B(l) \neq \emptyset$ for distinct $k \in A$ and $l \in B$. Assume the former happens. For $k \in A \cap B \cap n$ we have $g_A(k) = h(k) = g_B(k)$, hence $k \geq n$. Now assume that the latter happens. Then, since $g_A(k)$ are pairwise disjoint, we have $k \geq n$ and similarly $l \geq n$. In either case, $\{A, B\} \in L_0([n,\infty))$.

Therefore each $\mathcal{X}(g)$ is $L_0([n,\infty))$-homogeneous, and the desired conclusion follows.

For the second part, for each $A \in \mathcal{Y}$ fix $A' \in \mathcal{Z}$ such that $A \subseteq^* A'$. By Lemma A.7.3, there is $m(A) \geq n$ such that $g_A(n) = g_{A'}(j)$ for all $j \geq m(A)$. By replacing $\mathcal{Y}$ with an uncountable subset, we may assume there is $m$ such that $A \setminus m \subseteq A'$ and $g_A(j) = g_{A'}(j)$ for all $n \geq m$ and all $A \in \mathcal{Y}$. We choose $A'$ so that the function $A \mapsto A'$ is injective.

By refining $\mathcal{Y}$ further, we may assume that for some $s \subseteq m$ and $t \subseteq m$ we have $A \cap m = s$ and $A' \cap m = t$ for all $A \in \mathcal{Y}$. By another refinement, we may assume that $g_A \restriction s = g_B \restriction s$ and $g_{A'} \restriction t = g_{B'} \restriction t$ for all $A, B$ in $\mathcal{Y}$.

Since $\mathcal{P}(\mathbb{N})$ is second-countable, by yet another refinement we may assume that the intersection of each of the sets $\mathcal{Y}$ and $\{A' : A \in \mathcal{Y}\}$ with any given clopen subset of $\mathcal{P}(\mathbb{N})$ is either empty or uncountable. Since $\mathcal{Y}$ is $L_0$-homogeneous, for distinct $A$ and $B$ in $\mathcal{Y}$ some $j \geq m$ satisfies $g_A(j) \neq g_{B(j)} = g_{B'}(j)$. Each of the sets $\mathcal{Y}' = \{C \in \mathcal{Y} : g_C(j) = g_A(j)\}$ and $\mathcal{Z}' = \{A' : g_{A'}(j) = g_{B'(j)}\}$ is uncountable, and $\mathcal{Y}' \times \mathcal{Z}' \subseteq L_0([n,\infty))$ as required. □

**Definition A.7.9.** Let $\mathcal{P}$ be the poset of all $\langle s, k, F\rangle$, where

($\mathcal{P}1$) $k \in \mathbb{N}$ and $s \subseteq k$,
($\mathcal{P}2$) $F$ is a finite $L_0(s)$-homogeneous subset of $\mathcal{X}$.

We are using the convention that every $p \in \mathcal{P}$ is of the form

$$p = \langle s^p, k^p, F^p\rangle.$$

Define the order on $\mathcal{P}$ by letting $p \leq q$ if

($\mathcal{P}3$) $s^p \cap k^q = s^q$ and $F^p \supseteq F^q$.

The *working part* of a condition $p \in \mathcal{P}$ is $\langle s^p, k^p\rangle$. The working parts of $p, q$ are said to be *compatible* if $s^p \cap k^q = s^q \cap k^p$.

We will prove that $\mathcal{P}$ has a strong form of a countable chain condition that will assure ccc-ness of a certain amalgamation $\mathcal{P}_{\omega_1}$ of uncountably many copies of $\mathcal{P}$. The following lemma will help the reader internalise the definition of the poset $\mathcal{P}$ (if $F^p = \emptyset$ then $p$ is the maximal condition of $\mathcal{P}$).

**Lemma A.7.10.** *For conditions $p$ and $q$ in $\mathcal{P}$ the following are equivalent.*

(1) *$p$ and $q$ are compatible,*

(2) *With* $F = F^p \cap F^q$, *the conditions*

$$\langle s^p, k^p, F^p \setminus F\rangle \ \& \ \langle s^q, k^q, F^q \setminus F\rangle,$$

*are compatible.*

(3) $s^p \cap k^q = s^q \cap k^p$ *and, with* $k = \max(k^p, k^q)$, *for all* $A \in F^p \setminus F$ *and* $B \in F^q \setminus F$ *we have* $\{A, B\} \in L_0(s^p \cup s^q \cup [k, \infty))$.

