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THE TRACIAL TRANSFER PROPERTY

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ABSTRACT. We provide a model theoretic version of the "complemented partitions of unity" (CPoU) property for C*-algebras introduced by Castillejos, Evington, Tikuisis, White, and Winter. It is shown that this *tracial transfer property* is equivalent to CPoU. In particular, the tracial transfer property holds for all $\mathcal{Z}$-stable C*-algebras with compact trace simplex.

CONTENTS

INTRODUCTION

def:transfer **Definition A.** A C*-algebras A with $T(A)$ non-empty and compact has the *tracial transfer property* if

thm:main **Theorem B.** *A C^*-algebras A with $T(A)$ non-empty and compact has the tracial transfer property if and only if it has CPoU.*

Acknowledgments. We are indebted to Pieter Spaas for providing the references to [1] and to [5] used in the proof of Proposition 6.3.

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sec:tc-prelmin

1. Tracially complete C*-algebras

In this preliminary subsection, we briefly recall some background on tracially complete C*-algebras from [**?**]. For a C*-algebra A, let $T(A)$ denote the set of tracial states on A with the weak* topology inherited from the dual of A. We will typically assume $T(A)$ is compact—this is automatic, for example, if A is unital.

sec:tc-def

1.1. Basic definitions and notation. Given a non-empty set $X \subseteq T(A)$, define the *uniform 2-seminorm* on A by

eq:2u-seminorm

$$\|a\|_{2,X} \coloneqq \sup_{\tau \in X} \tau(a^*a)^{1/2}, \quad a \in A. \tag{1.1}$$

The notations $\|\cdot\|_{2,\mathrm{u}}$ and $\|\cdot\|_{2,\tau}$ stand for $\|\cdot\|_{2,X}$ in the cases when $X = T(A)$ (here "u" standards of "uniform") and $X = \{\tau\}$, respectively. Note that the function $T(A) \to \mathbb{R}\colon \tau \mapsto \tau(a^*a)$ is affine and weak*-continuous. Hence when X is weak*-compact and convex, the supremum in (1.1) is obtained at an extreme point of X. We let $\partial_e X \subseteq X$ denote the set of extreme points.

Following [**?**, Definition ?], a *tracially complete* C*-algebra $\mathsf{M} = (\mathcal{M}, X)$ is a pair consisting of a C*-algebra $\mathcal{M}$ and a weak*-compact, convex set $X \subseteq T(\mathcal{M})$ such that the uniform 2-seminorm $\|\cdot\|_{2,X}$ is a norm on $\mathcal{M}$ and the operator norm unit ball of $\mathcal{M}$ is $\|\cdot\|_{2,X}$-complete. Note that $\mathcal{M}$ is necessarily unital (see []). For $\tau \in X$, consider the GNS representation $\pi_\tau\colon \mathcal{M} \to \mathcal{B}(\mathcal{H}_\tau)$ corresponding to τ. We say M is of *type* II_1 if $\pi_\tau[\mathcal{M}]''$ is of type II_1 for all $\tau \in X$ ([]), and we say M is *factorial* if $\pi_\tau[\mathcal{M}]''$ is a factor for all $\tau \in \partial_e X$ ([,]). Note that by [,], M is factorial if and only if X is a face in $T(\mathcal{M})$ (i.e., for all $\tau_1, \tau_2 \in T(\mathcal{M})$ with $\frac{1}{2}(\tau_1 + \tau_2) \in X$, we have $\tau_1, \tau_2 \in X$.)

Given a C*-algebra A with a weak*-compact and convex set $X \subseteq T(A)$, Ozawa defined the *tracial completion* of A in [] (under the name *strict closure*) to be the C*-algebra $\overline{A}^X$ formed by adding a limit point to every $\|\cdot\|_{2,X}$-Cauchy, $\|\cdot\|$-bounded sequence in A. In this case, each trace $\tau \in X$ extends canonically to a trace on $\overline{A}^X$, still denoted τ, and after identifying X with the corresponding subset of $T\big(\overline{A}^X\big)$, the pair $\big(\overline{A}^X, X\big)$ is a tracially complete C*-algebra. In the case $X = T(A)$, we denote the tracial completion by $\big(\overline{A}^{\mathrm{u}}, T(A)\big)$. In the case $X = \{\tau\}$, the tracial completion can be identified with the tracial von Neumann algebra $(\pi_\tau[A]'', \tau)$.

sec:tc-ultraproduct

1.2. Ultraproducts. For a free ultrafilter $\mathcal{U}$ on an index set $\mathbb{J}$,[1] and let $\mathsf{M}_j \coloneqq (\mathcal{M}_j, X_j)$ be a tracially complete C*-algebra for each $j \in \mathbb{J}$. Let $\prod_{j\in\mathbb{J}} \mathcal{M}_j$ denote the ℓ^∞-product of the $\mathcal{M}_j$ (i.e., all $\|\cdot\|$-bounded, $\mathbb{J}$-indexed sequences with j-th entry in $\mathcal{M}_j$, and define

$$\prod^{\mathcal{U}} \mathcal{M}_j \coloneqq \prod_{j\in\mathbb{J}} \mathcal{M}_j \big/ \big\{(a_j)_{j\in\mathbb{J}} \in \prod_{j\in\mathbb{J}} \mathcal{M}_j : \lim_{j\to\mathcal{U}} \|a_j\|_{2,X_j} = 0\big\}. \tag{1.2}$$

[1] In most of our arguments, $\mathbb{J}$ will be $\mathbb{N}$.

We freely identify elements of $\prod_{j\in\mathbb{J}}\mathcal{M}_j$ with their equivalence classes in $\prod^{\mathcal{U}}\mathcal{M}_j$.

Any family of tracial states $(\tau_j)_{j\in\mathbb{J}}$ with $\tau_j \in X_j$ for all $j\in\mathbb{J}$ induces a tracial state $\bar{\tau}$ on $\prod^{\mathcal{U}}\mathcal{M}_j$ defined by

$$\bar{\tau}\big((a_j)_{j\in\mathbb{J}}\big) \coloneqq \lim_{j\to\mathcal{U}}\tau_j(a_j). \tag{1.3}$$

A tracial state on $\prod^{\mathcal{U}}\mathcal{M}_j$ which can be written this way is called a *limit tracial state*. The set of limit tracial states is convex but is not necessarily weak*-closed ([3, Theorem 1.3]). Let $\sum^{\mathcal{U}}X_j$ denote the weak*-closure of the limit tracial states in $T\big(\prod^{\mathcal{U}}\mathcal{M}_j\big)$.

The pair $\prod^{\mathcal{U}}\mathsf{M}_j \coloneqq \big(\prod^{\mathcal{U}}M_j, \sum^{\mathcal{U}}X_j\big)$ is then a tracial complete C*-algebra, called the *ultraproduct* of the collection $\{\mathsf{M}_j\}_{j\in\mathbb{J}}$. When $\mathbb{J}=\mathbb{N}$, this follows from [**?**]. In general, FINISH.

What is the most efficient way of doing this?

In the case of $\mathsf{M}_j = \mathsf{M}$ for all $j\in\mathbb{J}$, we write $\mathsf{M}^{\mathcal{U}}$ in place of $\prod^{\mathcal{U}}\mathsf{M}_j$ and call $\mathsf{M}^{\mathcal{U}}$ the *ultrapower* of M. Further, if $(\mathcal{M},X)\coloneqq\mathsf{M}$, then we write $(\mathcal{M}^{\mathcal{U}},X^{\mathcal{U}})\coloneqq\mathsf{M}^{\mathcal{U}}$. In this case, we identify $\mathcal{M}$ as a subset of $\mathcal{M}^{\mathcal{U}}$ by identifying $a\in\mathcal{M}$ with the equivalence class of the constant sequence $(a)_{j\in\mathbb{J}}$. Note that this embedding $\mathcal{M}\to\mathcal{M}^{\mathcal{U}}$ is isometric with respect to both the operator norm and the uniform 2-norm.

In the proof of Theorem A.1 of the appendix we show that, when considering tracially complete C*-algebras endowed with the model theoretic structure introduced in §**??**, the definition above agrees with the standard definition of ultraproducts of metric structures. In other words, we prove that $\sum^{\mathcal{U}}X_j$ (or equivalently $\prod^{\mathcal{U}}X_j$) gives rise to the ultraproduct of the norms $\{\|\cdot\|_{2,X_j}\}_{j\in\mathbb{J}}$, and that the definition of ultraproduct above provides the right notion of ultraproduct in the category of tracially complete C*-algebras.

ec:tc-central-sequence

1.3. **Central sequences.** Let $\mathsf{M}\coloneqq(\mathcal{M},X)$ be a tracially complete C*-algebra and let $\mathcal{U}$ be an ultrafilter on an index set $\mathbb{J}$. As in §1.2, let $\mathcal{M}^{\mathcal{U}} = (\mathcal{M}^{\mathcal{U}},X^{\mathcal{U}})$ denote the ultrapower of $\mathcal{M}$ and identify $\mathcal{M}$ as the subalgebra of $\mathcal{M}^{\mathcal{U}}$ consisting of (the equivalence classes of) constant sequences. We let

$$\mathcal{M}^{\mathcal{U}}\cap\mathcal{M}' \coloneqq \{a\in\mathcal{M}^{\mathcal{U}} : [a,b]=0 \text{ for all } b\in\mathcal{M}\} \tag{1.4}$$

denote the *central sequence algebras* of M. When M is not separable (i.e., $\mathcal{M}$ is not $\|\cdot\|_{2,X}$-separable, se will often consider C*-algebras of the form $\mathcal{M}^{\mathcal{U}}\cap S'$ for a $\|\cdot\|_{2,X}$-separable set $S\subseteq\mathcal{M}$ in place of $\mathcal{M}^{\mathcal{U}}\cap\mathcal{M}'$.

For our purposes, the following is the most important property of tracially complete C*-algebras. This was introduced in [] in the context of (uniform tracial completions of) C*-algebras and extended to tracially complete C*-algebras in [].

Definition 1.1. A factorial tracially complete C*-algebras $\mathsf{M}=(\mathcal{M},X)$ has *complemented partitions of unity (CPoU)* if for every $\|\cdot\|_{2,X}$-separable set

$S \subseteq \mathcal{M}$, $a_1, \ldots, a_n \in \mathcal{M}_+$, and $\delta > 0$ satisfying

$$\sup_{\tau \in X} \min_{1 \leq i \leq n} \tau(a_i) < \delta, \tag{1.5}$$

there are projections $p_1, \ldots, p_n \in \mathcal{M}^{\mathcal{U}} \cap S'$ such that $\sum_{i=1}^n p_i = 1_{\mathcal{M}^{\mathcal{U}}}$ with

$$\tau(a_i p_i) < \delta \tau(a_i), \quad \tau \in X^\omega, i = 1, \ldots, n. \tag{1.6}$$

We note that if A is a $\mathcal{Z}$-stable C*-algebras with $T(A)$ compact, then the uniform tracial completions of A, $\big(\overline{A}^{\mathrm{u}}, T(A)\big)$, has CPoU by []. More generally, by the same result, if A is a $\mathcal{Z}$-stable C*-algebra and $X \subseteq T(A)$ is a weak*-compact face, then $\big(\overline{A}^{X}, X\big)$ is a factorial tracially complete C*-algebra with CPoU.

sec:group-actions

2. Group actions on tracially complete C*-algebras

ADD SOME TEXT HERE. HAVE THESE BEEN CONSIDERED ELSEWHERE?

2.1. **Actions and completions.** Let G be a discrete group, let A be a C*-algebra, and let $\alpha : G \to \mathrm{Aut}(A)$ be an action. The action α induces a (left) affine G-action α^* on $T(A)$ defined as $\alpha^*_g(\tau) = \tau \circ \alpha_{g^{-1}}$. With a slight abuse of notation, we drop the dual symbol and we simply say that τ is α-*invariant* if $\alpha^*_g(\tau) = \tau$ for all $g \in G$. Similarly, we say that $X \subseteq T(A)$ is α-invariant if $\alpha^*_g[X] = X$ for all $g \in G$. Finally, we denote the set of all α-invariant tracial states in X by X^α.

Recall that by Day's fixed point theorem [31, Theorem 1.3.1], a group G is amenable if and only if every affine G-action has a fixed point. In particular, it follows that X^α is always nonempty when $X \subseteq T(A)$ is weak*-compact, convex and α-invariant, and G is amenable. Most statements in this paper will not require amenability of G, but will instead assume that there exists a normal subgroup of finite index $H \leq G$ such that the action of H on X is trivial. This condition also easily implies that X^α is non-empty.

A G-*tracially complete* C*-algebra is a triple $\mathsf{M} := (\mathcal{M}, X, \alpha)$ where $(\mathcal{M}, X)$ is a tracially complete C*-algebra, $\alpha : G \to \mathrm{Aut}(\mathcal{M})$ is an action (by C*-automorphisms), and $X \subseteq T(\mathcal{M})$ is α-invariant. We say M is factorial whenever $(\mathcal{M}, X)$ is factorial (i.e. X is a face in $T(\mathcal{M})$). Note that tracially complete C*-algebras can be recovered as a special case of G-tracially complete C*-algebras by imposing that G is the trivial group (or more generally, allowing G to be arbitrary and imposing that α is the trivial action).

Suppose A is a C*-algebra, $X \subseteq T(A)$ is a weak*-compact, convex set, and $\alpha\colon G \to \mathrm{Aut}(A)$ is an action such that $\alpha^*_g(X) \subseteq X$ for all $g \in G$. Then α_g extends to an endomorphism α^X_g of the tracially complete C*-algebra $\big(\overline{A}^{X}, X\big)$ by [**?**,]. Further,

$$\alpha^X_{gh} = \alpha^X_g \alpha^X_h, \quad g, h \in G, \tag{2.1}$$

as this equality holds when restricted to A and both sides of the equality are $\|\cdot\|_{2,X}$-contractive. Since $\alpha_1 = \mathrm{id}_A$, we have $\alpha_1^X = \mathrm{id}_{\mathcal{M}}$, and hence there is a group homomorphism

$$\alpha^X : G \to \mathrm{Aut}\big(\overline{A}^X\big) : g \mapsto \alpha_g^X. \tag{2.2}$$

Then the triple $\mathsf{M} := \big(\overline{A}^X, X, \alpha^X\big)$ is a G-tracially complete C*-algebra.

Specializing to the case when X is a point, if $\alpha : G \to \mathrm{Aut}(A)$ is an action of a group G on a C*-algebra A and $\tau \in T(A)$ is α-invariant, then the action α on A can be canonically extended to the von Neumann algebra $\pi_\tau[A]''$ obtained from the GNS representation π_τ associated to τ. We denote such extension by α^τ. Analogously to the non-equivariant case, the triple $(\mathcal{M}, \{\tau\}, \alpha)$, where $\mathcal{M}$ is a finite von Neumann algebra and τ is an α-invariant faithful tracial state will be abbreviated as $(\mathcal{M}, \tau, \alpha)$.

2.2. **Actions with small orbits.** This section contains some preliminary results concerning group actions on C*-algebras whose orbits on the tracial state space are small—in particular, we require each orbit is finite, and further, we require a uniform bound on the cardinality of the orbits. This ensures that the action of the group on the trace simplex factors through an action of a finite group. This subsection collects some technical results about such actions for use in the proof of an equivariant version of Theorem B (see Theorem **??**).

The following lemma is the main role of the small-orbit condition on group actions.

lemma:uniformlybounded

Lemma 2.1. *Suppose that G is a group and $(\mathcal{M}, X, \alpha)$ is a G-tracially complete C*-algebra such that the action induced by α on X has orbits whose cardinalities are uniformly bounded by a constant C. Then*

eq:inv-bound

$$\|\cdot\|_{2,X^\alpha}^2 \leq \|\cdot\|_{2,X}^2 \leq C\|\cdot\|_{2,X^\alpha}^2. \tag{2.3}$$

In particular, the norms $\|\cdot\|_{2,X}$ and $\|\cdot\|_{2,X^\alpha}$ are equivalent.

Proof. Fix $\tau \in X$ and let $G\cdot\tau$ denote the G-orbit of τ. Since X is an α-invariant, convex subset $T(\mathcal{M})$, the tracial state

$$\tau^\alpha := \frac{1}{|G\cdot\tau|}\sum_{\sigma\in G\cdot\tau}\sigma \tag{2.4}$$

belongs to X^α. Moreover, for every $b \in \mathcal{M}_+$ the inequality

$$|G\cdot\tau|\cdot\tau^\alpha(b) = \sum_{\sigma\in G\cdot\tau}\sigma(b) \geq \tau(b)$$

holds, and therefore $\|\cdot\|_{2,X} \leq C^{1/2}\|\cdot\|_{2,X^\alpha}$. Since $X^\alpha \subseteq X$, the other inequality follows. $\square$

It is not difficult to see that the conclusion of Lemma 2.1 cannot be improved to $\|\cdot\|_{2,X^\alpha}^2 = \|\cdot\|_{2,X}^2$. Take for example $\mathcal{M}$ to be the direct sum of $n \geq 2$ copies of a II$_1$ factor $\mathcal{N}$, $X = T(\mathcal{M})$, and the finite group G to be $\mathbb{Z}/n$

acting on $\mathcal{M}$ by cyclicly permuting the minimal summands. Then X^α is the average of the tracial states of the copies of $\mathcal{N}$, and the two norms clearly differ. In fact, this example also shows that the bounds in (2.3) are sharp; indeed, the minimal value of C is n, and then the left bound is obtained at $1_{\mathcal{M}}$ and the right bound is attained at a minimal central projection of $\mathcal{M}$.

The next proposition and its corollary provide some useful consequences about the extension of an action α on a C*-algebra A to the von Neumann algebra $\pi_\tau[A]''$ when τ is an α-invariant trace on A.

prop:trace-centre

Proposition 2.2. *Suppose that G is a group and $(\mathcal{M}, X, \alpha)$ is a G-tracially complete C*-algebra. Let $h \in G$ be such that $\sigma \circ \alpha_h = \sigma$ for every $\sigma \in X$. Then, for all $\tau \in X^\alpha$, the automorphism α_h^τ of $\pi_\tau[\mathcal{M}]''$ restricts ot the identify on the centre of $\pi_\tau[\mathcal{M}]''$.*

Proof. Let $h \in G$ be such that $\sigma \circ \alpha_h = \sigma$ for every $\sigma \in X$. Towards a contradiction, given $\tau \in X^\alpha$, suppose that the action α_h^τ is non-trivial on the centre $Z(\pi_\tau[\mathcal{M}]'')$. Since this is a von Neumann algebra, there is a non-zero projection $p \in Z(\pi_\tau[\mathcal{M}]'')$ such that $\alpha_h^\tau(p)p = 0$. Indeed, since $Z(\pi_\tau[\mathcal{M}]'')$ is the closed span of its projections, there is a non-zero projection $q \in Z(\pi_\tau[\mathcal{M}]'')$ such that $q \neq \alpha_h^\tau(q)$. The projection $p \coloneqq q - \alpha_h^\tau(q)q$ has the required property.

The functional $\sigma \coloneqq \frac{1}{\tau(p)}\tau(p\pi_\tau(\cdot))$ is a tracial state on $\mathcal{M}$. Further, as X is a face in $T(\mathcal{M})$, $\tau \in X$, and $\sigma \leq \tau(p)\tau$, we have that $\sigma \in X$. By assumption it follows that $\sigma \circ \alpha_h = \sigma$, which in turn implies, by density of $\pi_\tau[\mathcal{M}]$ in $\pi_\tau[\mathcal{M}]''$,

$$0 = \frac{1}{\tau(p)}\tau(p\alpha_h^\tau(p)) = \frac{1}{\tau(p)}\tau(p) = 1, \tag{2.5}$$

which is a contradiction. □

Corollary 2.3. *If G and $(\mathcal{M}, X, \alpha)$ are as in Proposition 2.2 and the action induced by α on X factors through a finite group action, then so does the restriction of the action α^τ to the centre of $\pi_\tau[\mathcal{M}]''$, for every $\tau \in X^\alpha$.* □

2.3. **Ultraproducts and equivarant CPoU.**

TO DO: Update

def:cpou

Definition 2.4 (Dynamical Complemented Partitions of Unity)**.** Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. Suppose that G is a discrete group and let $\mathsf{M} := (\mathcal{M}, X, \alpha)$ be a G-tracially complete C*-algebra such that $X^\alpha \neq \varnothing$. We say that M has *α-invariant complemented partitions of unity* (abbreviated *α-CPoU*) if for every collection $a_1, \dots a_n \in \mathcal{M}_+$, every $\delta > 0$ such that

$$\sup_{\tau \in X^\alpha} \min\{\tau(a_1), \dots, \tau(a_k)\} < \delta,$$

and every $\|\cdot\|_{2,X^{\mathcal{U}}}$-separable $S \subseteq \mathcal{M}_X^{\mathcal{U}}$, there are orthogonal projections $p_1, \dots, p_n \in \mathcal{M}_X^{\mathcal{U}} \cap S'$ such that

(1) $\sum_{i=1}^n p_i = 1$,
(2) $(\alpha^{\mathcal{U}})_g(p_i) = p_i$ for all $i \leq n$ and $g \in G$,
(3) $\tau(p_i a_i) \leq \delta\tau(p_i)$ for all $i \leq n$ and $\tau \in (X^\alpha)^{\mathcal{U}}$.

For a (not necessarily G-tracially complete) C*-algebra A, we say that A has α-CPoU if $(\overline{A}^{T(A)}, T(A), \alpha)$ has α-CPoU.

WHEN DOES α-CPoU HOLD? DO WE HAVE A GOOD NOTION OF α-Γ, FOR EXAMPLE?

TO DO: Move elsewhere.