Proof. After unravelling the definitions, the equivalence of (1) and (2) reduces to the fact that if the sets $X$ and $Y$ are $L_0$-homogeneous, then since $L_0 \cup L_1$ is a partition of pairs, $X \cup Y$ is $L_0$-homogeneous if and only if $(X \cup Y) \setminus (X \cap Y)$ is $L_0$-homogeneous. But this is a basic property of any partition of pairs.

The equivalence of (2) and (3) is also straightforward. □

**Definition A.7.11.** A pair of uncountable subsets $X$ and $Y$ of $\mathcal{P}$ such that every $p \in X$ is incompatible with every $q \in Y$ is called an *uncountable rectangle of incompatible conditions*. One similarly defines the notion of an *uncountable rectangle of compatible conditions*.

**Lemma A.7.12.** *The poset* $\mathcal{P}$ *can be partitioned into* $\mathcal{P}_n$, *for* $n \in \mathbb{N}$, *so that for every* $n$ *and all uncountable subsets* $X, Y$ *of* $\mathcal{P}_n$ *there are uncountable* $X' \subseteq X$ *and* $Y' \subseteq Y$ *such that every* $p \in X'$ *is compatible with every* $q \in Y'$. *In particular,* $\mathcal{P}$ *has countable chain condition.*

Proof. Since there are only countably many working parts, we can choose $\mathcal{P}_n$ so that all conditions in $\mathcal{P}_n$ have the same working part. Fix $n$ and an uncountable rectangle $X, Y$ in $\mathcal{P}_n$. For each $p \in X$ fix $A^p \in F^p$ and $n^p$ such that $A^p \setminus A \subseteq n^p$ for all $A \in F^p$. For each $p \in Y$ fix $B^p \in F^p$ and $m^p$ such that $B^p \setminus B \subseteq m^p$ for all $B \in F^p$. By replacing $X$ and $Y$ with their uncountable subsets and increasing $m^p$ or $n^p$ as needed, we may assume that there is $n$ such that $n^p = m^q = n$ for all $p \in X$ and all $q \in Y$.

By Lemma A.7.8, there are uncountable $X' \subseteq X$ and $Y' \subseteq Y$ such that

$$\{A^p : p \in X'\} \times \{B^q : q \in Y'\} \subseteq L_0([n, \infty)).$$

Fix $p \in X'$ and $q \in Y'$, and fix $A \in F^p$ and $B \in F^q$. Since $\{A^p, B^q\} \in L_0([n, \infty))$, $A \setminus A^p \subseteq n$, and $B \setminus B^q \subseteq n$, we have $\{A, B\} \in L_0([n, \infty))$. By Lemma A.7.10, $p$ and $q$ are compatible. □

**Lemma A.7.13.** *If* $X, Y$ *are uncountable subsets of* $\mathcal{P}$ *such that for all* $p, q \in X \cup Y$ *their working parts are compatible, then there is an uncountable rectangle* $X' \subseteq X$, $Y' \subseteq Y$ *of compatible conditions.*

Proof. The assumption asserts that $X \cup Y$ is included in $\mathcal{P}_n$ as defined in the proof of Lemma A.7.12, hence the conclusion follows from Lemma A.7.12. □

**A.7.2. Martin's Axiom and liftings, II. Poset $\mathcal{P}_{\omega_1}$.** The final component in the proof of the second part of Theorem 6.1.3 is an amalgamation of $\aleph_1$ copies of $\mathcal{P}$.

**Definition A.7.14.** A typical condition in the poset $\mathcal{P}_{\omega_1}$ has the form

$$p = \langle I, k, s(\xi)(\xi \in I), F(\xi)(\xi \in I)\rangle,$$

where $I$ is a finite subset of $\omega_1$ and $p(\xi) = \langle k, s(\xi), F(\xi)\rangle$ is in $\mathcal{P}$ for all $\xi \in I$.

The order on $\mathcal{P}_{\omega_1}$ is defined by letting $p \leq q$ if

($\mathcal{P}4$) $I^p \supseteq I^q$ and $p(\xi) \leq_{\mathcal{P}} q(\xi)$ for all $\xi \in I^q$, and
($\mathcal{P}5$) the sets $s^p(\xi) \setminus \{1, \dots, k\}$ ($\xi \in I^q$) are pairwise disjoint.