Subhomogeneous formulas were defined in the paragraph preceding Corollary 7.3. The following completes the proof of such Corollary.

Move this text and Prop. 2.5 elsewhere.

proposition:subhomog

Proposition 2.5. *Let G be a discrete group. Suppose that $\mathsf{M} := (\mathcal{M}, X, \alpha)$ is a factorial G-tracially complete C*-algebra, and that the action induced by α on X has orbits whose cardinalities are uniformly bounded by a constant C. Denote $(\mathcal{M}, X^\alpha, \alpha)$ by M^α. If $\psi(\bar{x})$ is a subhomogeneous max-formula, then for every tuple $\bar{a}$ in $\mathcal{M}$ of the appropriate sort*

$$\psi^{\mathsf{M}^\alpha}(\bar{a}) \leq \psi^{\mathsf{M}}(\bar{a}) \leq C\psi^{\mathsf{M}^\alpha}(\bar{a}).$$

Proof. Suppose that $\psi(\bar{x})$ is a quantifier-free $\mathcal{L}_{\|\cdot\|_2, G}$-formula of the form

$$\max\{h_1(\|Q_1(\bar{x})\|_2^2), \ldots, h_k(\|Q_k(\bar{x})\|_2^2)\},$$

where $Q_1(\bar{x}), \ldots, Q_m(\bar{x})$ are G-*-polynomials and each $h_j \colon \mathbb{R} \to \mathbb{R}$ is an increasing, convex, subhomogeneous function. For a tuple $\bar{a}$ in $\mathcal{M}$ of the appropriate sort, by using the inequalities $\|Q_j(\bar{a})\|_{2,X^\alpha}^2 \leq \|Q_j(\bar{a})\|_{2,X}^2$ for $j \leq m$ (Lemma 2.1) and the monotonicity of each h_j, we have

$$\psi^{\mathsf{M}^\alpha}(\bar{a}) \leq \psi^{\mathsf{M}}(\bar{a}).$$

For the other inequality, by Lemma 2.1 and the subhomogeneity of h_j for $j \leq k$, we have

$$\begin{aligned}\max\{h_1(\|Q_1(\bar{a})\|_{2,X}^2), \ldots, h_m(\|(Q_m(\bar{a})\|_{2,X}^2)\} \leq \\ \leq \max\{C \cdot h_1(\|Q_1(\bar{a})\|_{2,X^\alpha}^2), \ldots, C \cdot h_m(\|(Q_m(\bar{a})\|_{2,X}^2)\} \\ = C \cdot \max\{h_1(\|Q_1(\bar{a})\|_{2,X^\alpha}^2), \ldots, h_m(\|(Q_m(\bar{a})\|_{2,X}^2)\}.\end{aligned}$$

This proves the assertion for the quantifier-free formulas. Since the inequalities are preserved by taking infs and sups, the desired conclusion follows by induction on the number of quantifiers in ψ. $\square$

The following is an immediate consequence of Lemma 2.1.

What is the point of this? I don't think it belongs here in any case.

remark:EquivNorms

Corollary 2.6. *Suppose that G is a discrete group and that $(\mathcal{M}, X, \alpha)$ is a factorial G-tracially complete C*-algebra such that the action induced by α on X has orbits which are uniformly bounded in size by some constant $C \in \mathbb{N}$. Then the norms $\|\cdot\|_{2,X^\alpha}^2$ and $\|\cdot\|_{2,X}^2$ are equivalent. Therefore*

$(\mathcal{M}, X^\alpha, \alpha)$ is G-tracially complete, and an $\mathcal{L}_{\|\cdot\|_2,G}$-structure. Moreover, every ultrafilter $\mathcal{U}$ satisfies $\mathcal{M}^{\mathcal{U}}_X = \mathcal{M}^{\mathcal{U}}_{X^\alpha}$. □

NEEDS WORK

3. A strong form of CPoU

GENERAL EXPOSITION
Should highlight CPoU+ (currently Lemma **??** a bit more, I think.

3.1. Complemented partitions of unity.

3.2. Proof of Theorem ??.

TO DO:. Read/rewrite: text below copy-and-pasted from earlier draft.

lemma:cpou+.0

Lemma 3.1. *Suppose that $(\mathcal{M}, X)$ is a factorial tracially complete C*-algebra with CPoU. Suppose there are $\delta > 0$ and $a_{i,j} \in \mathcal{M}_+$ for $i \leq n$ and $j \leq m$ such that*

$$\sup_{\tau \in X} \min_{i \leq n} \max_{j \leq m} \tau(a_{i,j}) < \delta.$$

Then, for every $\|\cdot\|_{2,X^{\mathcal{U}}}$-separable $S \subseteq \mathcal{M}^{\mathcal{U}}_X$, there are projections $p_1, \ldots, p_n$ in $\mathcal{M}^{\mathcal{U}}_X \cap S'$ such that $\sum_{i=1}^n p_i = 1$ and

$$\tau(p_i a_{i,j}) \leq \delta \tau(p_i) \text{ for all } i \leq n, j \leq m \text{ and } \tau \in X^{\mathcal{U}}.$$

Lemma 3.1 is a special case of its dynamical analog, Lemma 3.2; hence we will prove only the latter.

lemma:cpou+

Lemma 3.2. *Let G be a discrete countable group and let $(\mathcal{M}, X, \alpha)$ be a factorial G-tracially complete C*-algebra with α-CPoU. Suppose that the action induced by α on X factors through a finite group action. Let $\delta > 0$ and $a_{i,j} \in \mathcal{M}_+$ for $i \leq n$ and $j \leq m$ be such that*

$$\sup_{\tau \in X^\alpha} \min_{i \leq n} \max_{j \leq m} \tau(a_{i,j}) < \delta.$$

Then, for every $\|\cdot\|_{2,X^{\mathcal{U}}}$-separable $S \subseteq \mathcal{M}^{\mathcal{U}}_X$, there are orthogonal $\alpha^{\mathcal{U}}$-invariant projections $p_1, \ldots, p_n \in \mathcal{M}^{\mathcal{U}}_X \cap S'$ such that $\sum_{i=1}^n p_i = 1$ and

$$\tau(p_i a_{i,j}) \leq \delta \tau(p_i) \text{ for all } i \leq n, j \leq m \text{ and } \tau \in (X^\alpha)^{\mathcal{U}}.$$

Proof. Fix, for the rest of the proof, a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$. Let $(a_{i,j})_{i=1,j=1}^{n,m} \subseteq \mathcal{M}_+$ and $\delta > 0$ be as in the assumption of the lemma. By Proposition 7.7, it suffices to provide a proof in the case when $S = \mathcal{M}$.

The existence of $p_1, \ldots, p_n$ as required can be expressed in terms of satisfiability of an n-type (see §4.5). The type, denoted $\mathbf{t}(\bar{x})$, consists of the formulas[2] (recall that τ^+ was defined in Definition 5.9 and that $\varphi_p(x) = \max\{\|x - x^*\|_2^2, \|x^2 - x\|_2^2\}$ as in Example 5.4)

$$\varphi_p(x_i), \quad \|[b, x_i]\|_2, \quad \|\alpha_g(x_i) - x_i\|_2, \quad \tau^+((\delta - a_{i,j})x_i)$$

[2]A formula ψ corresponds to the condition asserting $\psi = 0$, hence a set of formulas determines a type.

for all $i \leq n$, $j \leq m$, $g \in G$, and all b in a fixed countable dense subset of $\mathcal{M}$. A proof that $\mathbf{t}(\bar{x})$ is satisfiable, given in the following claim, comprises the bulk of the ongoing proof.

Cl.p1-pn **Claim 3.3.** *For any $\epsilon > 0$, any finite $F \subset \mathcal{M}$ and any finite $K \subseteq G$, there are positive contractions $p_1, \dots, p_n \in \mathcal{M}_X^{\mathcal{U}}$ such that*

(1) $\|\sum_{i=1}^n p_i - 1\|_{2,X^{\mathcal{U}}} \leq \epsilon$,
item:cpou+2 *(2)* $\|p_i - p_i^2\|_{2,X^{\mathcal{U}}} \leq \epsilon$ *for all* $i \leq n$,
item:cpou+3 *(3)* $\|p_i b - b p_i\|_{2,X^{\mathcal{U}}} \leq \epsilon$ *for all* $i \leq n$ *and* $b \in F$,
item:cpou+4 *(4)* $\|(\alpha^{\mathcal{U}})_g(p_i) - p_i\|_{2,X^{\mathcal{U}}} \leq \epsilon$ *for all* $i \leq n$ *and* $g \in K$,
item:cpou+5 *(5)* $\tau(a_{i,j} p_i) \leq \delta \tau(p_i) + \epsilon$ *for all* $i \leq n$, $j \leq m$ *and* $\tau \in (X^\alpha)^{\mathcal{U}}$.

Proof. Fix $\tau \in X^\alpha$. The von Neumann algebra $\pi_\tau[\mathcal{M}]''$ is finite hence it has a centre-valued tracial state ([34, Theorem V.2.6])

$$\mathrm{Tr} : \pi_\tau[\mathcal{M}]'' \to Z(\pi_\tau[\mathcal{M}]''),$$

such that every tracial state σ on $\pi_\tau[\mathcal{M}]''$ filters through Tr, i.e., $\sigma = \sigma \circ \mathrm{Tr}$. By our assumption, there exists a normal subgroup $H \leq G$ of finite index such that the action induced by H on X is trivial. By Proposition 2.2, α_h^τ acts like the identity on $Z(\pi_\tau[\mathcal{M}]'')$ for all $h \in H$. Pick a single element from each left coset of H in G, and let $\{g_1, \dots, g_N\}$ be the set obtained from this selection. For $i \leq n$ and $j \leq m$, define

eq:cij
$$c_{i,j}^\tau := \frac{1}{N} \sum_{s=1}^N \alpha_{g_s}^\tau(\mathrm{Tr}(\pi_\tau(a_{i,j}))). \tag{3.1}$$

Since Tr is completely positive, each $c_{i,j}^\tau$ is positive, $\|c_{i,j}^\tau\| \leq \|a_{i,j}\|$, it belongs to $Z(\pi_\tau[\mathcal{M}]'')$, it is α^τ-invariant, and $\sigma(c_{i,j}^\tau) = \sigma(\pi_\tau(a_{i,j}))$ for all $\sigma \in T(\pi_\tau[\mathcal{M}]'')^{\alpha^\tau}$.

We will now define a local approximation $(\tilde{p}_i^\tau)_{i \leq n}$ to the required tuple $(p_i)_{i \leq n}$. Since $Z(\pi_\tau[\mathcal{M}]'')$ is a von Neumann algebra, there exists the largest α^τ-invariant central projection q in $\pi_\tau[\mathcal{M}]''$ such that

$$q(\delta - c_{1,j}) \geq 0$$

for all $j \leq m$. Denote this projection by $\tilde{p}_1^\tau$.

By induction on $2 \leq i \leq n$, define $\tilde{p}_i^\tau$ as the largest α^τ-invariant projection in $Z(\pi_\tau[\mathcal{M}]'')$ below $1 - \sum_{h=1}^{i-1} \tilde{p}_h^\tau$ such that

Eq.p.c
$$\tilde{p}_i^\tau(\delta - c_{i,j}^\tau) \geq 0 \tag{3.2}$$

for all $j \leq m$. This construction produces orthogonal projections $\tilde{p}_1^\tau, \dots, \tilde{p}_n^\tau \in Z(\pi_\tau[\mathcal{M}]'')$ which we claim also satisfy $\sum_{i=1}^n \tilde{p}_i^\tau = 1$. Indeed, suppose that $q_0 := 1 - \sum_{i=1}^n \tilde{p}_i^\tau$ is non-zero. By maximality of $\tilde{p}_1^\tau$, there are a non-zero central projection $q_1 \leq q_0$ and $j_1 \leq m$ such that $q_1(\delta - c_{1,j_1}^\tau) < 0$. Since c_{1,j_1}^τ is α^τ-invariant, we can assume that q_1 is α^τ-invariant as well. By repeating this argument for all $i \leq n$, one finds non-zero, α^τ-invariant, central projections $q_1 \geq q_2 \geq \dots \geq q_n$ in $\pi_\tau[\mathcal{M}]''$ such that for every $i \leq n$ there is $j_i \leq m$

which satisfies $q_i(\delta - c^{\tau}_{i,j_i}) < 0$. This implies

$$\frac{1}{\tau(q_n)}\tau(q_n\pi_\tau(a_{i,j_i})) = \frac{1}{\tau(q_n)}\tau(q_n c^{\tau}_{i,j_i}) > \delta,$$

for all $i \leq n$. Notice that the restriction of $\frac{1}{\tau(q_n)}\tau(q_n\pi_\tau(\cdot))$ to $\mathcal{M}$ is a tracial state in X^α. Indeed, as X is a face it follows that it belongs to X, while it is invariant since $\tau \in X^\alpha$ and q_n is α^τ-invariant and central. This is a contradiction, since by the assumptions of this lemma we have

$$\sup_{\sigma\in X^\alpha}\min_{i\leq n}\max_{j\leq m}\sigma(a_{i,j}) < \delta.$$

For every $i \leq n$ and $j \leq m$, let

eq:bij
$$\tilde{b}^{\tau}_{i,j} := \tilde{p}^{\tau}_i(\delta - c^{\tau}_{i,j}). \tag{3.3}$$

By (3.2) we have $0 \leq \tilde{b}^{\tau}_{i,j} \leq \delta\tilde{p}^{\tau}_i$. As a product of α^τ-invariant elements, it is α^τ-invariant.

Also

$$z^{\tau}_{i,j} := \tilde{b}^{\tau}_{i,j} - \frac{1}{N}\sum_{s=1}^{N}\alpha^{\tau}_{g_s}(\delta\tilde{p}^{\tau}_i - \pi_\tau(a_{i,j})\tilde{p}^{\tau}_i)$$

belongs to the nullset of Tr. Therefore, by [13, Théorème 3.2], there are $\tilde{x}^{(i,j,\tau)}_1, \dots, \tilde{x}^{(i,j,\tau)}_{10N}, \tilde{y}^{(i,j,\tau)}_1, \dots, \tilde{y}^{(i,j,\tau)}_{10N}$ in $\pi_\tau[\mathcal{M}]''$ each of norm not greater than $12\|z^{\tau}_{i,j}\|$ and such that

$$\frac{1}{N}\sum_{s=1}^{N}\alpha^{\tau}_{g_s}(\delta\tilde{p}^{\tau}_i - \pi_\tau(a_{i,j})\tilde{p}^{\tau}_i) = \tilde{b}^{\tau}_{i,j} + \sum_{h=1}^{10N}[\tilde{x}^{(i,j,\tau)}_h, \tilde{y}^{(i,j,\tau)}_h].$$

We can estimate $\|z^{\tau}_{i,j}\|$ as follows. Since each $\tilde{p}^{\tau}_i$ is α^τ-invariant and central, by the equalities (3.1) and (3.3) we have

$$\begin{aligned} z^{\tau}_{i,j} &= \tilde{b}^{\tau}_{i,j} - \delta\tilde{p}^{\tau}_i + \frac{1}{N}\sum_{s=1}^{N}\alpha^{\tau}_{g_s}(\pi_\tau(a_{i,j}))\tilde{p}^{\tau}_i \\ &= (\delta - c^{\tau}_{i,j})\tilde{p}^{\tau}_i - \delta\tilde{p}^{\tau}_i + \frac{1}{N}\sum_{s=1}^{N}\alpha^{\tau}_{g_s}(\pi_\tau(a_{i,j}))\tilde{p}^{\tau}_i \\ &= \frac{1}{N}\sum_{s=1}^{N}\alpha^{\tau}_{g_s}(\pi_\tau(a_{i,j}) - \mathrm{Tr}(\pi_\tau(a_{i,j})))\tilde{p}^{\tau}_i. \end{aligned}$$

Since $\|\pi_\tau(a_{i,j}) - \mathrm{Tr}(\pi_\tau(a_{i,j}))\| \leq 2\|a_{i,j}\|$, this implies

$$\|z^{\tau}_{i,j}\| \leq 2\|a_{i,j}\|.$$

Fix $\epsilon_0 > 0$ to be specified later (impatient readers may want to take a peek at (3.4)). Since $\|\cdot\|_{2,\tau}$ induces the strong operator topology on bounded sets, by Kaplansky's Density Theorem there are $b^{\tau}_{i,j}$, p^{τ}_i, $x^{(i,j,\tau)}_h$, and $y^{(i,j,\tau)}_h$ in $\mathcal{M}$ such that

item:i (i) $\|\tilde{b}^{\tau}_{i,j} - \pi_\tau(b^{\tau}_{i,j})\|_{2,\tau} < \epsilon_0$ and $b^{\tau}_{i,j} \geq 0$ for all $i \leq n$ and $j \leq m$,

(ii) $\|\tilde{p}_i^\tau - \pi_\tau(p_i^\tau)\|_{2,\tau} < \epsilon_0$ and $0 \leq p_i^\tau \leq 1$ for all $i \leq n$,

(iii) $\|\tilde{x}_h^{(i,j,\tau)} - \pi_\tau(x_h^{(i,j,\tau)})\|_{2,\tau} < \epsilon_0$ for all $i \leq n$, $j \leq m$ and $h \leq 10N$,

(iv) $\|\tilde{y}_h^{(i,j,\tau)} - \pi_\tau(y_h^{(i,j,\tau)})\|_{2,\tau} < \epsilon_0$ for all $i \leq n$, $j \leq m$ and $h \leq 10N$.

item:v (v) $\|b_{i,j}^\tau\| \leq \|\tilde{b}_{i,j}\|$, $\|p_i^\tau\| \leq \|\tilde{p}_i\|$, $\|x_h^{(i,j,\tau)}\| \leq \|\tilde{x}_h^{(i,j,\tau)}\|$, and $\|y_h^{(i,j,\tau)}\| \leq \|\tilde{y}_h^{(i,j,\tau)}\|$.

Define $s_\tau \in \mathcal{M}_+$ as

$$\begin{aligned} s_\tau := {} & \left(1 - \sum_{i=1}^n p_i^\tau\right)^2 + \sum_{i=1}^n (p_i^\tau - (p_i^\tau)^2)^2 + \\ & + \sum_{b \in F} \sum_{i=1}^n (p_i^\tau b - b p_i^\tau)^2 + \sum_{g \in K} \sum_{i=1}^n (\alpha_g(p_i^\tau) - p_i^\tau)^2 + \\ & + \sum_{i=1}^n \sum_{j=1}^n \left(\frac{1}{N} \left[\sum_{s=1}^N \alpha_{g_s}^\tau (\pi_\tau(a_{i,j}) p_i^\tau - \delta p_i^\tau) \right] + b_{i,j}^\tau + \sum_{h=1}^{10N} [x_h^{(i,j,\tau)}, y_h^{(i,j,\tau)}] \right)^2 . \end{aligned}$$

We will prove that

Eq.tau(c)
$$\begin{aligned} \tau(s_\tau) \leq (n^2 + 9n + 4n|F| \max_{b \in F} \|b\|^2 + 4n|K| \\ + nm((\max_{i,j} \|a_{i,j}\| + \delta)^2 + 1 + 10N \cdot 96^2 \max_{i,j} \|a_{i,j}\|^2))\epsilon_0^2. \end{aligned} \tag{3.4}$$

The previously established norm estimates (i)–(v) entail that

$$\begin{aligned} & \|1 - \sum_i p_i^\tau\|_{2,\tau} \leq \|1 - \sum_i \tilde{p}_i^\tau\|_{2,\tau} + n\epsilon_0 \leq n\epsilon_0, \\ & \|p_i^\tau - (p_i^\tau)^2\|_{2,\tau} < 3\epsilon_0, \\ & \qquad \|[p_i^\tau, b]\|_{2,\tau} \leq 2\|b\|\epsilon_0 \text{ for all } b \in F, \end{aligned}$$

and (since α_g is an isometry) $\|\alpha_g(p_i^\tau) - p_i^\tau\|_{2,\tau} < 2\epsilon_0$ for all $g \in K$. Also, for $i \leq n$, $j \leq m$, and $h \leq 10N$ we have

$$\begin{aligned} & \|[x_h^{(i,j,\tau)}, y_h^{(i,j,\tau)}] - [\tilde{x}_h^{(i,j,\tau)}, \tilde{y}_h^{(i,j,\tau)}]\|_{2,\tau} \\ & \qquad < (2\|\tilde{x}_h^{(i,j,\tau)}\| + 2\|\tilde{y}_h^{(i,j,\tau)}\|)\epsilon_0 \leq 48\|z_{i,j}^\tau\|\epsilon_0 \leq 96 \max_{i,j} \|a_{i,j}\|\epsilon_0, \end{aligned}$$

and

$$\|\alpha_{g_s}^\tau(\pi_\tau(a_{i,j})p_i^\tau - \delta p_i^\tau) - \alpha_{g_s}^\tau(\pi_\tau(a_{i,j})\tilde{p}_i - \delta\tilde{p}_i)\|_{2,\tau} < (\|a_{i,j}\| + \delta)\epsilon_0.$$

Using $\tau(|x|^2) = \|x\|_{2,\tau}^2$ and $\|xy\|_{2,\tau} \leq \|x\|\|y\|_{2,\tau}$, and by adding up the previous estimates, the inequality (3.4) follows. By taking ϵ_0 small enough we can suppose that $\tau(s_\tau) < \epsilon^2/C$, where $C := [G : H]$.

Being a closed face of $T(\mathcal{M})$, X is weak*-compact, and so is its closed subset X^α. Thus, there exist $\ell \geq 1$ and $\tau_1, \ldots, \tau_\ell$ in X such that

$$\sup_{\tau \in X^\alpha} \min_{k \leq \ell} \tau(s_{\tau_k}) < \epsilon^2/C.$$

Since $(\mathcal{M}, X, \alpha)$ has α-CPoU, there are pairwise orthogonal projections $q_1, \dots, q_\ell \in \mathcal{M}^{\mathcal{U}}_X \cap \mathcal{M}'$ such that

(a) $\sum_{k=1}^{\ell} q_k = 1$,
(b) $(\alpha^{\mathcal{U}})_g(q_k) = q_k$ for all $g \in G$ and $k \leq \ell$,
item:cpou+:c (c) $\tau(s_{\tau_k} q_k) \leq (\epsilon^2/C)\tau(q_k)$ for all $k \leq \ell$ and $\tau \in (X^\alpha)^{\mathcal{U}}$.

For $i \leq n$, $j \leq m$, and $h \leq 10N$, let

$b_{i,j} := \sum_{k=1}^{\ell} q_k b_{i,j}^{\tau_k}$, $p_i := \sum_{k=1}^{\ell} q_k p_i^{\tau_k}$, $x_h^{(i,j)} := \sum_{k=1}^{\ell} q_k x_h^{(i,j,\tau_k)}$ and $y_h^{(i,j)} := \sum_{k=1}^{\ell} q_k y_h^{(i,j,\tau_k)}$.

Since q_k and $p_i^{\tau_k}$ are commuting projections, each p_i is a positive contraction.

Given a tracial state $\tau \in (X^\alpha)^{\mathcal{U}}$, since $(1 - \sum_{i=1}^n p_i^{\tau_k})^2 < s_{\tau_k}$ for all $k \leq \ell$, we have

$$\begin{aligned}\|1 - \textstyle\sum_{i=1}^n p_i\|_{2,\tau}^2 = \|\textstyle\sum_{k=1}^{\ell} q_k(1 - \sum_{i=1}^n p_i^{\tau_k})\|_{2,\tau}^2 &\leq \textstyle\sum_{k=1}^{\ell} \tau(q_k s_{\tau_k}) \\ &\leq \textstyle\sum_{k=1}^{\ell} \tau(q_k)(\epsilon^2/C) \\ &= \epsilon^2/C.\end{aligned}$$

Hence, by Lemma 2.1,

$$\|1 - \textstyle\sum_{i=1}^n p_i\|_{2,X^{\mathcal{U}}} \leq C^{1/2}\|1 - \sum_{i=1}^n p_i\|_{2,(X^\alpha)^{\mathcal{U}}} \leq \epsilon.$$

In a similar fashion, using the definition of s_{τ_k}, one checks that $p_1, \dots, p_n$ satisfy clauses (2)–(4). We finally check that clause (5) holds. Fix $\tau \in (X^\alpha)^{\mathcal{U}}$. Then, using the fact that for $i \leq n$, $j \leq m$ and $k \leq \ell$ we have

$$\left(\frac{1}{N}\left[\sum_{s=1}^{N} \alpha^\tau_{g_s}(\pi_\tau(a_{i,j})p_i^{\tau_k} - \delta p_i^{\tau_k})\right] + b_{i,j}^{\tau_k} + \sum_{h=1}^{10N}[x_h^{(i,j,\tau_k)}, y_h^{(i,j,\tau_k)}]\right)^2 < s_{\tau_k},$$

and that $q_1, \dots q_k$ are orthogonal $\alpha^{\mathcal{U}}$-invariant projections in $\mathcal{M}^{\mathcal{U}}_X \cap \mathcal{M}'$ adding to 1, along with the fact $\tau(q_k s_{\tau_k}) < (\epsilon^2/C)\tau(q_k)$ for all k, we obtain

$$\left\|\frac{1}{N}\left[\sum_{s=1}^{N}(\alpha^{\mathcal{U}})_{g_s}(a_{i,j}p_i - \delta p_i)\right] + b_{i,j} + \sum_{h=1}^{10N}[x_h^{(i,j)}, y_h^{(i,j)}]\right\|_{2,\tau} < \epsilon.$$

Therefore, for $i \leq n$ and $j \leq m$

$$\tau(a_{i,j}p_i - \delta p_i) = \tau\left(\frac{1}{N}\sum_{s=1}^{N}(\alpha^{\mathcal{U}})_{g_s}(a_{i,j}p_i - \delta p_i)\right) < \epsilon - \tau(b_{i,j}) \leq \epsilon,$$

where the first equality holds since τ is $\alpha^{\mathcal{U}}$-invariant, and the last one since $b_{i,j} \geq 0$, being a sum of positive elements. □

Claim 3.3 implies that every finite subset of $\mathbf{t}(\bar{x})$ is approximately satisfiable in $\mathcal{M}$. Since $\mathcal{M}^{\mathcal{U}}_X$ is countably saturated (Theorem 4.5), $\mathbf{t}(\bar{x})$ is satisfiable in $\mathcal{M}^{\mathcal{U}}_X$. Any tuple $p_1, \dots, p_n$ that satisfies $\mathbf{t}(\bar{x})$ is as required, and this concludes the proof. □

sec:language

4. The language of tracially complete C*-algebras

In the context of continuous model theory, C*-algebras and von Neumann algebras are customarily interpreted as structures in the languages introduced in [17, §2] (to which we refer for all basic definitions on continuous model theory for operator algebras; see also [16]). In this section we present two new languages, $\mathcal{L}_{\|\cdot\|_2}$ and $\mathcal{L}_{\|\cdot\|_2,G}$, which provide a suitable framework for tracially complete and G-tracially complete C*-algebras.

MORE HERE?

sec:LL

4.1. The language $\mathcal{L}_{\|\cdot\|_2}$.

The language $\mathcal{L}_{\|\cdot\|_2}$ has a single sort with countably many domains D_n, it contains two constant symbols 0 and 1, symbols for the algebraic operations $+, \cdot$, and *, a symbol λ for each $\lambda \in \mathbb{C}$ and it is equipped with a symbol for the tracial norm $\|\cdot\|_2$. The metric on the structures in this language that satisfy the theory of interest is canonically associated with the norm, and we therefore omit a symbol for this metric. The moduli of continuity assigned to the algebraic operations are chosen in the natural fashion.

When we define $\mathcal{L}_{\|\cdot\|_2,G}$, we may a big deal that G should be countable so that the language is countable. We can't include a symbol for each λ in $\mathbb{C}$ if we want a countable language!

Let $(\mathcal{M}, X)$ be a tracially complete C*-algebra. The pair $(\mathcal{M}, X)$ can be thought of as an $\mathcal{L}_{\|\cdot\|_2}$-structure, with the symbol $\|\cdot\|_2$ interpreted as $\|\cdot\|_{2,X}$, the operation symbols interpreted in the obvious way, and D_k interpreted as the k-ball in the operator norm. This defines an equivalence between appropriate categories (see Theorem A.1).

The language $\mathcal{L}_{\|\cdot\|_2}$ is equipped with an infinite set of variables. Each variable is associated with a domain D_k for some $k \geq 1$, and there are infinitely many variables associated with each D_k. The association of x with D_k signifies that in every interpretation, x ranges over the operator norm k-ball of the tracially complete C*-algebra under consideration. In order to relax the terminology, we will say that a tuple $\bar{a} = (a_1, \dots, a_n)$ is *of the appropriate sort* for a tuple of variables $\bar{x} = (x_1, \dots, x_n)$ if a_j belongs to the domain associated with x_j, for each $j = 1, \dots, n$.

I don't understand this sentence. The definition of TC in the appendix also seems wrong. Is the point that TC-algebras are axiomatizable in the language L2?

The terms in the langugage $\mathcal{L}_{\|\cdot\|_2}$ are *-polynomials in non-commuting variables. Formulas are defined by recursion on their complexity in the standard fashion, as in [17, §2.4]. Atomic formulas are of the form $\|t\|_2$, for a term t. The set of formulas is the smallest set that contains all atomic formulas and it has the following closure properties.

F.1
(F1) If $n \geq 1$, $g \colon \mathbb{R}^n \to \mathbb{R}$ is uniformly continuous, and φ_j is a formula for $j = 1, \dots, n$, then $g(\varphi_1, \dots, \varphi_n)$ is a formula.

F.2
(F2) If φ is a formula, x is a variable of the sort D_k for some $k \geq 1$ then $\sup_{\|x\| \leq k} \varphi$ and $\inf_{\|x\| \leq k} \varphi$ are formulas.

inition:interpretation

Definition 4.1 (Interpretation of formulas)**.** Given a *-polynomial $Q(\bar{x})$, if $\varphi(\bar{x}) = \|Q(\bar{x})\|_2$ is an atomic formula in $\mathcal{L}_{\|\cdot\|_2}$, $\mathsf{M} := (\mathcal{M}, X)$ is a tracially complete C*-algebra, and $\bar{a}$ is a tuple in $\mathcal{M}$ of the same sort as $\bar{x}$, then the interpretation of $\varphi(\bar{x})$ in M at $\bar{a}$ is

$$\varphi^{\mathsf{M}}(\bar{a}) := \|Q(\bar{a})\|_{2,X}. \tag{4.1}$$

Given an arbitrary $\mathcal{L}_{\|\cdot\|_2}$-formula $\varphi(\bar{x})$, the interpretation of $\varphi(\bar{x})$ in M at $\bar{a}$ is defined recursively, interpreting (F1) and (F2) in the natural way, and is denoted by $\varphi^{\mathsf{M}}(\bar{a})$ (see [16, §2.1] for details).

Suppose that A is a C*-algebra and $X \subseteq T(A)$ is a weak*-compact, convex set. If (A, X) is not tracially complete, then it is, according to the standard definition, not an $\mathcal{L}_{\|\cdot\|_2}$-structure. This is because in a metric structure each sort is required to be complete. However, if $\varphi(\bar{x})$ is an $\mathcal{L}_{\|\cdot\|_2}$-formula and $\bar{a}$ is a tuple in A of the appropriate sort, then $\varphi^{(A,X)}(\bar{a})$ can be defined as in Definition 4.1. If $\mathsf{M} \coloneqq (\overline{A}, X)$, then we have $\varphi^{(A,X)}(\bar{a}) = \varphi^{\mathsf{M}}(\bar{a})$ for all tuples $\bar{a}$ in A of the appropriate sort. Because of this, all of our transfer results can be stated in terms of tracially complete C*-algebras or in terms of C*-algebras with a weak*-compact, convex set of traces.

In the special case of a C*-algebra A and a trace $\tau \in T(A)$, when A is clear from context, we write

$$\varphi^{\tau}(\bar{a}) \coloneqq \varphi^{(\pi_\tau[A]'',\tau)}(\pi_\tau(\bar{a})), \tag{4.2}$$

where $(\pi_\tau[A]'', \tau)$ is viewed as a tracially complete C*-algebra with the set of distinguished traces being the singleton $\{\tau\}$ and where, for a tuple $\bar{a} = (a_1, \ldots, a_n)$ in A,

$$\pi_\tau(\bar{a}) \coloneqq (\pi_\tau(a_1), \ldots, \pi_\tau(a_n)). \tag{4.3}$$

Remark 4.2. The described language of strict closures of C*-algebras is a reduct of the language of tracial von Neumann algebras described in [17, §2.3.2]. The latter language in addition contains function symbols tr^r and tr^i for the real and imaginary parts of the tracial state. To a certain extent, in tracially complete C*-algebras the role of tr^r is played by the definable predicate τ^+ (Definition 5.9). Note that tr^i is redundant, being definable from tr^r as $\mathrm{tr}^i(a) = -i\mathrm{tr}^r(ia)$.

sec:LLG

4.2. **The language $\mathcal{L}_{\|\cdot\|_2,G}$.** Fix a countable group G. The language $\mathcal{L}_{\|\cdot\|_2,G}$ is the expansion $\mathcal{L}_{\|\cdot\|_2,G} \coloneqq \mathcal{L}_{\|\cdot\|_2} \cup \{\alpha_g : g \in G\}$, where each α_g is a unary function symbol. The modulus of continuity for each α_g assures that it is 1-Lipshitz (i.e., a contraction). Countability of G is used only to assure the countability of the language and the separability of the space of formulas of the language (with respect to the norm defined in §4.3).

Given a tuple $\bar{y} = (y_1, \ldots, y_n)$ of variables in $\mathcal{L}_{\|\cdot\|_2,G}$ and $g \in G$, we denote the tuple $(\alpha_g(y_1), \ldots, \alpha_g(y_n))$ by $\alpha_g(\bar{y})$. For a tuple of non-commuting variables $\bar{y} = (y_1, \ldots, y_n)$, we use the abbreviation *G-*-polynomial* in the variables $\bar{y}$ to refer to a *-polynomial in the non-commuting variables $(\alpha_g(\bar{y}))_{g\in G}$. The G-*-polynomials are the terms of $\mathcal{L}_{\|\cdot\|_2,G}$. This granted, the formulas of $\mathcal{L}_{\|\cdot\|_2,G}$ are defined recursively, as in (F1) and (F2) above.

Consider a G-tracially complete $(\mathcal{M}, X, \alpha)$. The triple $(\mathcal{M}, X, \alpha)$ can be thought of as an $\mathcal{L}_{\|\cdot\|_2,G}$-structure after interpreting $(\mathcal{M}, X)$ as an $\mathcal{L}_{\|\cdot\|_2}$ structure as in §4.1 and interpreting the unitary symbol α_g as the automorphism α_g for $g \in G$.

In analogy to Definition 4.1, if $\varphi(\bar{x})$ is an $\mathcal{L}_{\|\cdot\|_2,G}$-formula, $\mathsf{M} := (\mathcal{M}, X, \alpha)$ is a G-tracially complete C*-algebra, and $\bar{a}$ is a tuple in $\mathcal{M}$, the evaluation of φ at $\bar{a}$ in M, again denoted $\varphi^{\mathsf{M}}(\bar{a})$, is defined recursively like in the non-equivariant case, and it is concretely obtained by interpreting $\|\cdot\|_2$ as $\|\cdot\|_{2,X}$.

Given a C*-algebra A and an action $\alpha : G \to \mathrm{Aut}(A)$, in the case when $\tau \in T(A)^\alpha$ and A and α are clear from the context, we write

$$\varphi^\tau(\pi_\tau(\bar{a})) := \varphi^{(\pi_\tau[A]'',\tau,\alpha^\tau)}(\pi_\tau(\bar{a})). \tag{4.4}$$

S.Theories

4.3. **Theories, axiomatizable classes.** Let $\mathcal{L} \in \{\mathcal{L}_{\|\cdot\|_2}, \mathcal{L}_{\|\cdot\|_2,G}\}$, where G is a countable group. For a fixed tuple $\bar{x}$ of variables, all $\mathcal{L}$-formulas with the free variables included among $\bar{x}$ form an $\mathbb{R}$-algebra (see (F1)), denoted $\mathfrak{F}^{\bar{x}}_{\mathcal{L}}$. The $\mathcal{L}$-sentences (i.e., formulas with no free variables) form an $\mathbb{R}$-algebra, denoted $\mathfrak{F}^{\langle\rangle}_{\mathcal{L}}$.

The *theory* of an $\mathcal{L}$-structure M (such as G-tracially complete C*-algebras), denoted $\mathrm{Th}(\mathsf{M})$, is the kernel of the character[3] on $\mathfrak{F}^{\langle\rangle}_{\mathcal{L}}$ defined by $\varphi \mapsto \varphi^{\mathsf{M}}$. A class $\mathcal{C}$ of $\mathcal{L}$-structures is *elementary* (or *axiomatizable*) if there is $T \subseteq \mathfrak{F}^{\langle\rangle}_{\mathcal{L}}$ such that $\mathsf{M} \in \mathcal{C}$ if and only if $T \subseteq \mathrm{Th}(\mathsf{M})$. The set T is called the *theory* of $\mathcal{C}$.

A class of structures in the same language is axiomatizable if and only if it is closed under ultraproducts, ultraroots, and isomorphisms ([16, Theorem 2.4.1]). This is used in the appendix to show that the class of all tracially complete C*-algebras, as well as some other relevant subclasses of this category, are axiomatizable. We will often implicitly use this fact throughout the paper, starting from the next subsection.

S.expand

4.4. **Expanding the language.** It is often possible (and convenient) to expand the language by adding function and predicate symbols. Towards this end, following [16, §3.1], we first introduce definable predicates.

Def.Definable

Definition 4.3. Let $\mathcal{L} \in \{\mathcal{L}_{\|\cdot\|_2}, \mathcal{L}_{\|\cdot\|_2,G}\}$ for a countable group G and fix an elementary class of $\mathcal{L}$-structures $\mathcal{C}$ and a tuple of variables $\bar{x}$ in $\mathcal{L}$. On the $\mathbb{R}$-algebra $\mathfrak{F}^{\bar{x}}_{\mathcal{L}}$ of all $\mathcal{L}$-formulas in the variables $\bar{x}$, define the seminorm

$$\|\varphi(\bar{x})\|_{\mathcal{C}} := \sup_{\mathsf{M},\bar{a}} |\varphi^{\mathsf{M}}(\bar{a})|, \tag{4.5}$$

where the supremum is taken over all M in $\mathcal{C}$ and all $\bar{a}$ of the appropriate sort in M. By $\mathfrak{W}^{\bar{x}}_{\mathcal{C}}$ we denote the Banach algebra obtained by quotienting and completing $\mathfrak{F}^{\bar{x}}_{\mathcal{L}}$ with respect to $\|\cdot\|_{\mathcal{C}}$. The elements of this algebra are the *definable predicates.*

Definable functions are defined analogously by recursively using the closure properties (F1) and (F2). The only instances of definable functions needed in the present paper are given by applications of continuous functional calculus, as discussed in [16, §3.4] and recorded in the following result.

[3]It is sometimes convenient to identify the theory with this character.

L.def.exp

Lemma 4.4. *Let $\mathcal{L} = \mathcal{L}_{\|\cdot\|_2}$ (resp., $\mathcal{L} = \mathcal{L}_{\|\cdot\|_2,G}$ for a coutnable group G) and let $\mathcal{L}'$ is the expansion of $\mathcal{L}$ obtained by adding the following.*

(1) A symbol exp *for the exponential function (with the appropriate modulus of uniform continuity).*
(2) For each continuous $f\colon \mathbb{R} \to \mathbb{C}$, a symbol for f to be interpreted as $a \mapsto f((a + a^*)/2)$.[4]

Then the formulas in $\mathcal{L}'$ are all definable predicates in the theory of tracially complete C^*-algebras (resp. G-tracially complete C^*-algebras).

Proof. This is a consequence of the Stone–Weierstrass theorem, by which the *-polynomials (which are terms in $\mathcal{L}$) are uniformly dense in $C(X)$ for every compact $X \subseteq \mathbb{R}$, and can thus approximate any given $f \in C(X)$. □

Since the set of normal elements is not definable in the theory of C*-algebras, in case of arbitrary C*-algebras this lemma cannot be extended to accommodate the full continuous functional calculus (see [16, §3.4]).

S.Conditions

4.5. **Conditions and (model-theoretic) types.** Fix a language $\mathcal{L}$. A *condition* is an expression of the form $\varphi(\bar{x}) = r$ for an $\mathcal{L}$-formula $\varphi(\bar{x})$ (cf. [17, §2.4]). If the formula φ belongs to the language expanded by constants for elements of a fixed structure M,, we then say that $\varphi(\bar{x}) = r$ is a condition *over* M. A condition $\varphi(\bar{x}) = r$ is *satisfied* by $\bar{a}$ in a structure M if $\varphi^{\mathsf{M}}(\bar{a}) = r$. Since the condition $\varphi(\bar{x}) = r$ is equivalent to the condition $\varphi(\bar{x}) - r = 0$, we will consider only conditions of the form $\varphi(\bar{x}) = 0$.

"cf." or "see" or "see . . . , for example"?

This will be an uncountable language. Is that a problem?

It is convenient (and harmless) to allow more general conditions of the form $\varphi(\bar{x}) = r$, where φ is a definable predicate. A particularly useful case is when the terms in φ use continuous functional calculus (see Lemma 4.4). Such condition can be uniformly approximated by standard conditions.

Let $\bar{x} = (x_1, \dots, x_n)$. An *$n$-type* (or a *type*[5] if n is understood) $\mathbf{t}(\bar{x})$ is a set of conditions in the variables $\bar{x}$, and it is a *type over a structure* M if all of its conditions are conditions over M. Given that all conditions in any type considered in this paper will be of the form $\varphi(\bar{x}) = 0$, the type can be identified with the set of these formulas $\varphi(\bar{x})$.[6] A type $\mathbf{t}(\bar{x})$ over a structure M is *approximately satisfiable* if for every finite subset of $\mathbf{t}(\bar{x})$ and every $\epsilon > 0$, there exists $\bar{b}$ in M of the appropriate sort such that $|\varphi^{\mathcal{M}}(\bar{b})| < \epsilon$ for every $\varphi(\bar{x})$ in the subset. Thus an n-type over M is approximately satisfiable if it is a weak*-limit of (naturally defined) types of n-tuples in M of the appropriate sort. A type $\mathbf{t}(\bar{x})$ over M is *satisfiable*

[4]The self-adjoint part $\mathcal{M}_{\mathrm{sa}}$ is a definable subset of every tracially complete C*-algebra $\mathcal{M}$, hence we could interpret f as a function on $\mathcal{M}_{\mathrm{sa}}$.

[5]All types considered here will be partial types in the sense of [16]. Since we will have no use for complete types, we will consistently write "type" in place of "partial type".

[6]In the setting of [16, §4.1], types are identified with real linear functionals on a subset of the algebra of formulas, and the set of formulas considered here is the kernel of this functional. Note that this kernel will typically contain definable predicates that are not formulas themselves.

if there exists some $\bar{b}$ in the domain of M of the appropriate sort such that $\varphi^{\mathsf{M}}(\bar{b}) = 0$ for all $\varphi(\bar{x})$ in the type.