We state an immediate consequence of the definitions.

**Lemma A.7.15.** *Two conditions $p$ and $q$ in $\mathcal{P}_{\omega_1}$ are compatible if and only if with $I = I^p \cap I^q$ the conditions*

$$\langle I, s^p_\xi, k^p_\xi, F^p_\xi : \xi \in I\rangle \ \ \& \ \ \langle I, s^q_\xi, k^q_\xi, F^q_\xi : \xi \in I\rangle,$$

*are compatible.*

*In particular, if $I^p$ and $I^q$ are disjoint, then $p$ and $q$ are compatible.* □

**Lemma A.7.16.** *The poset $\mathcal{P}_{\omega_1}$ has countable chain condition.*

Proof. Let $p_\alpha$ ($\alpha < \omega_1$) be an uncountable subset of $\mathcal{P}_{\omega_1}$. We can assume that the sets $I^\alpha = I^{p_\alpha}$ form a $\Delta$-system with root $\bar{I}$.

By Lemma A.7.15 we may assume that

$$I^\alpha = \bar{I} = \{\xi_1, \dots, \xi_l\}$$

for all $p_\alpha$. Uniformizing further, we may assume that for every fixed $i \in \{1, \dots, l\}$ the working parts of all conditions $p^\alpha(\xi_i)$ ($\alpha < \omega_1$) are equal, say

$$s^\alpha_{\xi_i} = \bar{s}_i \text{ and } k^\alpha_{\xi i} = \bar{k}_i, \text{ for all } \alpha \text{ and } i \in \{1, \dots, l\}.$$

We can, moreover, assume that the sets $F^\alpha_{\xi_i} = F^{p_\alpha}_{\xi_i}$ form a $\Delta$-system with root $F_{\xi_i}$.

By Lemma A.7.10, removing the root $F_{\xi_i}$ from $F^\alpha \dots \xi_i$ does not affect the compatibility of these conditions. We can therefore remove the root and assume that the sets $F^\alpha_{\xi_i}$ ($\alpha < \omega_1$) are pairwise disjoint for every fixed $i \in \{1, \dots, l\}$. We can apply Claim A.7.8 to $p_\alpha(\xi_1)$ ($\alpha < \omega_1$) and get subsets $X_1, Y_1$ of $\omega_1$ such that $p_\alpha(\xi_1)$ ($\alpha \in X_1$) and $p_\beta(\xi_1)$ ($\beta \in Y_1$) form an uncountable rectangle of compatible conditions. Let $\bar{k}_1' > \bar{k}_1$ and $\bar{s}_1' \subseteq \bar{k}_1'$ be such that

$$\langle \bar{s}_1', \bar{k}_1', F^\alpha_{\xi_1} \cup F^\beta_{\xi_1}\rangle$$

is a joint extension of $p_\alpha(\xi_1)$ and $p_\beta(\xi_1)$ for all $\alpha \in X_1$ and $\beta \in Y_1$. Now we can extend every $p_\alpha(\xi_2)$ ($\alpha \in X_1 \cup Y_1$) to

$$p'_\alpha(\xi_2) = \langle \bar{s}_2, \bar{k}_2', F^\alpha(\xi_2)\rangle$$

so that $\bar{k}_2' > \bar{k}_1$. Another application of Claim A.7.8 gives sets $X_2 \subseteq X_1$ and $Y_2 \subseteq Y_1$ such that $p'_\alpha(\xi_2)$ ($\alpha \in X_1$) and $p'_\beta(\xi_2)$ ($\beta \in Y_2$) is an uncountable rectangle of compatible conditions. Note that, if

$$\langle \bar{s}_2', \bar{k}_2'', F^\alpha_{\xi_2} \cup F^\beta_{\xi_2}\rangle$$

is a joint extension of $p'_\alpha(\xi_2)$ and $p'_\beta(\xi)$, then by our choice of $\bar{k}_2'$ the condition ($\mathcal{P}7$) is satisfied between $\bar{s}_1'$ and $\bar{s}_2'$. By continuing this construction for $i = 3, \dots, l$, we get an uncountable rectangle of compatible conditions in $\mathcal{P}_{\omega_1}$. □

Proof of the second part of Theorem 6.1.3. Assume $\mathrm{OCA_T}$ and MA and let $\Phi\colon \mathcal{P}(\mathbb{N})/\operatorname{Fin} \to \mathcal{P}(\mathbb{N})/\operatorname{Fin}$ be a homomorphism. We need to prove that $\Phi$ has a completely additive almost lifting.