A structure M is *countably saturated* if every countable type that is approximately satisfiable in M is satisfiable in M.[7] A structure M is *quantifier-free countably saturated* if every countable quantifier-free type that is approximately satisfiable in M is satisfiable in M.

The following is well-known.

What does the theorem below mean? Is this in any language $\mathcal{L}$? In the theory of tracially complete C*-algebras? Is the relative commutant defined in this level of generality?

T.saturated

Theorem 4.5.

(1) *Every ultraproduct associated with a free ultrafilter on* $\mathbb{N}$ *is countably saturated.*
(2) *Every reduced product associated with the Fréchet filter is countably saturated.*
(3) *If* M *is countably saturated and* $S \subseteq \mathsf{M}$ *is separable, then the relative commutant* $\mathsf{M} \cap S'$ *is countably quantifier-free saturated.*

Proof. For the first part see [2] or [14, Theorem 16.4.1]. The second is [14, Theorem 16.5.1]. In the case of C*-algebras with the usual language (i.e. where the norm is interpreted as the uniform norm instead of the tracial norm, like in this paper), the third part is [14, Corollary 16.5.3]; the proof of the general case is analogous. □

sec:transfer-def

5. The tracial transfer property

In this section we introduce the notion of tracial transfer property (§5.2) Before doing so, we isolate a specific class of formulas to which our transfer results apply in §5.1 and set out some further prerequisites in §§**??**–**??**.

REVISIT

S.convex.increasing

5.1. **Max and convex formulas and their zero sets.** In the following definition, and later on in the paper, for every $n \geq 1$ we will need a norm on the n-th power of each of the normed spaces considered. We will be using the max-norm, and for a tuple $\bar{x} = (x_1, \ldots, x_n)$ in a normed space, we write

Eq.Maxnorm

$$\|\bar{x}\| := \max\{\|x_1\|, \ldots, \|x_n\|\}. \tag{5.1}$$

Needless to say, $\|\cdot\|$ will often stand for one of the 2-norms associated with a tracially complete C*-algebras or tracial von Neumann algebras.

Readers familiar with the recursive definition of formulas in logic of metric structures will find some of the following definitions familiar. We follow the convention according to which inf corresponds to the existential quantifier and sup corresponds to the universal quantifier.

def.Aconvex

Definition 5.1. Let G be a countable group.

[7] A type is *countable* if it is countable as a set of conditions. Equivalently, one can consider types that are separable as subsets of the Banach algebra $\mathfrak{W}_{\mathcal{C}}^{\bar{x}}$ defined in §4.4.

1.def.Aconvex

(1) Let $\varphi(\bar{x})$ be a quantifier-free $\mathcal{L}_{\|\cdot\|_2}$-formula (or an $\mathcal{L}_{\|\cdot\|_2,G}$-formula) of the form
$$h(\|Q_1(\bar{x})\|_2^2, \dots, \|Q_k(\bar{x})\|_2^2),$$
where $k \geq 1$, Q_j for $1 \leq j \leq k$ are *-polynomials (or G-*-polynomials) in the non-commuting variables $\bar{x}$, and $h\colon \mathbb{R}^k \to \mathbb{R}$ is continuous. If h is a convex function, then φ is said to be a *quantifier-free convex* formula.

1.def.max

(2) A $\mathcal{L}_{\|\cdot\|_2}$-formula (or an $\mathcal{L}_{\|\cdot\|_2,G}$-formula) $\varphi(\bar{x})$ is a *quantifier-free max-formula* if it has the form
$$\max\{h_1(\|Q_1(\bar{x})\|_2^2), \dots, h_k(\|Q_k(\bar{x})\|_2^2)\}, \tag{5.2}$$
where $k \geq 1$, Q_j for $1 \leq j \leq k$ are *-polynomials (or G-*-polynomials) in the non-commuting variables $\bar{x}$, and $h_j\colon \mathbb{R} \to \mathbb{R}$ for $1 \leq j \leq k$ are continuous, increasing, convex functions. Note that every quantifier-free max-formula is automatically (equivalent to) a quantifier-free convex formula, since the connective max is convex.

(3) A formula of the form $\inf_{\|\bar{x}\|\leq 1} \varphi(\bar{x})$, for some quantifier-free convex formula φ, is called $\exists$-*convex* formula. The $\exists$-*max* formulas are defined analogously.

(4) If ψ is an $\exists$-convex formula, then $\sup_{\|\bar{x}\|\leq 1} \psi(\bar{x})$ is called an $\forall\exists$-*convex* formula. $\forall\exists$-*max* formulas are defined analogously.

(5) By induction on the complexity of a formula, we define *convex* formulas as follows. If φ is convex and $\bar{x}$ is a tuple of variables, then both $\sup_{\|\bar{x}\|\leq 1} \varphi(\bar{x})$ and $\inf_{\|\bar{x}\|\leq 1} \varphi(\bar{x})$ are convex. The *max-formulas* are defined analogously.

6.def.Aconvex

(6) A predicate which is definable in the theory of tracially complete C*-algebras is *convex-definable* (resp., *max-definable*, $\exists$-*max-definable*, $\forall\exists$-*convex-definable*, etc.) if it is a uniform limit, in such theory, of formulas that are convex (resp., max, $\exists$-max, $\forall\exists$-convex, etc.).

In particular, in the definitions of quantifier-free max and convex formulas, the *-polynomials Q_j can be replaced by terms in the expanded language (see §4.4), and therefore possibly include, e.g., the exponential function, or the square root function applied to a positive operator (see Lemma 4.4).

1.Ex.formula

Example 5.2. The formula $\max\{\|x - x^*\|_2^2, \|x - x^2\|_2^2\}$ is a quantifier-free max formula. The predicate $\inf_{\|y\|\leq 1} \|x - \exp(2\pi i y^* y)\|_2^2$ is an $\exists$-max definable predicate.

Def.convex.condition

Definition 5.3. A condition (see §4.5) of the form $\varphi(\bar{x}) = 0$ is *convex* if the formula $\varphi(\bar{x})$ is convex. Conditions are said to be $\forall\exists$-*convex, max*, etc. if the the formula $\varphi(\bar{x})$ has the corresponding property.

A type is *convex* (resp., *max*, $\forall\exists$-*convex*, etc.) if all of its conditions are convex (resp., max, $\forall\exists$-convex, etc.).

Be aware that many elementary formulas are left out by the classes of formulas we just defined. For instance, atomic formulas are not necessarily convex, since in our definitions, we consider exclusively formulas where the norm $\|\cdot\|_2$ appears squared. This is needed for some computations appearing later in the paper (e.g., in the proof of Theorem 6.2). These restrictions are often only formal, as in many cases, like in the atomic one, the kernel (or zero-set) of a positive formula does not change when its atomic subformulas are replaced by their squares. This is important since a formula often serves primarily as a mean for defining its kernel.

For a language $\mathcal{L} \in \{\mathcal{L}_{\|\cdot\|_2}, \mathcal{L}_{\|\cdot\|_2,G}\}$ and a $\mathcal{L}$-structure $\mathsf{M} := (\mathcal{M}, X)$ or $\mathsf{M} := (\mathcal{M}, X, \alpha)$, the *zero set* of an $\mathcal{L}$-formula $\varphi(x_1, \dots, x_n)$ in M is denoted

$$Z^{\mathsf{M}}(\varphi(\bar{x})) := \{\bar{a} \in \mathcal{M}^n : \varphi^{\mathsf{M}}(\bar{a}) = 0\}, \tag{5.3}$$

where we implicitly assume that $\bar{a}$ ranges over all tuples of the appropriate sort.

Ex.formulas **Example 5.4.** Each of the following sets is the zero set of a quantifier-free, max-formula (see [16, Example 3.2.7]).

(1) The set of all self-adjoint elements is the zero-set of

$$\|x - x^*\|_2^2 \tag{5.4}$$

.

3.Ex (2) The set of all projections is the zero-set of

$$\max\{\|x - x^*\|_2^2, \|x^2 - x\|_2^2\}. \tag{5.5}$$

(3) The set of all unitaries is the zero-set of

$$\max\{\|x^*x - 1\|_2^2, \|xx^* - 1\|_2^2\} \tag{5.6}$$

(the second expression in the maximum is redundant in stably finite C*-algebras, and therefore in tracially complete C*-algebras).

4.Ex (4) For $n \geq 2$, the set of all n^2 tuples that are the M_n-matrix units is the zero-set of

$$\max_{\substack{i,j \leq n \\ k \neq k' \leq n}} \{\|x_{ij} - x_{ji}^*\|_2^2, \|x_{ij} - x_{ik}x_{kj}\|_2^2, \|x_{ik}x_{k'j}\|_2^2\}. \tag{5.7}$$

(5) As in [16, §3.4], we can expand the language to accommodate continuous functional calculus (see also Lemma 4.4), by adding symbols for certain continuous functions. With this convention applied to the exponential function, the formula

$$\inf_{\|z\| \leq 1} \|x - \exp(2\pi i z^* z)\|_2^2 \tag{5.8}$$

is $\exists$-max, and its zero-set is the set of all elements that can be approximated by unitaries that have a positive logarithm of the minimal possible norm, 2π.

A word of caution is warranted: to an $\mathcal{L}_{\|\cdot\|_2}$- or $\mathcal{L}_{\|\cdot\|_2,G}$-formula φ one can associate a formula of the language of C*-algebras φ', by replacing all instances of $\|\cdot\|_2$ with $\|\cdot\|$. In every tracially complete C*-algebra $(\mathcal{M}, X)$ we then have $Z^{(\mathcal{M},X)}(\varphi) = Z^{\mathcal{M}}(\varphi')$. However, the behaviours of φ and φ' can be very different with respect to the definability properties. For example, the set of projections is definable in the theory of C*-algebras but not in the theory of tracially complete C*-algebras.[8]

why? I don't think we have any counterexamples to definability.

S_TTP

5.2. **Tracial Transfer Properties.** Here we introduce a central notion for this paper, which formalizes in a precise model-theoretic form the general (and often observed) phenomenon of transfer of properties from the fibres of a tracially complete C*-algebra to the whole algebra.

Isn't this just classification of projections?

def.ttf0

Should the definition be restricted to the factorial setting?

Definition 5.5. Suppose that $\mathsf{M} := (\mathcal{M}, X)$ is a tracially complete C*-algebra. We say that M has the *tracial transfer property* if for every $\forall\exists$-max $\mathcal{L}_{\|\cdot\|_2}$-formula ψ and every tuple $\bar{a}$ of elements of $\mathcal{M}$ of the appropriate sort,

eq:TTP

$$\psi^{\mathsf{M}}(\bar{a}) = \sup_{\tau \in X} \psi^{\tau}(\pi_\tau(\bar{a})). \tag{5.9}$$

We say that M has the *factorial tracial transfer property* if for every $\forall\exists$-max $\mathcal{L}_{\|\cdot\|_2}$-formula ψ and every tuple $\bar{a}$ of elements of $\mathcal{M}$ of the appropriate sort,

eq:fTTP

$$\psi^{\mathsf{M}}(\bar{a}) = \sup_{\tau \in \partial_e X} \psi^{\tau}(\pi_\tau(\bar{a})). \tag{5.10}$$

The name "factorial tracial transfer property" was chosen since when $(\mathcal{M}, X)$ is a factorial tracially complete C*-algebra, $\pi_\tau[\mathcal{M}]''$ is a factor whenever $\tau \in \partial_e X$. Therefore, the factorial tracial transfer property states that in order to compute $\psi^{\mathsf{M}}(\bar{a})$, it is enough to consider factorial von Neumann algebra completions of $\mathcal{M}$.

We will also consider the following equivariant version of the tracial transfer property.

Should we only define this when there is a uniform bound on the cardinality of the orbits in the action on the traces?

def.ttf

Definition 5.6. Suppose that G is a countable group and $\mathsf{M} := (\mathcal{M}, X, \alpha)$ is a G-tracially complete C*-algebra such that $X^\alpha \neq \varnothing$. Denote $(\mathcal{M}, X^\alpha, \alpha)$ by M^α. We say that M has the *tracial transfer property* if for every $\forall\exists$-max $\mathcal{L}_{\|\cdot\|_2,G}$-formula ψ and every tuple $\bar{a}$ of elements in $\mathcal{M}$ of the appropriate sort,

eq:TTP-alpha

$$\psi^{\mathsf{M}^\alpha}(\bar{a}) = \sup_{\tau \in X^\alpha} \psi^{\tau}(\pi_\tau(\bar{a})). \tag{5.11}$$

We say that M has the *factorial tracial transfer property* if $X = X^\alpha$ and for every $\forall\exists$-max $\mathcal{L}_{\|\cdot\|_2,G}$-formula ψ and every tuple $\bar{a}$ of elements in $\mathcal{M}$ of the appropriate sort,

eq:fTTP-alpha

$$\psi^{\mathsf{M}^\alpha}(\bar{a}) = \sup_{\tau \in \partial_e X} \psi^{\tau}(\pi_\tau(\bar{a})). \tag{5.12}$$

We note that the tracial transfer property defines the following stronger version of itself where the formulas are replaced with definable predicates (6) of Definition 5.1). This is immediate since any $\forall\exists$-max definable predicate is uniformly approximated by $\forall\exists$-max formulas.

[8]It is fortunately definable in the theory of factorial tracially complete C*-algebras with the tracial transfer property; see [18].

Proposition 5.7. *Suppose $\mathcal{M} := (\mathcal{M}, X)$ is a tracially complete C^*-algebra and ψ is an $\forall\exists$-max definable predicate in the language $\mathcal{L}_{\|\cdot\|_2}$. If M has the tracial transfer property and $\bar{a}$ is a tuple of $\mathcal{M}$ of the appropriate sort, then*

$$\psi^{\mathsf{M}}(\bar{a}) = \sup_{\tau \in X} \psi^\tau(\pi_\tau(\bar{a})), \tag{5.13}$$

and if M has the factorial tracial transfer property and $\bar{a}$ is a tuple in $\mathcal{M}$ of the appropriate sort, then

$$\psi^{\mathsf{M}}(\bar{a}) = \sup_{\tau \in \partial_e X} \psi^\tau(\pi_\tau(\bar{a})), \tag{5.14}$$

The analogous statement holds in the equivariant setting, replacing $\mathcal{L}_{\|\cdot\|_2}$ with $\mathcal{L}_{\|\cdot\|_2,G}$ and $(\mathcal{M}, X)$ with $(\mathcal{M}, X, \alpha)$ for a countable group G.

remark:limits **Remark 5.8.** It is of course possible to make sense of definitions of transfer properties for classes of formulas which are larger than $\forall\exists$-max formulas (or predicates). One should note however that some requirements need to be imposed, as otherwise the equality in (5.9) could fail for trivial reasons. Indeed, let

$$\psi(\bar{x}) = h(\|Q(\bar{x})\|_2^2)$$

be a quantifer-free formula where Q is a $*$-polynomial and $h\colon \mathbb{R} \to \mathbb{R}$ is a continuous function. Given a tracially complete C*-algebra $(\mathcal{M}, X)$ and a tuple $\bar{a}$ in $\mathcal{M}$ of the appropriate sort, claiming that

$$h(\|Q(\bar{a})\|_{2,X}^2) = \sup_{\tau \in X} h(\|Q(\pi_\tau(\bar{a}))\|_{2,\tau}^2),$$

is the same as saying

$$h\Big(\sup_{\tau \in X} \|Q(\bar{a})\|_{2,\tau}^2\Big) = \sup_{\tau \in X} h(\|Q(\bar{a})\|_{2,\tau}^2),$$

which will generally fail when h does not commute with sup, for instance when h is strictly decreasing.

Nevertheless, the class of max-formulas turns out to be quite versatile and rich, and it often suffices in applications, as shown in §8.

There is a possibility that for every $\mathcal{L}_{\|\cdot\|_2}$ or $\mathcal{L}_{\|\cdot\|_2,G}$ formula φ, the evaluation φ^{M} is determined by the behaviour of the functions $\tau \mapsto \psi^\tau$ where ψ ranges over a prescribed set of formulas that 'control' φ. Such Feferman–Vaught-style result is conjectured and briefly discussed in §9.1.

TO DO: MOVE ELSEWHERE.

This subsec seems out of place.

5.3. Definability of τ^+.

Def:Tau+ **Definition 5.9.** We expand our language $\mathcal{L}_{\|\cdot\|_2}$ by adding a relational symbol τ^+, interpreted as a 1-Lipshitz (i.e., contractive) predicate. If $\mathsf{M} := (\mathcal{M}, X)$ is a tracially complete C*-algebra, and $b \in \mathcal{M}$, we then interpret $(\tau^+)^{\mathsf{M}}(b)$ as $\sup_{\tau \in X} \mathrm{Re}(\tau(b))$. This interpretation also applies to G-tracially complete C*-algebras.

Lemma 5.10 shows that the predicate τ^+ is quantifier-free definable in the language $\mathcal{L}_{\|\cdot\|_2}$ (and therefore in $\mathcal{L}_{\|\cdot\|_2,G}$ as well) on tracially complete C*-algebras.

lemma:Tau+Definable

Lemma 5.10. *The predicate τ^+ is a quantifier-free max-definable predicate in the theory of tracially complete C*-algebras (and therefore in the theory of G-tracially complete C*-algebras for any countable group G).*

Proof. We need to prove that for every $N \in \mathbb{N}$ there exists a sequence $\{\varphi_n(x)\}_{n\in\mathbb{N}}$ of quantifier-free max formulas in $\mathcal{L}_{\|\cdot\|_2}$ such that, for every tracially complete C*-algebra $\mathsf{M} := (\mathcal{M}, X)$ and every $b \in \mathcal{M}$ of operator norm smaller than N

$$(\tau^+)^{\mathsf{M}}(b) = \lim_{n\to\infty} (\varphi_n)^{\mathsf{M}}(b),$$

where the convergence is uniform.

Fix M and fix for a moment $\tau \in X$. Since $\|b\|_{2,\tau} = \tau(b^*b)^{1/2}$, for $b \geq 0$ we have $\tau(b) = \|b^{1/2}\|_{2,\tau}^2$. If $b \in \mathcal{M}_{\mathrm{sa}}$ belongs to the N-ball, then $b + N \geq 0$ and therefore $\tau(b) = \|(b+N)^{1/2}\|_{2,\tau}^2 - N$. It follows that for a self-adjoint b we have

$$\tau^+(b) = \|(b+N)^{1/2}\|_{2,X}^2 - N.$$

By the Stone-Weierstrass Theorem, we can approximate the square root function uniformly on $[0, 2N]$ by polynomials, thus yielding the desired sequence of formulas $(\varphi_n(x))_{n\in\mathbb{N}}$.

If b is not self-adjoint, then $\tau^+(b) = \tau^+((b+b^*)/2)$, and together with the previous paragraph this gives a definition of $\tau^+(b)$. $\square$

Section4

6. Measurable decomposition and selection

REWRITE

The factorial tracial transfer property will reduce to the tracial transfer property (see Definition 5.5) via a direct integral argument. Roughly, for a factorial tracially complete C*-algebra $\mathsf{M} := (\mathcal{M}, X)$ with X metrizable and a trace $\tau \in X$, there is a unique probability measure μ_τ on $\partial_e X$ with barycentre τ, and there is a canonical identification

$$\pi_\tau[\mathcal{M}]'' \cong \int_{\partial_e X}^{\oplus} \pi_\sigma[\mathcal{M}]''\, d\mu_\tau(\sigma). \tag{6.1}$$

This will allow us to deduce information about all tracial von Neumann algebra completions of M from the factorial ones.

The main theorems of this section are stated in §6.1. After recalling some preliminaries on disintegration theory in §**??**, MORE HERE.

The proof of this theorem is entirely von Neumann-algebraic, and so is the main result behind Theorem 6.2 (see Theorem 9.1 below).

sec:convex.transfer

6.1. **The reduction theorems.** Using such a decomposition, we will prove the following result, which immediately gives the equivalence of the factorial tracial transfer property and the non-factorial version.

P.convex.transfer.0

Theorem 6.1. *Let $\varphi(\bar{x})$ be an $\forall\exists$-convex $\mathcal{L}_{\|\cdot\|_2}$-formula or an $\forall\exists$-convex definable predicate. Let $\mathsf{M} := (\mathcal{M}, X)$ be a factorial tracially complete C^*-algebra such that $\mathcal{M}$ is $\|\cdot\|_{2,X}$-separable . Then, for every tuple $\bar{a}$ in $\mathcal{M}$ of the appropriate sort,*

$$\sup_{\sigma\in X} \varphi^\sigma(\pi_\sigma(\bar{a})) = \sup_{\sigma\in\partial_e X} \varphi^\sigma(\pi_\sigma(\bar{a})). \tag{6.2}$$

In particular, M has the tracial transfer property if and only if M has the factorial tracial transfer property

The above theorem follows immediately from the equivariant version below by taking G to be the trivial group. Note that although the action of G on X is required to be trivial, the action of G on $\mathcal{M}$ could be highly non-trivial: e.g., if $\mathcal{M}$ is the hyperfinite II_1 factor, all groups admit an outer action on $\mathcal{M}$.

P.convex.transfer

Theorem 6.2. *Let G be a countable group and let $\varphi(\bar{x})$ be an $\forall\exists$-convex $\mathcal{L}_{\|\cdot\|_2,G}$-formula or an $\forall\exists$-convex definable predicate. Let $\mathsf{M} := (\mathcal{M}, X, \alpha)$ be a factorial G-tracially complete C^*-algebra such that $\mathcal{M}$ is $\|\cdot\|_{2,X}$-separable and G acts trivially on X. Then, for every tuple $\bar{a}$ in $\mathcal{M}$ of the appropriate sort,*

$$\sup_{\sigma\in X} \varphi^\sigma(\pi_\sigma(\bar{a})) = \sup_{\sigma\in\partial_e X} \varphi^\sigma(\pi_\sigma(\bar{a})). \tag{6.3}$$

In particular, M has the tracial transfer property if and only if M has the factorial tracial transfer property.