By Proposition 6.3.1, the ideal $\mathcal{J}_{\mathrm{dec}}$ of all $A$ such that the restriction of $\mathcal{P}(\mathbb{N})$ to $\mathcal{P}(A)$ has a completely additive lifting is ccc over Fin. For each $A \in \mathcal{J}_{\mathrm{dec}}$ fix $g_A\colon A \to \operatorname{Fin}$ such that $X \mapsto g_A[X]$ defines a completely additive lifting of the restriction of $\Phi$ to $\mathcal{P}(A)$.

By Lemma A.7.5, we need to prove that with the open partition $[\mathcal{J}_{\mathrm{dec}}]^2 = L_0 \cup L_1$ the ideal $\mathcal{J}_{\mathrm{dec}}$ is $\sigma$-$L_1$-homogeneous.

If not, then $\mathrm{OCA_T}$ implies that $\mathcal{J}_{\mathrm{dec}}$ has an uncountable $L_0$-homogeneous subset. By Lemma A.7.7, there is an uncountable $L_0$-homogeneous subset of $\mathcal{J}_{\mathrm{dec}}$ which is well-ordered by $\subseteq^*$. Let $A_\xi$, for $\xi < \omega_1$, be the $\subseteq^*$-increasing enumeration of this set.

By Lemma A.7.12 and Lemma A.7.16, each of the posets $\mathcal{P}$ and $\mathcal{P}_{\omega_1}$ has the countable chain condition. By Lemma A.6.4 some $p \in \mathcal{P}$ forces that the set $\{\xi : A_\xi \in F^p \text{ for some } p \in \dot{G}_{\mathcal{P}}\}$ is uncountable. There is a condition $q$ in $\mathcal{P}_{\omega_1}$ which forces that the set $\dot{Z} = \{\eta : p^\eta \leq p\}$ is uncountable. Let $\mathcal{D}_{\xi,\eta}$ be the set of conditions $r \leq q$ in $\mathcal{P}_{\omega_1}$ such that $r$ decides the $\xi$-th element of $\dot{Z}$ is $\xi'$ and for some $\eta' \geq \eta$ we have $A_{\eta'} \in F^{r_{\xi'}}$. Each of these sets is clearly open, and it is dense below $q$ by the choice of $q$.

Let $G$ be a filter of $\mathcal{P}_{\omega_1}$ that intersects all $\mathcal{D}_{\xi,\eta}$, for $\xi, \eta < \omega_1$. and for $\xi < \omega_1$ define

$$D_\xi = \bigcup_{p \in G} s^p_\xi \ \ \& \ \ \mathcal{X}_\xi = \bigcup_{p \in G} F^p_\xi.$$

Then the sets $D_\xi$ are pairwise almost disjoint and $\mathcal{X}_\xi$ is $L_0(D_\xi)$-homogeneous for all $\xi$. Moreover, each $\mathcal{X}_\xi$ is uncountable. Then we can choose a suitable family of $\aleph_1$ dense open subsets of $\mathcal{P}_{\omega_1}$ so that if $G$ is a filter of $\mathcal{P}_{\omega_1}$ containing $p$ and intersecting all these dense open sets, then for uncountably many $\xi$ the set $\mathcal{X}_\xi$ is uncountable. By Lemma A.7.5, for such $\xi$ the set $D_\xi$ is not in $\mathcal{J}_{\mathrm{dec}}$ and therefore not in $\mathcal{J}_{\mathrm{cont}}$. But this contradicts the conclusion of Proposition 6.3.1. This concludes the proof of the second part of Theorem 6.1.3. □

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# List of Symbols

*Warning:* Interpretations of some of the symbols, such as supp and the ideals of the form $\mathcal{J}^{\mathcal{K}}_{\circledcirc}(\Phi)$, differ between Chapter 9 (from page 145 on) and the earlier chapters. This is the case with symbols that involve the Boolean algebra $\mathbb{B} = \prod_n \mathbb{B}_n$ explicitly or implicitly.

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