We only prove Theorem 6.2 as Theorem 6.1 then follows immediately. The rest of this section is devoted to the proof of Theorem 6.2.

sec:disintegration

6.2. **Disintegration.** This subsection recalls some elements of the disintegration theory of finite von Neumann algebras and representations (see [34, §IV.8], [4, §III.1.6], [24, §14], or [32, §1.1.4] for a quick overview). The main tool going forward will be the selection result stated as Proposition 6.3 below, which will be used (via Lemma 6.6) in proving the existential case of Theorem 6.2 (i.e., the case when ψ is an $\exists$-convex $\mathcal{L}_{\|\cdot\|_2,G}$-formula or definable predicate).

Let us set the notation. Fix a separable C*-algebra A, a countable dense subset $\{a_n\}_{n\in\mathbb{N}}$ of A and a positive Radon measure μ on the state space $S(A)$ with support contained in some Borel subset $T \subseteq S(A)$. For each $\sigma \in T$, let $(\pi_\sigma, H_\sigma, \eta_\sigma)$ be the cyclic representation obtained from the GNS-construction. The association $\sigma \in T \to H_\sigma$ produces a measurable field of

Hilbert spaces $\{H_\sigma\}_{\sigma\in T}$, where $\xi = (\xi_\sigma)_{\sigma\in T}$ in $\prod_{\sigma\in T} H_\sigma$ is a *measurable vector field* if and only if the maps

$$\sigma \mapsto \langle \pi_\sigma(a_n)\eta_\sigma, \xi_\sigma\rangle$$

are μ-measurable for every $n \in \mathbb{N}$. In addition, the map $\sigma \mapsto \|\xi_\sigma\|$ is μ-measurable. The Hilbert space $H_\mu = \int_T^\oplus H_\sigma \, d\mu(\sigma)$ is the space of all measurable vector fields $\xi = (\xi_\sigma)_{\sigma\in T}$ such that

$$\|\xi\| = \Big(\int_T^\oplus \|\xi_\sigma\|^2 d\mu(\sigma)\Big)^{1/2} < \infty.$$

Vectors in H_μ are written as $\int_T^\oplus \xi_\sigma \, d\mu(\sigma)$. We let $\int_T^\oplus \pi_\sigma \, d\mu(\sigma)$ denote the direct integral representation obtained from the measurable field of representations $\{\pi_\sigma\}_{\sigma\in T}$.

An operator $a \in B(H_\mu)$ is called *decomposable* if there is a function $\sigma \mapsto a_\sigma$ such that $a_\sigma \in B(H_\sigma)$ for almost all σ, and for every measurable vector field $\xi := \int_T^\oplus \xi_\sigma d\mu(\sigma)$ in H_μ the image $a(\xi) = \int_T^\oplus a_\sigma(\xi_\sigma) d\mu(\sigma)$ is a measurable vector field. If a is decomposable and there is an L^∞-scalar function $\sigma \mapsto \lambda_\sigma$ such that $a_\sigma = \lambda_\sigma 1_{H_\sigma}$ for almost all σ, then a is called *diagonalizable*. An operator in $B(H_\mu)$ is decomposable if and only if it commutes with all diagonalizable operators ([34, Corollary IV.8.16])

Given a $\|\cdot\|_{2,X}$-separable factorial tracially complete C*-algebra $(\mathcal{M}, X)$, the measure used to disintegrate a representation π_τ for $\tau \in X$ will come from the decomposition of τ as an integral of extremal tracial states. If $T(\mathcal{M})$ is weak*-compact and metrizable and $X \subseteq T(\mathcal{M})$ is a closed face, then for every $\tau \in X$ there exists a unique probability measure μ on $\partial_e X$ such that $f(\tau) = \int_{\partial_e X} f(\sigma)\, d\mu(\sigma)$ for every continuous affine function $f \colon X \to \mathbb{R}$ (this is the second part of [34, Theorem IV.6.15]). Such μ is called the *representing measure* of τ on X, and τ is called the *barycenter* of μ on X. By a result of Choquet, the extreme boundary of a metrizable Choquet simplex is a G_δ subset (this is the first part of [34, Theorem IV.6.15]), and therefore a Polish space with respect to the subspace topology.

The last sentence of the following result is known among the experts, but it does not seem to appear explicitly in the existing literature.

P.disintegration **Proposition 6.3.** *Suppose that $\mathsf{M} = (\mathcal{M}, X)$ is a factorial tracially complete C*-algebra which is $\|\cdot\|_{2,X}$-separable. Suppose that $\tau \in X$. Then the representing measure μ satisfies the following.*

1.P.disintegration *(1) π_τ is spatially equivalent to $\int_{\partial_e X}^\oplus \pi_\sigma \, d\mu(\sigma)$.*

2.P.disintegration *(2) $\pi_\tau[\mathcal{M}]'' \cong \int_{\partial_e X}^\oplus \pi_\sigma[\mathcal{M}]''\, d\mu(\sigma)$.*

In particular $\pi_\tau[\mathcal{M}]''$ can be disintegrated as a direct integral of II$_1$ factors and its elements can be identified with decomposable operators in $\int_{\partial_e X}^\oplus B(H_\sigma)\, d\mu(\sigma)$.

Proof. In order to prepare the grounds for the disintegration theory, let A be a (norm-)separable subalgebra of $(\mathcal{M}, \|\cdot\|)$ which is dense in the $\|\cdot\|_{2,X}$-norm. Note that by [6, Corollary 2.23], with the set $X(A) = \{\tau \restriction A : \tau \in$

$X\} \subseteq T(A)$, we have an isomorphism between $(\overline{A}^{X(A)}, X(A))$ and $(\mathcal{M}, X)$. In what follows we shall thus identify states in X with their restrictions to A.

The first key point in the proof of Proposition 6.3 is given by [1, Lemma 4.1] (see also [5, Lemma 3.4] for an alternative presentation of this result closer to our setup). By these results, if $\tau \in X$ and μ is a measure supported on $\partial_e X$ representing τ, then, by uniqueness of the GNS-representation, π_τ is spatially equivalent to a subrepresentation of $\int_{\partial_e X}^{\oplus} \pi_\sigma \, d\mu(\sigma)$, and the *-homomorphism on $\pi_\tau[A]$

$$\pi_\tau(a) \mapsto \int_{\partial_e X}^{\oplus} \pi_\sigma(a) \, d\mu(\sigma)$$

extends to an injective and normal *-homomorphism $\Phi \colon \pi_\tau[A]'' \to \int_{\partial_e X}^{\oplus} \pi_\sigma[A]'' \, d\mu(\sigma)$. By [34, Theorem IV.8.31], the representation π_τ is unitarily equivalent to $\int_{\partial_e X}^{\oplus} \pi_\sigma \, d\mu(\sigma)$ if and only if the measure μ is *orthogonal* in the sense of [34, Definition IV.6.20]. A practical equivalent formulation of μ being orthogonal is taken from [34, Theorem IV.6.19]. Consider the ultraweakly continuous linear positive map $\theta \colon L^\infty(\partial_e X, \mu) \to \pi_\tau[A]'$ such that $\theta(1) = 1$ and such that, for $f \in L^\infty(\partial_e X, \mu)$, satisfies

eq:trace

$$\tau(\theta(f)\pi_\tau(a)) = \int_{\partial_e X}^{\oplus} f(\sigma)\sigma(a) \, d\mu(\sigma). \tag{6.4}$$

Such a map exists and it is unique, by [34, Proposition IV.6.18]. The measure μ is orthogonal if and only if θ is multiplicative. The fact that in the tracial setting θ is a $*$-homomorphism is a consequence [29, Lemma 10], where it is proved that θ is an isomorphism between $L^\infty(\partial_e X, \mu)$ and $\mathcal{Z}(\pi_\tau[A]'')$. We can thus conclude that π_τ and $\int_{\partial_e X}^{\oplus} \pi_\sigma \, d\mu(\sigma)$ are spatially equivalent representations of A. Since the spatial equivalence between π_τ and $\int_{\partial_e X}^{\oplus} \pi_\sigma \, d\mu(\sigma)$ from [34, Theorem IV.8.31] is given by $\int_{\partial_e X}^{\oplus} \pi_\sigma(a)\eta_\sigma \mapsto \pi_\tau(a)\eta_\tau$ for all $a \in A$, there is a unitary $U \colon \int_{\partial_e X}^{\oplus} H_\sigma \to H_\tau$ such that $UaU^* = \Phi(a)$ for all $a \in \pi_\tau[A]''$. This, along with equation (6.4), entails that $\Phi[\mathcal{Z}(\pi_\tau[A]'')]$ corresponds to the set of diagonalizable operators in $B(\int_{\partial_e X}^{\oplus} H_\sigma)$, which in turn implies that $\int_{\partial_e X}^{\oplus} \pi_\sigma[A]'' \, d\mu(\sigma)$ is equal to the image of $\pi_\tau[A]''$ via Φ (see e.g. [12, Lemma 8.4.1]). Therefore Φ is surjective.

Since A is $\|\cdot\|_{2,X}$-dense in $\mathcal{M}$, we have $\pi_\sigma[A]'' = \pi_\sigma[\mathcal{M}]''$ for all $\sigma \in X$ and therefore $\mathrm{Ad}(U)$ can be extended to an isomorphism (again denoted Φ) $\Phi \colon \pi_\tau[\mathcal{M}]'' \to \int_{\partial_e X}^{\oplus} \pi_\sigma[\mathcal{M}]'' \, d\mu(\sigma)$. □

In [29, Theorem 11] it is shown that if $(\mathcal{M}, X)$ is a continuous W*-bundle and $\tau \in \partial_e X$ then $\pi_\tau[\mathcal{M}] = \pi_\tau[\mathcal{M}]''$. We do not know whether this conclusion holds in every factorial tracially complete C*-algebra for every $\tau \in \partial_e X$. Note however that if τ is faithful then $\pi_\tau[\mathcal{M}]$ is a von Neumann algebra if and only if $\mathcal{M}$ is, hence the conclusion can fail for non-extremal tracial states.

If G is a discrete group acting on a tracially complete C*-algebra $(\mathcal{M}, X)$ via an action α such that $X = X^\alpha$, and $\tau \in X$, then the isomorphism Φ from the previous proof acts equivariantly between $(\pi_\tau[\mathcal{M}], \tau, \alpha^\tau)$ and $(\int_{\partial_e X}^{\oplus} \pi_\sigma[\mathcal{M}]'', \tau, \int_{\partial_e X}^{\oplus} \alpha^\sigma)$. Indeed, for every $a \in \mathcal{M}$ and $g \in G$ we have

$$\begin{aligned}\Phi(\alpha_g^\tau(\pi_\tau(a))) = \Phi(\pi_\tau(\alpha_g(a))) &= \int_{\partial_e X}^{\oplus} \pi_\sigma(\alpha_g(a))\, d\mu(\sigma)\\ &= \int_{\partial_e X}^{\oplus} \alpha_g^\sigma(\pi_\sigma(a))\, d\mu(\sigma)\end{aligned}$$

The following then immediately follows from Proposition 6.3.

disintegration.dynamic **Proposition 6.4.** *Suppose that G is a discrete countable group and $(\mathcal{M}, X, \alpha)$ is a factorial G-tracially complete C*-algebra which is $\|\cdot\|_{2,X}$-separable and such that $X = X^\alpha$. Suppose that $\tau \in X$. Then the representing measure $\mu = \mu_\tau$ satisfies the following.*

(1) π_τ is unitarily equivalent to $\int_{\partial_e X}^{\oplus} \pi_\sigma\, d\mu(\sigma)$.
(2) $\pi_\tau[\mathcal{M}]'' \cong \int_{\partial_e X}^{\oplus} \pi_\sigma[\mathcal{M}]''\, d\mu(\sigma)$.
(3) α_g^τ is conjugate to $\int_{\partial_e X}^{\oplus} \alpha_g^\sigma\, d\mu(\sigma)$, for all $g \in G$, via the isomorphism mapping $\pi_\tau(a) \mapsto \int_{\partial_e X}^{\oplus} \pi_\sigma(a)\, d\mu(\sigma)$ for all $a \in \mathcal{M}$.

In particular $\pi_\tau[\mathcal{M}]''$ can be disintegrated as a direct integral of II$_1$ factors of which G is acting and the elements of $\pi_\tau[\mathcal{M}]''$ can be identified with decomposable operators in $\int_{\partial_e X}^{\oplus} B(H_\sigma)\, d\mu(\sigma)$. □

With Propositions 6.3 and 6.4 under our belt, we now turn to the other component of the proofs of this section, measurable selection. Let $(\mathcal{M}, X, \alpha)$ be a separable factorial G-tracially complete C*-algebra as in the assumptions of Theorem 6.2. Consider $\mathcal{M}$ with the topology induced by the norm $\|\cdot\|_{2,X}$ and X as a subspace of the dual of $\mathcal{M}$ with the weak*-topology. With $\mathcal{M}_k$ denoting the ball of radius $k > 0$ of $\mathcal{M}$ with respect to the norm topology, both $\mathcal{M}_k$ and X are Polish spaces, namely each one is separable and admits a compatible complete metric (for $\mathcal{M}_k$ the metric is given by the $\|\cdot\|_{2,X}$-norm and for X see [25, §1.3]).

Because of the requirement $X = X^\alpha$, proving Theorem 6.2 under the assumption that the action α is trivial on X is nearly identical to the general case. Therefore, in order to lighten the notation, we will drop α and provide only proofs for tracially complete C*-algebras for the remaining part of this section.

lem:Continuity **Lemma 6.5.** *Suppose that $\varphi(\bar{x}, \bar{y})$ is an $\mathcal{L}_{\|\cdot\|_2}$-formula, that $(\mathcal{M}, X)$ is a tracially complete C*-algebra and that $\bar{a}$ is a tuple of the same sort as $\bar{x}$ in $\mathcal{M}$.*

2.L.Evaluation

(1) If φ is quantifier-free, then the evaluation function[9] $\mathrm{Ev}_{\varphi,\bar{a}} : X \times \mathcal{M}^n \to \mathbb{R}$

$$\mathrm{Ev}_{\varphi,\bar{a}}(\tau, \bar{b}) \mapsto \varphi^\tau(\pi_\tau(\bar{a}, \bar{b}))$$

is jointly continuous with respect to the induced weak-topology on X and the operator norm topology of $\mathcal{M}$.*

1.L.Evaluation

(2) If $\mathcal{M}$ is in addition $\|\cdot\|_{2,X}$-separable, then the evaluation function of a not necessarily quantifier-free formula is Borel-measurable.

Proof. (1) If $\varphi(\bar{x}, \bar{y})$ is atomic, say $\|Q(\bar{x}, \bar{y})\|_2$ for a *-polynomial $Q(\bar{x}, \bar{y})$, then the map $\mathrm{Ev}_{\varphi,\bar{a}}$ is the composition of the continuous functions $\bar{y} \mapsto Q(\bar{a}, \bar{y})$ and $(\tau, z) \mapsto \|\pi_\tau(z)\|_{2,\tau}$.

Suppose that $\varphi(\bar{x}, \bar{y}) = f(\psi_1(\bar{x}, \bar{y}), \dots, \psi_n(\bar{x}, \bar{y}))$ for a continuous function f and ψ_j, for $j \leq n$, and that the evaluation function $\mathrm{Ev}_{\psi_j,\bar{a}}$ is continuous for each ψ_j. Then $\mathrm{Ev}_{\varphi,\bar{a}} = f(\mathrm{Ev}_{\psi_1,\bar{a}}, \dots, \mathrm{Ev}_{\psi_n,\bar{a}})$ is continuous, as a composition of continuous functions.

(2) This is proved by induction on the complexity of φ. The only non-trivial cases are when

$$\varphi(\bar{x}, \bar{y}) = \inf_{\|z\| \leq N} \psi(\bar{x}, \bar{y}, z) \text{ or } \varphi(\bar{x}, \bar{y}) = \sup_{\|z\| \leq N} \psi(\bar{x}, \bar{y}, z),$$

for some $N \in \mathbb{N}$. Fix a countable $\|\cdot\|_{2,X}$-dense subset $\{a_n\}_{n \in \mathbb{N}}$ in the N-ball of $\mathcal{M}$. If $\psi(\bar{x}, \bar{y}, z)$ is such that the evaluation function $\mathrm{Ev}_{\psi,\bar{a}}$ is Borel-measurable, then for all $\bar{b}$ in $\mathcal{M}$ of the appropriate sort, we have

$$\sup_{\|z\| \leq N} \psi^\tau(\pi_\tau(\bar{a}, \bar{b}), z) = \sup_{n \in \mathbb{N}} \psi^\tau(\pi_\tau(\bar{a}, \bar{b}, a_n)),$$

$$\inf_{\|z\| \leq N} \psi^\tau(\pi_\tau(\bar{a}, \bar{b}), z) = \inf_{n \in \mathbb{N}} \psi^\tau(\pi_\tau(\bar{a}, \bar{b}, a_n)).$$

Thus the evaluations of $\sup_{\|z\| \leq N} \psi(\bar{x}, \bar{y}, z)$ and $\inf_{\|z\| \leq N} \psi(\bar{x}, \bar{y}, z)$ are pointwise limits of sequences of Borel functions, and therefore Borel-measurable themselves. □

L.disintegration

Lemma 6.6. *Suppose that $(\mathcal{M}, X)$ is a factorial tracially complete C*-algebra such that $\mathcal{M}$ is $\|\cdot\|_{2,X}$-separable. Suppose that $n \geq 1$, $k \geq 1$, and Z is a Borel subset of $\partial_e X \times \mathcal{M}_k^n$ and let*

$$\tilde{Z} := \{(\sigma, \pi_\sigma(\bar{c})) : (\sigma, \bar{c}) \in Z\}.$$

Suppose in addition that $\tau \in X$ and that μ is the representing measure for τ. Then there exists

$$\bar{b} = \int_{\partial_e X}^{\oplus} \bar{b}_\sigma \, d\mu(\sigma) \in (\pi_\tau[\mathcal{M}]'')_k^n,$$

such that $(\sigma, \bar{b}_\sigma) \in \tilde{Z}$ for μ-almost all $\sigma \in \partial_e X$ for which the section $Z_\sigma := \{\bar{a} : (\sigma, \bar{a}) \in Z\}$ is nonempty

[9] See Definition 4.1 for the notation $\varphi^\tau(\pi_\tau(\bar{a}, \bar{b}))$.

Proof. Let $Y \subseteq \partial_e X$ be the set of all $\sigma \in \partial_e X$ for which Z_σ is nonempty, and let $\pi_1 \colon Z \to Y$ be the first coordinate projection. By [34, Theorem A.16], there is a μ-measurable cross-section $\sigma \mapsto (\sigma, f(\sigma))$ from Y into Z. In particular, the map $\sigma \mapsto f(\sigma)$ is a μ-measurable function from Y into $\mathcal{M}_k^n$, which we extend to $\partial_e X$ by setting $f(\sigma)$ to be the n-sequence constantly equal to 0, for all $\sigma \in \partial_e X \setminus Y$. By μ-measurability of f and the fact that the operators in its range are uniformly bounded in the operator norm, it follows that the function $\sigma \mapsto \bar{b}_\sigma := \pi_\sigma(f(\sigma))$ defines a decomposable operator

$$\bar{b} = \int_{\partial_e X}^{\oplus} \bar{b}_\sigma \, d\mu(\sigma) \in \left(\int_{\partial_e X}^{\oplus} \pi_\sigma[\mathcal{M}]'' \, d\mu(\sigma) \right)_k^n .$$

Finally, Proposition 6.4 implies that $\bar{b} \in (\pi_\tau[\mathcal{M}]'')_k^n$. □

We are finally ready to prove the main result of this section.

Proof of Theorem 6.2. Suppose that G is a discrete, countable group, $\mathsf{M} := (\mathcal{M}, X, \alpha)$ is a separable factorial G-tracially complete C*-algebra, the induced action of G on X is trivial and $\varphi(\bar{x})$ is an $\forall\exists$-convex $\mathcal{L}_{\|\cdot\|_2, G}$-formula. Clearly

$$\sup_{\sigma \in X} \varphi^\sigma(\pi_\sigma(\bar{a})) \geq \sup_{\sigma \in \partial_e X} \varphi^\sigma(\pi_\sigma(\bar{a}))$$

and it suffices to prove the reverse inequality.

Let $r \in \mathbb{R}$ be such that

$$\sup_{\tau \in \partial_e X} \varphi^\sigma(\pi_\sigma(\bar{a})) < r.$$

We need to prove that $\varphi^\tau(\pi_\tau(\bar{a})) < r$ for all $\tau \in X$.

Fix $\tau \in X$ and let μ be its representing measure supported on $\partial_e X$. Let us first assume that $\varphi(\bar{x}) = \|Q(\bar{x})\|_2^2$ for some *-polynomial $Q(\bar{x})$. We have

$$\varphi^\tau(\pi_\tau(\bar{a})) = \|Q(\bar{a})\|_{2,\tau}^2 = \int_{\partial_e X} \|Q(\bar{a})\|_{2,\sigma}^2 \, d\mu(\sigma) < \int_{\partial_e X} r \, d\mu(\sigma) = r.$$

Suppose now that $\varphi(\bar{x})$ is a quantifier-free convex formula, hence there are $m \geq 1$, some G-*-polynomials $Q_1(\bar{x}), \ldots, Q_m(\bar{x})$ and a convex function $g \colon \mathbb{R}^m \to \mathbb{R}$ such that

$$\varphi(\bar{x}) = g(\|Q_1(\bar{x})\|_2^2, \ldots, \|Q_m(\bar{x})\|_2^2).$$

Jensen's inequality allows us to settle this case as follows

$$\begin{aligned}
\varphi^\tau(\pi_\tau(\bar{a})) &= g\left(\int_{\partial_e X} \|Q_1(\bar{a})\|_{2,\sigma}^2 \, d\mu(\sigma), \ldots, \int_{\partial_e X} \|Q_m(\bar{a})\|_{2,\sigma}^2 \, d\mu(\sigma) \right) \\
&\leq \int_{\partial_e X} g(\|Q_1(\bar{a})\|_{2,\sigma}^2, \ldots, \|Q_m(\bar{a})\|_{2,\sigma}^2) \, d\mu(\sigma) \\
&= \int_{\partial_e X} \varphi^\sigma(\pi_\sigma(\bar{a})) \, d\mu(\sigma) < r.
\end{aligned}$$

Consider the case when $\varphi(\bar{x}) = \inf_{\|\bar{y}\| \leq 1} \psi(\bar{x}, \bar{y})$, for a quantifier-free, convex, formula ψ and an n-tuple of variables $\bar{y}$.

The function $(\bar{b}, \sigma) \mapsto \varphi^\sigma(\pi_\sigma(\bar{a}, \bar{b}))$ is Borel-measurable by Lemma 6.5. Therefore the preimage of $[0, r)$ under this function,

$$\{(\sigma, \bar{b}) \in X \times \mathcal{M}_k^m : \varphi^\sigma(\pi_\sigma(\bar{a}, \bar{b})) < r\},$$

is also Borel. By Lemma 6.6 there exists $\bar{b} = \int_{\partial_e X}^{\oplus} b_\sigma \, d\mu(\sigma) \in (\pi_\tau[\mathcal{M}]'')_1^n$ such that $\psi^\sigma(\pi_\sigma(\bar{a}), \bar{b}_\sigma) < r$ for a μ-co-null set of $\sigma \in \partial_e X$. Since $\psi(\bar{x}, \bar{y})$ is quantifier-free, by the quantifier-free case proved above we obtain $\psi^\tau(\pi_\tau(\bar{a}), \bar{b}) < r$.

Finally, suppose that $\varphi(\bar{x}) = \sup_{\|\bar{y}\| \leq 1} \inf_{\|\bar{z}\| \leq 1} \psi(\bar{x}, \bar{y}, \bar{z})$, for ψ quantifier-free and convex. For every tuple $\bar{b}$ of the appropriate sort in $\mathcal{M}_1$ we have that

$$\sup_{\sigma \in \partial_e X} \inf_{\|\bar{z}\| \leq 1} \psi^\sigma(\pi_\sigma(\bar{a}, \bar{b}, \bar{z})) < r.$$

It follows from the previous case that, for every tuple $\bar{b}$ of the appropriate sort $\sup_{\sigma \in X} \inf_{\|\bar{z}\| \leq 1} \psi^\sigma(\pi_\sigma(\bar{a}, \bar{b}, \bar{z})) < r$. By $\| \cdot \|_{2,\sigma}$-density of $\pi_\sigma[\mathcal{M}]$ in $\pi_\sigma[\mathcal{M}]''$, we can conclude that $\sup_{\sigma \in X} \varphi^\sigma(\pi_\sigma(\bar{a})) < r$, as desired. □

6.3. **Theories of fibres and limiting examples.** We conclude this section with a few limiting examples.

6.3.1. *Theories of fibres.* Following [16, §2.1.1, 3.1.1], we identify the theory of a tracial von Neumann algebra with a bounded linear functional on the real Banach algebra $\mathfrak{W}_{\mathcal{L}_{\|\cdot\|_2}}$ of $\mathcal{L}_{\|\cdot\|_2}$-sentences. This equips the set of all theories with the weak*-topology, known as the logic topology. The following is a consequence of the second part of Lemma 6.5.

C.Borel **Corollary 6.7.** *If A is a tracial, unital, separable C*-algebra and $\tau \in T(A)$, then the evaluation of the theory of $\pi_\tau[A]''$, as a function from $T(A)$ into $\mathfrak{W}^*_{\mathcal{L}_{\|\cdot\|_2}}$,*

$$\tau \mapsto \mathrm{Th}(\pi_\tau[A]'')$$

is Borel-measurable. □

The conclusion of Corollary 6.7 is optimal, as the following example shows that even the evaluation of $\exists$-formulas is not necessarily continuous (it is not difficult to show that it is lower semicontinuous).

We provide a C*-algebra A such that $T(A)$ is homeomorphic to $[0, 1]$ and an $\exists$-formula φ such that the evaluation of φ on $T(A)$ is not continuous. Consider the dimension drop algebra

$$A := \{f \colon [0, 1] \to M_2(\mathbb{C}) : f(0) \in \mathbb{C} \cdot 1_2\}.$$

Let $\varphi = \inf_{\|x\| \leq 1} \|x^* - x\|_2 + \|x^2 - x\|_2 + |1/2 - \|x\|_2|$. Then $\varphi^\tau = 0$ for every $\tau \in (0, 1]$, since in this case $\pi_\tau[A]'' \cong M_2(\mathbb{C})$ contains a projection of trace 1/2. On the other hand $\varphi^{\mathbb{C}} > 0$, since the contrary would imply the existence of a non-trivial projection (this is the case since in $\mathbb{C}$ the norm $\| \cdot \|_2$ coincides with the uniform norm).

ss.failure

6.3.2. *The failure of transfer.* An obvious example of a sentence that does not transfer from factorial tracial fibers $(\mathcal{M}_\sigma, \sigma)$ to the tracial von Neumann algebra $\mathcal{M} = \int^{\oplus} \mathcal{M}_\sigma \, d\mu(\sigma)$, with the (non-factorial) tracial state $\tau = \int^{\oplus} \sigma \, d\mu(\sigma)$ is the sentence that axiomatizes factorial tracial von Neumann algebras ([17, Proposition 3.4]). In the usual language of tracial von Neumann algebras (of which our $\mathcal{L}_{\|\cdot\|_2}$ from §**??** is a reduct, see §A.2), the sentence is (there is a typo in the definition of $\xi(a)$ in [17] where $\mathrm{tr}^2(a)$ should be replaced with $\mathrm{tr}(a)\overline{\mathrm{tr}(a)}$)[10]

eq:factor

$$\sup_{\|x\|\leq 1} \sup_{\|y\|\leq 1} \sqrt{\|x\|_2^2 - \mathrm{tr}(x)\overline{\mathrm{tr}(x)}} \mathbin{\dot{-}} \|[x,y]\|_2. \tag{6.5}$$

Here tr is interpreted as the tracial state. The class of tracial von Neumann algebras $(\mathcal{M}, \tau)$ is axiomatizable in $\mathcal{L}_{\|\cdot\|_2}$ as the class of those structures for which the predicate τ^+ from Definition 5.9 is linear (see Theorem A.2). Within the class of tracial von Neumann algebras, seen as $\mathcal{L}_{\|\cdot\|_2}$-structures, the predicate tr is definable. On the positive elements it can be recovered as $\mathrm{tr}(x^*x) = \|(x^*x)^{1/2}\|_2^2 = \tau^+(x^*x)$. Hence in this class the formula (6.5) can be rephrased as

$$\sup_{\|x\|\leq 1} \sup_{\|y\|\leq 1} \left(\|x^*x\|_2^2 - \|(x^*x)^{1/2}\|_2^4\right) \mathbin{\dot{-}} \|[x^*x,y]\|_2^2.$$

One reason why this sentence fails to transfer from the fibers to $\mathcal{M}$ is that its quantifier-free core is not convex. A more subtle reason is that the definition of tr does not respect the disintegration. In fact, when applied fiberwise in $\int^{\oplus} \mathcal{M}_\sigma \, d\mu(\sigma)$, it results in the center-valued trace (assuming $\mathcal{M}_\sigma$ is a factor for μ-a.e. σ; in general it will be 'diagonalizable operator-valued') instead of the scalar-valued τ.

7. Proof of Theorem B

CUT AND PASTE FROM SECTION 5.

7.1. **Main transfer results.** We prove that CPoU (§**??**) and the tracial transfer property (§5.2) are two sides of the same coin.

thm:MainThm.0

Theorem 7.1. *Suppose that $(\mathcal{M}, X)$ is a factorial tracially complete C^*-algebra. Then the following are equivalent.*

item1:mt *(1) $(\mathcal{M}, X)$ has the Tracial Transfer Property.*

item2:mt *(2) $(\mathcal{M}, X)$ has CPoU.*

If moreover $\mathcal{M}$ is $\|\cdot\|_{2,X}$-separable, the above conditions are equivalent to

item3:mt *(3) $(\mathcal{M}, X)$ has the Factorial Tracial Transfer Property.*

Theorem 7.1 is a special case of the following.

[10] The notation $x \dot{-} y$ is an abbreviation for $\max\{0, x-y\}$.

thm:MainThm **Theorem 7.2.** *Let G be a discrete countable group and let $(\mathcal{M}, X, \alpha)$ be a factorial G-tracially complete C*-algebra such that the action induced by α on X factors through a finite group action. Then the following are equivalent.*

item1:mt-alpha *(1) $(\mathcal{M}, X, \alpha)$ has the Tracial Transfer Property.*

item2:mt-alpha *(2) $(\mathcal{M}, X, \alpha)$ has α-CPoU.*

Moreover, if the action induced by α on X is trivial and $\mathcal{M}$ is $\|\cdot\|_{2,X}$-separable, the above conditions are equivalent to

item3:mt-alpha *(3) $(\mathcal{M}, X, \alpha)$ has the Factorial Tracial Transfer Property.*

Proofs of Theorem 7.1 and Theorem 7.2. The implication (2) $\Rightarrow$ (1) is proved in Section **??**, while (1) $\Rightarrow$ (2) is proved in Section **??**. Finally, in Section 6 we show that condition (3) is equivalent to conditions (1)–(2) when the action induced by α on X is trivial and $\mathcal{M}$ is $\|\cdot\|_{2,X}$-separable. $\square$

In case when the action induced by α on X is nontrivial, the Tracial Transfer Property may fail, but Corollary 7.3 below is a 'poor man's version' of the Tracial Transfer Property that suffices for some applications. In it we need to restrict ourselves to a certain subclass of max-formulas. In particular, we say that a qunatifier-free max-formula

$$\varphi(\bar{x}) = \max\{h_1(\|Q_1(\bar{x})\|_2^2), \dots, h_k(\|Q_k(\bar{x})\|_2^2)\},$$

is *subhomogeneous* if, on top of all conditions required in item 2 of Definition 5.1, we also have that $h_j(Cz) \leq Ch_j(z)$ for all $C \geq 0$, $z \in \mathbb{R}$ and $j \leq k$. *Subhomogeneous* $\exists$-max, $\forall\exists$-max and max-formulas are defined following the pattern of Definition 5.1.

cor:extremeTraces **Corollary 7.3.** *Let G be a discrete countable group and let $\mathsf{M} := (\mathcal{M}, X, \alpha)$ be a factorial G-tracially complete C*-algebra such that the action induced by α on X factors through a finite group action. Suppose that $(\mathcal{M}, X, \alpha)$ has α-CPoU. Then there is a constant $C \in \mathbb{N}$ such that for every subhomogeneous $\forall\exists$-max $\mathcal{L}_{\|\cdot\|_2,G}$-formula ψ, and any tuple $\bar{a}$ from $\mathcal{M}$*

$$\psi^{\mathsf{M}}(\bar{a}) \leq C \sup_{\tau \in X^\alpha} \psi^\tau(\pi_\tau(\bar{a})).$$

Proof. This follows from Theorem 7.2 and Proposition 2.5 proved below. $\square$

cor:Applications **Corollary 7.4.** *If A is a $\mathcal{Z}$-stable separable C*-algebra for which $T(A)$ is nonempty and compact, then $(\overline{A}^{T(A)}, T(A))$ has the Tracial Transfer Property.*

Proof. Use the fact that $\mathcal{Z}$-stability of A implies that A has CPoU (this was proved with the additional assumption of nuclearity in [8], and in general in [6]), in combination with Theorem 7.1. $\square$

For a type $\mathbf{t}(\bar{x})$ over a tracially complete C*-algebra $(\mathcal{M}, X)$ and $\tau \in X$ let $\mathbf{t}^\tau(\bar{x})$ be the type over $\pi_\tau[M]''$ obtained by replacing every parameter a with $\pi_\tau(a)$. The following is another consequence of Theorem 7.1 (see Definition 5.3).

cor.types

Corollary 7.5. *Suppose that* $\mathbf{t}(\bar{x})$ *is a countable quantifier-free max-type in* $\mathcal{L}_{\|\cdot\|_2}$. *If a factorial tracially complete C*-algebra* $\mathsf{M} := (\mathcal{M}, X)$ *has CPoU, then the following are equivalent.*

item:type1 *(1)* $\mathbf{t}$ *is approximately satisfiable in* M.

4.cor.types *(2)* $\mathbf{t}$ *is satisfied in some (every) ultrapower* $\mathsf{M}^{\mathcal{U}}$ *associated with a free ultrafilter* $\mathcal{U}$ *on* $\mathbb{N}$.

item:type3 *(3)* $\mathbf{t}^\tau$ *is approximately satisfiable in* $(\pi_\tau[\mathcal{M}]'', \tau)$ *for every* $\tau \in X$.

If moreover $\mathcal{M}$ *is* $\|\cdot\|_{2,X}$*-separable, the conditions above are equivalent to*

item:type4 *(4)* $\mathbf{t}^\tau$ *is approximately satisfiable in* $(\pi_\tau[\mathcal{M}]'', \tau)$ *for every* $\tau \in \partial_e X$.

Proof. Fix a finite number of quantifier-free max-formulas $\varphi_1(\bar{x}), \dots, \varphi_k(\bar{x})$ in $\mathbf{t}(\bar{x})$. Set

$$\psi := \inf_{\|\bar{x}\| \leq 1} \max\{\varphi_1(\bar{x}), \dots, \varphi_k(\bar{x})\}.$$

The formula ψ is an $\exists$-max formula, hence, by Theorem 7.1, if $\mathsf{M} = (\mathcal{M}, X)$ is a factorial tracially complete C*-algebra with CPoU, we have that

eq:TTP--

$$\psi^{\mathsf{M}} = \sup_{\tau \in X} \psi^\tau. \tag{7.1}$$

(1) is equivalent to saying that the left-hand side of the equation is zero, regardless of the finite set of formulas in $\mathbf{t}(\bar{x})$ that we started with. Similarly, (3) corresponds to the right-hand side being zero. The equality (7.1) thus grants the equivalence between (1) and (3). An analogous argument provides the equivalence between (1) and (4).

(2) is equivalent to (1) by Łoś's Theorem and countable saturation of $\mathsf{M}^{\mathcal{U}}$. $\square$

The statement of Corollary 7.5 feels incomplete. If a type $\mathbf{t}$ is satisfied in M by $\bar{a}$, then $\mathbf{t}^\tau$ is satisfied in $\pi_\tau[M]''$ by $\pi_\tau(a)$ for all $\tau \in X$. It is however not clear whether if each $\mathbf{t}^\tau$ is satisfied in $\pi_\tau[M]''$ then $\mathbf{t}$ is satisfied in M, assuming CPoU (otherwise there are easy counterexamples). One instance of this question is the case when $\mathbf{t}(x)$ consists of a single formula, $\max(\|x - x^*\|, \|u - \exp(ix)\|_2)$. If u is a unitary in M, then for every τ there is a such that $0 \leq a \leq 2\pi$ and $\pi_\tau(u) = \exp(ia)$. It is however not known, even assuming that M has CPoU, whether every unitary in M has a logarithm.

Corollary 7.5 can be generalized to the dynamical setting by an analogous proof, using Theorem 7.2.

Corollary 7.6. *Suppose that* $\mathbf{t}(x)$ *is a countable quantifier-free max-type in* $\mathcal{L}_{\|\cdot\|_{2,G}}$. *If a factorial G-tracially complete C*-algebra* $\mathsf{M} := (\mathcal{M}, X, \alpha)$ *has* α*-CPoU, then, setting* $\mathsf{M}^\alpha = (\mathcal{M}, X^\alpha, \alpha)$*, the following are equivalent.*

(1) $\mathbf{t}$ *is approximately satisfiable in* M^α.

(2) $\mathbf{t}$ *is satisfied in some (every) ultrapower* $(\mathsf{M}^\alpha)^{\mathcal{U}}$ *associated with a free ultrafilter* $\mathcal{U}$ *on* $\mathbb{N}$.

(3) $\mathbf{t}^\tau$ *is approximately satisfiable in* $(\pi_\tau[\mathcal{M}]'', \tau)$ *for every* $\tau \in X^\alpha$.

Moreover, if $X = X^\alpha$ is trivial and $\mathcal{M}$ is $\|\cdot\|_{2,X}$-separable, the above conditions are equivalent to

(4) $\mathbf{t}^\tau$ is approximately satisfiable in $(\pi_\tau[\mathcal{M}]'', \tau)$ for every $\tau \in \partial_e X$.

As promised in §**??**, we state and prove the (well-known) fact that the choice of the ultrafilter and (modulo mild restrictions) the choice of the $\|\cdot\|_2$-separable set S do not affect the existence of complemented partitions of unity.

T.CPoUs **Proposition 7.7.** *For any tracially complete C^*-algebra $\mathsf{M} = (\mathcal{M}, X)$, for every collection $a_1, \dots a_n \in \mathcal{M}_+$, every $\delta > 0$ such that*

$$\sup_{\tau \in X} \min\{\tau(a_1), \dots, \tau(a_n)\} < \delta,$$

and any free ultrafilter $\mathcal{U}$ on $\mathbb{N}$, the following are equivalent.

1.CPoU *(1) For every $F \Subset \mathcal{M}$ there are orthogonal projections $p_1, \dots, p_n$ in $\mathcal{M}^{\mathcal{U}}_X \cap F'$ such that*
1.a.CPoU *(a) $\sum_{i=1}^n p_i = 1$ and*
1.b.CPoU *(b) $\tau(p_i a_i) \le \delta\tau(p_i)$ for all $i \le n$ and $\tau \in X^{\mathcal{U}}$.*
1.5.CPoU *(2) For every $F \Subset \mathcal{M}$ and every $\varepsilon > 0$ there are positive contractions $e_1, \dots, e_n$ in $\mathcal{M}$ such that*
(a) $\sum_{i=1}^n e_i = 1$,
(b) $\max_{i \le n} \|e_i - e_i^2\|_2 < \varepsilon$.
(c) $\max_{b \in F, i \le n} \|[b, e_i]\|_2 < \varepsilon$,
1.5.CPoU.d *(d) $\tau(e_i a_i) \le \delta\tau(e_i) + \varepsilon$ for all $i \le n$ and $\tau \in X$.*
2.CPoU *(3) For every $\|\cdot\|_{2,X^{\mathcal{U}}}$-separable $S \subseteq \mathcal{M}^{\mathcal{U}}_X$ there are orthogonal projections $p_1, \dots, p_n$ in $\mathcal{M}^{\mathcal{U}}_X \cap S'$ that satisfy* (1a) *and* (1b).

The assertions (1) *and* (3) *are also equivalent to the analogous statements obtained by replacing $\mathcal{U}$ with another free ultrafilter on $\mathbb{N}$.*

Proof. Clearly (3) implies (1), and (2) implies (1) by countable saturation of $\mathcal{M}^{\mathcal{U}}_X$.

We prove that (1) implies (3). Fix a $\|\cdot\|_{2,X^{\mathcal{U}}}$-separable $S \subseteq \mathcal{M}^{\mathcal{U}}_X$ and, using the formula φ_p whose zero-set is the set of projections (see Example 5.4), and Definition 5.9 for τ^+, let

$$\psi(x_1, \dots, x_n) = \max\Big\{\max_{j \le n} \varphi_p(x_j), \max_{j \le n} \tau^+(x_j a_j - \delta x_j), \Big\|1 - \sum_{j \le n} x_j\Big\|_2^2\Big\}.$$

By Lemma 5.10, the predicate ψ is quantifier-free max-definable. Consider the type $\mathbf{t}_S(x_1, \dots, x_n)$ consisting of ψ together with the conditions $\|[x_i, b]\|_2 = 0$ where b ranges over a countable dense subset of S. We show that this type is approximately satisfiable in $\mathcal{M}^{\mathcal{U}}_X$. By the countable saturation of $\mathsf{M}^{\mathcal{U}} = (\mathcal{M}^{\mathcal{U}}_X, \Sigma^{\mathcal{U}} X)$ (see Theorem 4.5), this implies that $\mathbf{t}_S$ is realized in $\mathcal{M}^{\mathcal{U}}_X$, hence (3) follows.

Let $\mathbf{t}_0$ be a finite subset of $\mathbf{t}_S$ and let $b_1, \dots, b_m$ be the elements of S mentioned in the formulas of $\mathbf{t}_0$. Consider the following formulas in $\bar{y} =$

$(y_1, \dots, y_m)$ and $\bar{x}$

$$\eta_m(\bar{x}, \bar{y}) := \max\{\psi(\bar{x}), \max_{i \leq m, j \leq n} \|[y_i, x_j]\|_2^2\},$$
$$\theta_m(\bar{y}) := \inf_{\|\bar{x}\| \leq 1} \eta_m(\bar{x}, \bar{y}).$$

For every $\bar{a}$ in $\mathcal{M}$ of the same sort as $\bar{b} := (b_1, \dots, b_m)$, by (1) we have $\theta_m^{\mathsf{M}^{\mathcal{U}}}(\bar{a}) = 0$. By Łoś's Theorem, $\theta_m^{\mathsf{M}}(\bar{a}) = 0$. Since $\bar{a}$ was arbitrary, we have $(\sup_{\|\bar{y}\| \leq 1} \theta_m(y_1, \dots, y_m))^{\mathsf{M}} = 0$. Using Łoś again, this implies $(\sup_{\|\bar{y}\| \leq 1} \theta_m(y_1, \dots, y_m))^{\mathsf{M}^{\mathcal{U}}} = 0$, and thus $\theta_m(b_1, \dots, b_m)^{\mathsf{M}^{\mathcal{U}}} = 0$. Since $\bar{b}$ was arbitrary, this implies that $\mathbf{t}_0$ is approximately satisfiable in $\mathcal{M}_X^{\mathcal{U}}$ and therefore, by repeating this argument for every finite subset of $\mathbf{t}_S$, that $\mathbf{t}_S$ is approximately satisfiable in $\mathcal{M}_X^{\mathcal{U}}$.

To prove that (1) implies (2), first note that the proof that (1) implies (3) shows that each of these assertions is equivalent to having

$$(\sup_{\|\bar{y}\| \leq 1} \theta_m(y_1, \dots, y_m))^{\mathsf{M}} = (\sup_{\|\bar{y}\| \leq 1} \inf_{\|\bar{x}\| \leq 1} \eta_m(\bar{x}, \bar{y}))^{\mathsf{M}} = 0$$

for all $m \geq 1$.

It suffices to consider the case when $F \cup \{a_1, \dots, a_n\}$ is a subset of $\mathcal{M}_1$. Let ε and $F = \{c_1, \dots, c_m\}$ be as in (2). Let $\varepsilon' := \varepsilon/(4n)$. By (1) we have $(\inf_{\|\bar{x}\| \leq 1} \eta_m(\bar{x}, \bar{c}))^{\mathsf{M}} = 0$. Fix $\bar{e}' = (e_1', \dots, e_n')$ in $\mathcal{M}_1$ which satisfies $\eta_m(\bar{e}', \bar{c}) < \varepsilon'$. Then $e_i'' := (e_i' + (e_i')^*)/2$ are self-adjoint and satisfy $\|e_i'' - e_i'\|_2 < \varepsilon'$ and $\|e_i'' - (e_i'')^2\|_2 < 4\varepsilon'$. Let $e_i := (e_i'')_+$, for $i < n$, and $e_n := 1 - \sum_{i<n} e_i$. Easy computations show that for all i and j we have $\|e_i - e_i'\|_2 < \varepsilon$, $\|e_i - e_i^2\|_2 \leq \|e_i'' - (e_i'')^2\|_2 < \varepsilon$, (using $\|c_j\| \leq 1$) $\|[e_i, c_j]\|_2 < \varepsilon$, and (using $\|a_i\| \leq 1$) that $\tau^+(e_i a_i - e_i) < \varepsilon$, which is equivalent to (2d). Therefore $e_1, \dots, e_n$ are as required.

Since (2) depends only on the theory of M, replacing $\mathcal{U}$ with another free ultrafilter $\mathcal{V}$ in (1) or (3) results in equivalent statements. □

In model-theoretic terms, the proof that (1) implies (2) in Proposition 7.7 relies on the fact that the set of partitions of unity into n positive contractions is definable in the theory of tracially complete C*-algebras. Notably, the set of partitions of unity into n projections (even with $n = 2$) is not definable in the theory of tracially complete C*-algebras, but it is definable in the theory of tracially complete C*-algebras with CPoU ([18]).

The dynamic version of Proposition 7.7 and its proof are analogous, and therefore omitted.

FROM OLD DRAFT.

In this section we prove (1) $\Rightarrow$ (2) of Theorem 7.2, that the Tracial Transfer Property implies (α-)CPoU.

Proof of Theorem 7.2 (1) $\Rightarrow$ (2). Suppose that $\mathsf{M} = (\mathcal{M}, X, \alpha)$ as in the statement of the Theorem satisfies the Tracial Transfer Property. Let $a_1, \dots, a_n \in$

$\mathcal{M}_+$ and $\delta > 0$ be such that

$$\sup_{\tau \in X^\alpha} \min\{\tau(a_1), \dots \tau(a_n)\} < \delta.$$

By Łoś's Theorem and countable saturation of $\mathcal{M}_X^{\mathcal{U}}$ (for some free ultrafilter $\mathcal{U}$ on $\mathbb{N}$), it suffices to prove that for every finite $F \subset \mathcal{M}$, every finite $K \subseteq G$ and any $\epsilon > 0$, there exist $p_1, \dots, p_n$ in the unit ball of $\mathcal{M}$ such that for all $i \le n$, $b \in F$ and $g \in K$ (with the abbreviation τ^+ justified in Lemma 5.10):

eq:CPoUFormula0 (1) $\|[p_i, b]\|_{2,X} < \epsilon$,
(2) $\|p_i - p_i^*\|_{2,X} < \epsilon$,
(3) $\|p_i - p_i^2\|_{2,X} < \epsilon$,
(4) $\|p_1 + \dots + p_k - 1\|_{2,X} < \epsilon$,
(5) $\|\alpha_g(p_i) - p_i\|_{2,X} < \epsilon$,
eq:CPoUFormula2 (6) $\tau^+(p_i a_i - \delta p_i) < \epsilon$.

For $\ell = |F|$, We define a quantifier-free formula in $\bar{x} = (x_1, \dots, x_n)$, $\bar{y} = (y_1, \dots, y_\ell)$ and $\bar{z} = (z_1, \dots, z_n)$ as follows

$$\varphi(\bar{x}, \bar{y}, \bar{z}) := \max_{i \le n, j \le \ell, g \in K} \Big\{ \|[x_i, y_j]\|_2^2, \|x_i - x_i^*\|_2^2, \|x_i - x_i^2\|_2^2,$$
$$\|x_1 + \dots, + x_k - 1\|_2^2, \|\alpha_g(x_i) - x_i\|_2^2, \tau^+(x_i z_i - \delta x_i) \Big\}$$

and let $\psi(\bar{y}, \bar{z}) := \inf_{\|\bar{x}\| \le 1} \varphi(\bar{x}, \bar{y}, \bar{z})$. Then, with F enumerated as $\bar{b} = (b_1, \dots, b_\ell)$, we have $\psi(\bar{b}, \bar{a}) = 0$ if and only if for some n-tuple $\bar{p}$ the conditions (1)–(6) are satisfied.

Lemma 5.10 implies that φ is quantifier-free max-definable predicate, and therefore ψ is $\exists$-max definable predicate.

For each $\tau \in X^\alpha$, by hypothesis there is $k \le n$ such that $\tau(a_k) < \delta$. Thus setting $p_i^\tau = 1$ if $i = k$ and zero otherwise, we obtain

$$\varphi^\tau(p_1^\tau, \dots, p_n^\tau, \pi_\tau(\bar{b}, \bar{a})) = 0.$$

This shows that $\psi^\tau(\pi_\tau(\bar{b}, \bar{a})) \le 0$. Using the Tracial Transfer Property, it follows that $\psi^{\mathsf{M}^\alpha}(\bar{b}, \bar{a}) \le 0$. Since by Corollary 2.6 the norms $\|\cdot\|_{2,X}$ and $\|\cdot\|_{2,X^\alpha}$ are equivalent, this implies the existence of $p_1, \dots, p_n$ satisfying the conditions (1)–(6), as required. $\square$

This section is devoted to the proof of (2) $\Rightarrow$ (1) in Theorem 7.2, that (α-)CPoU imply the Tracial Transfer Property. We start by showing that each of CPoU and α-CPoU imply an apparently stronger technical variation of themselves. The conclusion of the following lemma reduces to the definition of CPoU in the case when $m = 1$.

Proof of Theorem 7.2, (2) $\Rightarrow$ (1). Assume that $\mathsf{M} := (\mathcal{M}, X, \alpha)$ is a factorial G-tracially complete C*-algebra with α-CPoU and that the action induced by α on X factors through a finite group action. We will prove that for every $\forall\exists$-max formula ψ in $\mathcal{L}_{\|\cdot\|_2,G}$ and any tuple $\bar{a}$ in $\mathcal{M}$ of the appropriate sort (writing $\mathsf{M}^\alpha := (\mathcal{M}, X^\alpha, \alpha)$) we have

eq:TPP2
$$\psi^{\mathsf{M}^\alpha}(\bar{a}) = \sup_{\tau \in X^\alpha} \psi^\tau(\pi_\tau(\bar{a})).^{11} \tag{7.2}$$

The formula $\psi(\bar{x})$ has the form

$$\psi(\bar{x}) = \sup_{\|\bar{z}\|\leq 1} \inf_{\|\bar{y}\|\leq 1} \max\{h_1(\|Q_1(\bar{x},\bar{y},\bar{z})\|_2^2), \dots, h_m(\|Q_m(\bar{x},\bar{y},\bar{z})\|_2^2)\}$$

for $m \geq 1$, G-*-polynomials (possibly, as in §4.4, involving continuous functional calculus) $Q_1(\bar{x},\bar{y},\bar{z}), \dots, Q_m(\bar{x},\bar{y},\bar{z})$, and convex, increasing, continuous functions $h_1, \dots, h_m \colon \mathbb{R} \to \mathbb{R}$.

We start by proving the inequality $\geq$ in (7.2). This inequality does not require CPoU. Using that the h_j's are increasing and continuous in the third line of the chain of equalities below, and thus commute with the sup, for tuples $\bar{a}, \bar{b}, \bar{c}$ of elements in $\mathcal{M}$ of the appropriate sort we have

eq:qfree
$$\begin{aligned} \varphi^{\mathsf{M}^\alpha}(\bar{a},\bar{b},\bar{c}) &= \max_{j\leq m} h_j(\|Q_1(\bar{a},\bar{b},\bar{c})\|_{2,X^\alpha}^2) \\ &= \max_{j\leq m} h_j(\sup_{\tau\in X^\alpha} \|Q_1(\pi_\tau(\bar{a},\bar{b},\bar{c}))\|_{2,\tau}^2) \\ &= \sup_{\tau\in X^\alpha} \max_{j\leq m} h_j(\|Q_j(\pi_\tau(\bar{a},\bar{b},\bar{c}))\|_{2,\tau}^2) \\ &= \sup_{\tau\in X^\alpha} \varphi^\tau(\pi_\tau(\bar{a},\bar{b},\bar{c})). \end{aligned} \tag{7.3}$$

This gives (7.2) for quantifier-free max-formulas. Moreover, this implies that for tuples $\bar{a}, \bar{c}$ of the appropriate sorts in $\mathcal{M}$ we have

$$\begin{aligned} \inf_{\|\bar{y}\|\leq 1} \varphi^{\mathsf{M}^\alpha}(\bar{a},\bar{y},\bar{c}) &= \inf_{\|\bar{y}\|\leq 1} \sup_{\tau\in X^\alpha} \varphi^\tau(\pi_\tau(\bar{a},\bar{y},\bar{c})) \\ &\geq \sup_{\tau\in X^\alpha} \inf_{\|\bar{y}\|\leq 1} \varphi^\tau(\pi_\tau(\bar{a},\bar{y},\bar{c})). \end{aligned}$$

Since $\bar{c}$ was arbitrary, this in turn implies $\psi^{\mathsf{M}^\alpha}(\bar{a}) \geq \sup_{\tau\in X^\alpha} \psi^\tau(\pi_\tau(\bar{a}))$. Note that the proof of the inequality $\geq$ in (7.2) did not require CPoU, (see Corollary 7.8 stated below).

To prove the that the converse inequality holds in (7.2), we will first consider the case when ψ is of the form

Eq.psi
$$\psi(\bar{x}) = \inf_{\|\bar{y}\|\leq 1} \varphi(\bar{x},\bar{y}), \tag{7.4}$$

where φ is a quantifier-free max-formula.

By replacing ψ with $\psi - r$ (the latter is still an $\exists$-max formula), where $r := \sup_{\tau\in X^\alpha} \psi^\tau(\pi_\tau(\bar{a}))$, we may assume that the right-hand side of (7.2) is equal to 0. Fix a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$ and form the ultrapower $(\mathsf{M}^\alpha)^\mathcal{U}$. By Łoś's Theorem, the value of φ in M^α is equal to its value in $(\mathsf{M}^\alpha)^\mathcal{U}$ and therefore it suffices to find, for any given $\epsilon > 0$, a tuple $\bar{b}$ of contractions in

[11] Remember that ψ^τ is an abbreviation for $\psi^{(\pi_\tau[A]'',\tau,\alpha^\tau)}$.

$\mathcal{M}^{\mathcal{U}}_{X^\alpha}$ (which is by Corollary 2.6 equal to $\mathcal{M}^{\mathcal{U}}_X$) of the appropriate sort and such that

$$\varphi^{(\mathsf{M}^\alpha)^{\mathcal{U}}}(\bar{a},\bar{b})<\epsilon.$$

In order to verify this condition for some $\bar{b}$, by (7.3) applied to $(\mathsf{M}^\alpha)^{\mathcal{U}}$ it is enough to show that

eq:TracialTransfer1

$$\varphi^\tau(\pi_\tau(\bar{a},\bar{b}))<\epsilon,\quad \forall\tau\in(X^\alpha)^{\mathcal{U}}. \tag{7.5}$$

Let $\epsilon>0$ be given. By $\sup_{\tau\in X^\alpha}\psi^\tau(\pi_\tau(\bar{a}))=0$, for each $\tau\in X^\alpha$, there exists a tuple $\bar{b}'_\tau$ in the unit ball of $\pi_\tau[\mathcal{M}]''$ such that $\varphi^\tau(\pi_\tau(\bar{a}),\bar{b}'_\tau)<\epsilon$. By Kaplansky's density theorem, we can find a tuple $\bar{b}_\tau$ in the unit ball of $\mathcal{M}$ such that $\varphi^\tau(\pi_\tau(\bar{a},\bar{b}_\tau))<\epsilon$.

By the joint continuity of the evaluation function for quantifier-free formulas (Lemma 6.5), it follows that there is an open neighborhood U_τ of τ in X^α such that for $\sigma\in U_\tau$,

eq:TracialTransfer2

$$\varphi^\sigma(\pi_\sigma(\bar{a},\bar{b}_\tau))<\epsilon. \tag{7.6}$$

The open neighborhoods U_τ can be assumed to be of the form

$$U_\tau=\{\sigma:\sigma(s^\tau_j)<1\text{ for }j=1,\dots,N_\tau\}$$

for some $s^\tau_1,\dots,s^\tau_{N_\tau}\in\mathcal{M}_+$. To see this, let V_τ be the open set $\{\sigma:\varphi^\sigma(\pi_\sigma(\bar{a},\bar{b}_\tau))<\epsilon\}$. Recall that $\varphi(\bar{x},\bar{y})$ has the form

$$\max_{j\le m}h_j(\|Q_j(\bar{x},\bar{y})\|_2^2)$$

for some increasing convex functions $h_j\colon\mathbb{R}\to\mathbb{R}$ and some G-*-polynomials $Q_j(\bar{x},\bar{y})$ for $j\le m$. In particular, given $\tau\in X^\alpha$, as the h_j's are increasing and continuous, there exists $\gamma>0$ such that if σ satisfies

$$\max_{j\le m}\|Q_j(\bar{a},\bar{b}_\tau)\|^2_{2,\sigma}-\|Q_j(\bar{a},\bar{b}_\tau)\|^2_{2,\tau}<\gamma,$$

then $\sigma\in V_\tau$. If we set $\tilde{s}^\tau_j=Q_j(\bar{a},\bar{b}_\tau)Q_j(\bar{a},\bar{b}_\tau)^*$, we have that if

$$\sigma(\tilde{s}^\tau_j)<\|Q_j(\bar{a},\bar{b}_\tau)\|^2_{2,\tau}+\gamma,\ \forall j\le m,$$

then $\sigma\in V_\tau$. Multiplying $\tilde{s}^\tau_j$ by the inverse of the scalar on the right-hand side of the inequality above, gives the desired s^τ_j's and U_τ.

By compactness of X^α, there exists a finite subcover $U_{\tau_1},\dots,U_{\tau_k}$ of X^α. By adding dummy variables if necessary, we may assume that $N_{\tau_i}=N$ for all i. We also obtain, by finiteness, a $\delta<1$ such that for every $\tau\in X^\alpha$ some i satisfies

$$\max_{j\le N}\tau(s^{\tau_i}_j)<\delta.$$

By Lemma 3.2 and the α-CPoU, there exist pairwise orthogonal projections $p_1,\dots,p_k$ in $\mathcal{M}^{\mathcal{U}}_X\cap\mathcal{M}'\cap\{\bar{b}_{\tau_1},\dots,\bar{b}_{\tau_k}\}'$ such that

(a) $\sum_{i=1}^k p_i=1$,
(b) $\alpha^{\mathcal{U}}(p_i)=p_i$ for all $i\le k$,
(c) $\tau(p_is^{\tau_i}_j)\le\delta\tau(p_i)$ for all $i\le k$, $j\le r$, and τ in $(X^\alpha)^{\mathcal{U}}$.

Set

$$\bar{b} := \sum_{i=1}^{k} p_i \bar{b}_{\tau_i}.$$

In order to prove (7.5), fix a tracial state $\tau \in (X^\alpha)^{\mathcal{U}}$. For every $i \leq k$ such that $\tau(p_i) \neq 0$, set $\sigma_i := \frac{1}{\tau(p_i)}\tau(p_i \cdot)$. If $\tau(p_i) = 0$, simply put $\sigma_i = \tau$. Each σ_i induces a tracial state on $\mathcal{M}$ since p_i commutes with all elements in $\mathcal{M}$. Since X is a closed face, $\tau \in (X^\alpha)^{\mathcal{U}}$ and the p_i's are $\alpha_{\mathcal{U}}$-invariant, in each case σ_i is a tracial state in X^α.[12] Notice that, for every G-*-polynomial $Q(\bar{x}, \bar{y})$, using the properties of the p_i's we have

$$\begin{aligned}
\|Q(\bar{a}, \bar{b})\|_{2,\tau}^2 &= \tau(Q(\bar{a}, \bar{b})Q(\bar{a}, \bar{b})^*) \\
&= \sum_{i=1}^{k} \tau(p_i Q(\bar{a}, \bar{b}_{\tau_i})Q(\bar{a}, \bar{b}_{\tau_i})^*) \\
&= \sum_{i=1}^{k} \tau(p_i)\|Q(\bar{a}, \bar{b}_{\tau_i})\|_{2,\sigma_i}^2.
\end{aligned}$$

As $\sum_{i=1}^{k} \tau(p_i) = 1$ and φ as in (7.4) is convex, we get that

eq:TracialTransfer3

$$\varphi^\tau(\pi_\tau(\bar{a}, \bar{b})) \leq \sum_{i=1}^{k} \tau(p_i)\varphi^{\sigma_i}(\pi_{\sigma_i}(\bar{a}, \bar{b}_{\tau_i})). \tag{7.7}$$

Also, in the case when $\tau(p_i) \neq 0$, since $\tau(p_i s_j^{\tau_i}) \leq \delta\tau(p_i) < \tau(p_i)$, we have that $\sigma_i(s_j^{\tau_i}) < 1$, so that $\sigma_i \in U_{\tau_i}$. It follows by (7.6) that for all $i \leq k$ we have

$$\varphi^{\sigma_i}(\pi_{\sigma_i}(\bar{a}, \bar{b}_{\tau_i})) \leq \epsilon.$$

Combining this with (7.7), we conclude

$$\varphi^\tau(\pi_\tau(\bar{a}, \bar{b})) \leq \sum_{i=1} \tau(p_i)\epsilon = \epsilon,$$

as required.

It remains to consider the general case, when

$$\psi(\bar{x}) = \sup_{\|\bar{z}\|\leq 1} \inf_{\|\bar{y}\|\leq 1} \varphi(\bar{x}, \bar{y}, \bar{z}).$$

For any tuple of parameters $\bar{c}$ of the same sort as $\bar{z}$, the formula $\inf_{\|\bar{y}\|\leq 1} \varphi(\bar{x}, \bar{y}, \bar{c})$ is of the form handled in (7.4) and therefore satisfies the converse inequality in (7.2). Since $\bar{c}$ is arbitrary, this completes the proof. $\square$

The proof of Theorem 7.2 yields the following.

[12] If $\tau = (\tau_n)_{n\in\mathbb{N}}$ and $p_i = (p_i^n)_{n\in\mathbb{N}}$, then $\sigma_i = \lim_{n\to\mathcal{U}} \frac{1}{\tau_n(p_i^n)}\tau_n(p_i^n \cdot)$, hence σ_i is a limit of elements which belong to X, since the latter is a face.

L.TTP

Corollary 7.8. *Let $\mathsf{M} = (\mathcal{M}, X, \alpha)$ be a factorial G-tracially complete C*-algebra. Then M has α-CPoU if and only if for every $\exists$-max $\mathcal{L}_{\|\cdot\|_2,G}$-formula $\psi(\bar{x})$ and every tuple $\bar{a}$ of elements of $\mathcal{M}$ of the appropriate sort we have*

$$\psi^{\mathsf{M}^\alpha}(\bar{a}) \leq \sup_{\tau \in X^\alpha} \psi^\tau(\pi_\tau(\bar{a})),$$

where M^α is as in Definition 5.6.

Proof. The reason why it is suffices to state this for $\exists$-formulas instead of $\forall\exists$-formulas can be recovered from the proof of the implication (1) $\Rightarrow$ (2) in Theorem 7.2. In a nutshell, instead of proving $(\sup_{\|\bar{x}\|\leq 1} \inf_{\|\bar{y}\|\leq 1} \varphi(\bar{x}, \bar{y}))^{\mathsf{M}} = 0$, it suffices to prove $(\inf_{\|\bar{y}\|\leq 1} \varphi(\bar{a}, \bar{y}))^{\mathsf{M}} = 0$ for every tuple of parameters $\bar{a}$ in $\mathcal{M}$ of the appropriate sort. In particular, the formula $\psi(\bar{y}, \bar{z})$ defined in the proof of Theorem 7.2, ensuring the existence of CPoU, is an $\exists$-max formula. $\square$

S.applications S.easy

8. Applications

8.1. **Easy applications.** We start by listing the most obvious (and already known) applications of Theorem, 7.1.

C.uct.unitary

Corollary 8.1. *In every factorial tracially complete C*-algebra $(\mathcal{M}, X)$ with CPoU, the unitaries of the form $\exp(ia)$ for $0 \leq a \leq 2\pi$, are $\|\cdot\|_{2,X}$-dense in the unitary group.*

Proof. In every von Neumann algebra, by the Borel functional calculus every unitary is of the form $\exp(ia)$ for some $0 \leq a \leq 2\pi$. The definable predicate (see Example **??**) $\varphi(x) := \inf_{\|z\|\leq 1} \|x - \exp(2\pi i z^* z)\|_2^2$ is $\exists$-max.

Fix a unitary u in $\mathcal{M}$. Then $\pi_\sigma(u)$ is a unitary for every $\sigma \in X$, and therefore $\varphi^\sigma(\pi_\sigma(u)) = 0$. By Theorem 7.1, $\mathcal{M}$ has the Tracial Transfer Property, and therefore $\varphi^{(\mathcal{M},X)}(u) = 0$, which implies that u can be approximated arbitrarily well in the $\|\cdot\|_{2,X}$-norm by $\exp(iz)$ for some $0 \leq z \leq 2\pi$. Since u was arbitrary, this concludes the proof. $\square$

It is not known whether the conclusion of Corollary 8.1 can be improved to the assertion that in a tracially closed C*-algebra with CPoU every unitary has a logarithm, or even that the unitary group of $(\mathcal{M}, X)$ is path-connected. We however have the following.

.uct.unitary.saturated

Corollary 8.2. *If $(\mathcal{M}, X)$ is an ultraproduct or a reduced product associated to the Fréchet filter of factorial tracially complete C*-algebras with CPoU, or a relative commutant of a separable C*-subalgebra of such ultraproduct or reduced product, then every unitary in $(\mathcal{M}, X)$ is of the form $\exp(ia)$ for $0 \leq a \leq 2\pi$.*

Proof. If $\mathsf{M} = (\mathcal{M}, X)$ is an ultraproduct (or a reduced product) of factorial tracially complete C*-algebras with CPoU, then it is itself is factorial, tracially complete and has CPoU (see [6] or [18]). If $u \in \mathcal{M}$ is a unitary and $\varphi(x) := \inf_{\|z\|\leq 1} \|x - \exp(2\pi i z^* z)\|_2^2$ is the formula from Corollary 8.1, the proof of this corollary shows that $\varphi^{\mathsf{M}}(u) = 0$. As a consequence, the type composed by a unique condition $\|u - \exp(2\pi i z^* z)\|_2^2 = 0$ (with the understanding that z ranges over the unit ball), is approximately satisfiable in $(\mathcal{M}, X)$. By Theorem 4.5, $(\mathcal{M}, X)$ is countably quantifier-free saturated, hence there is $b \in \mathcal{M}$ such that, setting $a = 2\pi b^* b$, we have $u = \exp(ia)$, as desired.

If $(\mathcal{M}, X)$ is a commutant of a separable subalgebra S of an ultraproduct (or a reduced product) of factorial tracially complete C*-algebras with CPoU, then one obtains the conclusion by considering the type determined by the set of formulas (the point is that $(\mathcal{M}, X)$ is countably quantifier-free saturated, see [14, Corollary 16.5.4])

$$\{\|u - \exp(2\pi i z^* z)\|_2^2, \|[z, b]\|_2^2 : b \in D\},$$

where D is a countable dense subset of S ($(\mathcal{M}, X)$ is still countably saturated by Theorem 4.5). ☐

In the proof of Corollary 8.2 the ultraproduct or reduced product of M can be replaced by any countably quantifier-free saturated model of the theory of factorial tracially closed C*-algebras with CPoU, as well as with the relative commutant of any of its separable C*-subalgebras. This includes many reduced powers associated with free filters on $\mathbb{N}$, but not all of them (e.g., a reduced product of C*-algebras with respect to the asymptotic density zero filter is not countably saturated, see [14, p. 420]).

The conclusion of the following is an approximate comparison of projections.

C.comparison

Corollary 8.3. *Suppose that $(\mathcal{M}, X)$ is a factorial tracially complete C*-algebra with CPoU. If p and q are projections in $\mathcal{M}$ such that $\sigma(p) \leq \sigma(q)$ for every $\sigma \in \partial_e X$, then for every $\epsilon > 0$ there exists $v \in \mathcal{M}$ such that $\|p - v^* v p\|_{2,X} < \epsilon$ and $\|q - vv^*\|_{2,X} < \epsilon$.*

Proof. The formula $\psi(y, z) := \inf_{\|x\| \leq 1} \max\{\|y - x^* x y\|_2^2, \|z - x x^*\|_2^2\}$ is $\exists$-max. If p and q are projections as in the statement of this corollary, then since finite von Neumann algebras satisfy the comparison of projections we have $\psi^\sigma(\pi_\sigma(p, q)) = 0$ for all $\sigma \in X$. By Theorem 7.1 $(\mathcal{M}, X)$ satisfies the Tracial Transfer Property, and the conclusion follows. ☐

As in the case of Corollary 8.2, more can be said in the presence of saturation.

cor:MvN

Corollary 8.4. *Suppose that $(\mathcal{M}, X)$ is an ultraproduct or a reduced product associated to the Fréchet filter of factorial tracially complete C*-algebras with CPoU, or a relative commutant of a separable C*-subalgebra of such ultraproduct or reduced product. If p and q are projections in $\mathcal{M}$ such that $\sigma(p) \leq \sigma(q)$ for every $\sigma \in \partial_e X$, then there exists a partial isometry v in $\mathcal{M}$ such that $p \leq v^* v$ and $q = vv^*$.*

Proof. As in the proof of Corollary 8.2, $(\mathcal{M}, X)$ is countably quantifier-free saturated. Fix p and q as in the assumption and consider the type with conditions $\|p - x^*xp\|_{2,X} = 0$ and $\|q - xx^*\|_{2,X} = 0$ (with the understanding that the variable x ranges over the unit ball). This quantifier-free type is approximately realizable in $(\mathcal{M}, X)$ by Corollary 8.3, and therefore realized by some v. Clearly v is as required. $\square$

A more involved argument allows one to refine Corollary 8.3 and obtain the conclusion of Corollary 8.4, that there is $v \in \mathcal{M}$ such that $p \leq v^*v$ and $q = vv^*$, in every factorial tracially complete C*-algebra with CPoU, saturated or not (see [6]).

8.2. **Uniform Property Γ.** A II_1 factor $\mathcal{N}$ is said to have *property* Γ when its central sequence algebra $\mathcal{N}^{\mathcal{U}} \cap \mathcal{N}'$ is non-trivial. Dixmier proved that Property Γ is equivalent to $\mathcal{N}^{\mathcal{U}} \cap \mathcal{N}'$ being *diffuse* ([11]), a property which asserts, for all $n \in \mathbb{N}$, the existence of projections $p_1, \dots, p_n \in \mathcal{N}^{\mathcal{U}} \cap \mathcal{N}'$ such that $\sum_{j \leq n} p_j = 1$ and $\tau_{\mathcal{N}^{\mathcal{U}}}(p_i) = 1/n$ for all $j \leq n$. This property was generalized to tracial ultrapowers of C*-algebras in [8, Definition 2.1], and the following is the obvious analog of the definition in case of tracially complete C*-algebras.

def:gamma **Definition 8.5** (Uniform Property Γ)**.** Let $(\mathcal{M}, X)$ be a tracially complete C*-algebra. We say that $(\mathcal{M}, X)$ has *uniform property* Γ if for every $n \in \mathbb{N}$ and every free ultrafilter $\mathcal{U}$ on $\mathbb{N}$ there are projections $p_1, \dots p_n \in \mathcal{M}^{\mathcal{U}}_X \cap \mathcal{M}'$ such that

(1) $\sum_{i=1}^n p_i = 1$,
(2) $\tau(ap_i) = \frac{1}{n}\tau(a)$ for all $a \in \mathcal{M}$, $\tau \in \prod^{\mathcal{U}} X$ and $i \leq n$.

For a (not necessarily tracially complete) C*-algebra A, we say that A has uniform property Γ if $(\overline{A}^{T(A)}, T(A))$ has uniform property Γ. This latter definition is equivalent to [8, Definition 2.1].

A standard argument shows that the definition does not depend on the choice of $\mathcal{U}$ (see Corollary 8.9).

It is immediate to see that, for II_1 factors, the notion of uniform property Γ in Definition 8.5, and the von Neumann algebraic property Γ from [11], are the same. We will implicitly take advantage of this fact later in Corollary 8.8.

A theory is said to be ∀∃-*max* if all of its sentences are. (Needless to say, one can analogously consider theories with any of the properties defined in §5.1, but we will need only this particular variation.) The definition of uniform property Γ apparently requires quantification over projections. Since the set of projections is not definable in tracially complete C*-algebras, or even in factorial tracially complete C*-algebras ([18]), the following result may seem somewhat surprising. The trick lies in the fact that the projections are required to exist in the ultrapower of the C*-algebra in question.

T.Gamma **Theorem 8.6.** *Uniform property* Γ *is* $\forall\exists$*-max axiomatizable in the theory of tracially complete* C^**-algebras.*

Prior to proving this theorem we consider some of its corollaries. Since every $\forall\exists$-axiomatizable class is closed under inductive limits (this is the easy direction of [16, Proposition 2.4.4 (3)]), Theorem 8.6 implies the following.

C.Gamma **Corollary 8.7.** *Uniform property* Γ *is preserved in inductive limits of tracially complete* C^**-algebras.*

The Tracial Transfer Property also applies to uniform property Γ, the latter being $\forall\exists$-axiomatizable. In particular we have the following application of Theorem 7.1 where, as in Definition **??**, we say that a C*-algebra A with a nonempty and compact simplex of tracial states has CPoU if the factorial tracially complete C*-algebra $(\overline{A}^{T(A)}, T(A))$ has CPoU.

theorem:Gamma **Corollary 8.8.** *Suppose that* A *is a separable* C^**-algebra such that* $T(A)$ *is nonempty and compact, and with CPoU. Then* A *has uniform property* Γ *if and only if* $\pi_\tau[A]''$ *has property* Γ *for every extremal tracial state* τ.

Proof of Theorem 8.6. Using the predicate τ^+ (Definition 5.9), let

$$\tau^\dagger(a) := \max\{\tau^+(a), 0\}.$$

Since τ^+ is quantifier-free max-definable (Lemma 5.10), so is $\tau^\dagger$. In addition, $\tau^\dagger$ is positive and for self-adjoint a and b in any tracially complete C*-algebra $(\mathcal{M}, X)$ we have $\tau^\dagger(a-b) = 0$ if and only if $\tau(a) \leq \tau(b)$ for all $\tau \in X$.

Fix $m \geq 1$ and $n \geq 2$. Let $\psi_{m,n}(\bar{x}, \bar{y})$ denote the following formula in an m-tuple of variables $\bar{x}$ and an n-tuple of variables $\bar{y}$ (for the quantifier-free max formula $\varphi_p(y_j)$, whose zero set in a given algebra is the set of its projections, see Example 5.4(2))

$$\max\Big\{ \max_{j\leq n} \varphi_p(y_j), \|1 - \sum_{j\leq n} y_j\|_2^2, \max_{i\leq m, j\leq n} \{\|[x_i, y_j]\|_2^2,$$
$$\tau^\dagger(y_j x_i^* x_i - (1/n)x_i^* x_i)), \tau^\dagger((1/n)x_i^* x_i - y_j x_i^* x_i)\}\Big\}$$

and let

$$\varphi_{\Gamma,m,n} := \sup_{\|\bar{x}\|\leq 1} \inf_{\|\bar{y}\|\leq 1} \psi_{m,n}(\bar{x}, \bar{y}).$$

Fix a tracially complete C*-algebra $\mathsf{M} = (\mathcal{M}, X)$. We claim that M has uniform property Γ if and only if $\varphi^{\mathsf{M}}_{\Gamma,m,n} = 0$ for all m and n.

Before proceeding to prove this, note that $\psi^{\mathsf{M}}(\bar{b}, \bar{p}) = 0$ if and only if, with $a_j := b_j^* b_j$, we have that $\bar{p} = (p_1, \dots, p_n)$ are projections in the relative commutant of $\bar{a}$ such that $\sum_{j\leq n} p_j = 1$ and $\tau(a_i p_j) = \frac{1}{n}\tau(a_i)$ for all $i \leq m$ and all $j \leq n$.

For the converse implication assume $\varphi^{\mathsf{M}}_{\Gamma,m,n} = 0$ for all m and n and consider an ultrapower $\mathsf{M}^{\mathcal{U}}$. By Łoś's Theorem we also have $\varphi^{\mathsf{M}^{\mathcal{U}}}_{\Gamma,m,n} = 0$. For each m let $\bar{a}(m,j)$, for $j \in \mathbb{N}$, enumerate a dense set of m-tuples in $\mathcal{M}^m$.

For a fixed n, consider the type $\mathbf{t}(\bar{y})$ consisting of all conditions of the form $\psi_{m,n}(\bar{a}(m,j),\bar{y})=0$. By assumption $\mathbf{t}(\bar{y})$ is consistent and by saturation it is realized in $\mathsf{M}^{\mathcal{U}}$. Its realization is an n-tuple of projections as required.

Conversely, suppose that M has uniform property Γ and fix an m-tuple $\bar{a}$ in the unit ball of $\mathcal{M}$ and an n-tuple of projections $\bar{p}$ in $\mathcal{M}^{\mathcal{U}}\cap\mathcal{M}'$ as in the definition of uniform property Γ. Then $\psi_{m,n}^{\mathsf{M}^{\mathcal{U}}}(\bar{a},\bar{p})=0$. By Łoś's Theorem, $\inf_{\|\bar{y}\|\leq 1}\psi_{m,n}^{\mathsf{M}}(\bar{a},\bar{y})=0$. Since $\bar{a}$ was an arbitrary m-tuple in the unit ball, we conclude that $\varphi_{\Gamma,m,n}^{\mathsf{M}}=0$. □

Arguing like we did in Proposition 7.7 for CPoU, the proof of Theorem 8.6 can be used to infer a well-known fact, that if the definition of uniform Property Γ is modified by requiring the projections p_j to belong to the relative commutant of an arbitrary separable subset of the ultrapower, the resulting property is equivalent to uniform property Γ itself.

cor:gammaU **Corollary 8.9.** *A tracially complete C*-algebra $(\mathcal{M},X)$ has uniform property Γ if and only if for every $n\in\mathbb{N}$, every free ultrafilter $\mathcal{U}$ on $\mathbb{N}$ and every $\|\cdot\|_{2,X}$-separable $S\subseteq\mathcal{M}_X^{\mathcal{U}}$, there are projections $p_1,\dots p_n\in\mathcal{M}_X^{\mathcal{U}}\cap S'$ such that*

(1) $\sum_{i=1}^n p_i=1$,
(2) $\tau(ap_i)=\frac{1}{n}\tau(a)$ *for all* $a\in S$, $\tau\in\prod^{\mathcal{U}}X$ *and* $i\leq n$.

The fact that uniform property Γ is axiomatizable by an $\forall\exists$-max predicate has the following useful consequence.

Corollary 8.10. *Uniform property Γ is preserved under reduced products of tracially complete C*-algebras.*

Proof. Every max-formula belongs to the class of *conditional formulas* (see [22, Definition 3.5]; take $n=0$), which are preserved by reduced products (see [22, Theorem 3.9] and [27]). □

8.3. **Projectionization.** We will need the following well-known consequence of [10, §2 and Corollary 6.4] (stated explicitly in [7, Proposition 1.2] but without the lower bound $-\epsilon$, and in [26, Theorem 9.3], under the unnecessary simplicity assumption on the C*-algebra in question).

theorem:CuntzPedersen **Theorem 8.11** (Cuntz–Pedersen)**.** *If A is a C*-algebra such that $T(A)$ is nonempty and compact, then every weak*-continuous, affine, real-valued function f on $T(A)$ can be represented as the evaluation at a self-adjoint element a, in the sense that $f(\tau)=\tau(a)$ for all $\tau\in T(A)$. In addition, if $f\geq 0$ then we can assure that $-\epsilon\leq a\leq\|f\|_\infty+\epsilon$ for any prescribed $\epsilon>0$.* □

An argument using countable saturation can be used to show that in an ultraproduct a as in the conclusion of Theorem8.11 can be chosen so that $0\leq a\leq\|f\|_\infty$. In the following we go a step further.

prop:projection **Proposition 8.12.** *Suppose that A is a C^*-algebra such that $T(A)$ is nonempty and compact, A has CPoU, that $f\colon T(A) \to [0,1]$ is weak*-continuous and affine, and that $\mathcal{U}$ is a free ultrafilter on $\mathbb{N}$. Then the following are equivalent.*

1.lemma:projection-1 *(1) There exists a projection $p \in A^{\mathcal{U}}$ such that $f(\tau) = \tau(p)$ for all $\tau \in T(A)$.*

2.lemma:projection-1 *(2) Every extremal tracial state τ such that $\pi_\tau[A]''$ is a type I_n factor for some n satisfies $nf(\tau) \in \mathbb{N}$.*

*Moreover, if $\pi_\tau[A]''$ is McDuff for all $\tau \in \partial_e T(A)$ (in this case (**??**) is vacuous), then p as in (**??**) can be chosen in $A^{\mathcal{U}} \cap S'$ for every separable $S \subseteq A^{\mathcal{U}}$.*

Proof. Assume that (1) holds. Fix $\tau \in \partial_e T(A)$ and let $(\pi_\tau, H_\tau, \eta_\tau)$ the corresponding GNS representation. Let $H_{\mathcal{U}}$ be the metric ultrapower of H_τ, that is the set of all bounded sequences in $\prod_{n\in\mathbb{N}} H_\tau$ modulo the ideal

$$c_{0,H_\tau} := \Big\{(\xi_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} H_\tau : \lim_{n\to\mathcal{U}} \|\xi_n\| = 0\Big\}.$$

It follows that $H_{\mathcal{U}}$ is a Hilbert space with scalar product, given two vectors $(\xi_n)_{n\in\mathbb{N}}$,$(\eta_n)_{n\in\mathbb{N}} \in H_{\mathcal{U}}$, defined as

$$\langle(\xi_n)_{n\in\mathbb{N}}, (\eta_n)_{n\in\mathbb{N}}\rangle_{\mathcal{U}} = \lim_{n\to\mathcal{U}} \langle\xi_n, \eta_n\rangle.$$

Let $\pi_\tau^{\mathcal{U}} : A^{\mathcal{U}} \to B(H_{\mathcal{U}})$ be the representation mapping each $(a_n)_{n\in\mathbb{N}}$ in $A^{\mathcal{U}}$ to the operator sending a given vector $(\xi_n)_{n\in\mathbb{N}} \in H_{\mathcal{U}}$ to the vector $(\pi_\tau(a_n)\xi_n)_{n\in\mathbb{N}}$. It is immediate to see that the constant sequence $\bar\eta := (\eta_\tau)_{n\in\mathbb{N}}$ is a cyclic vector for $\pi_\tau^{\mathcal{U}}$, and that

$$\langle\pi_\tau^{\mathcal{U}}((a_n)_{n\in\mathbb{N}})\bar\eta, \bar\eta\rangle_{\mathcal{U}} = \lim_{n\to\mathcal{U}} \langle\pi_\tau(a_n)\eta_\tau, \eta_\tau\rangle = \lim_{n\to\mathcal{U}} \tau(a_n).$$

By uniqueness of the GNS representation, it follows that $\pi_\tau^{\mathcal{U}}$ is spatially equivalent to the GNS representation associated to the limit tracial state mapping each $(a_n)_{n\in\mathbb{N}} \in A^{\mathcal{U}}$ to $\lim_{n\to\infty} \tau(a_n)$. By assumption we have $\pi_\tau[A] \cong M_n$ for some $n \in \mathbb{N}$. By compactness of the unit ball of M_n, it follows that the map

$$\begin{aligned} \Theta : \pi_\tau[A] &\to \pi_\tau^{\mathcal{U}}[A^{\mathcal{U}}] \\ b &\mapsto (b)_{n\in\mathbb{N}}, \end{aligned}$$

is a surjective isomorphism. It follows that $\pi_\tau^{\mathcal{U}}[A^{\mathcal{U}}]'' \cong \pi_\tau^{\mathcal{U}}[A] \cong M_n$.

Suppose that p is a projection in $A^{\mathcal{U}}$ such that $f(\sigma) = \sigma(p)$ for every $\sigma \in T(A)$. We conclude that $\pi_\tau^{\mathcal{U}}(p)$ is a projection in $M_n(\mathbb{C})$. Therefore $\tau(p) = j/n$ for some $j \leq n$, and (2) follows.

To prove that (2) implies (1), fix f as in the statement. Fix $\epsilon > 0$. By Theorem 8.11 there exists a self-adjoint $a \in A$ such that $\tau(a) = f(\tau)$ for all $\tau \in T(A)$ and $-\epsilon \leq a \leq \|f\|_\infty + \epsilon$. The formula

$$\varphi(y) := \inf_{\|x\|\leq 1} \max\{\|x^2 - x\|_2^2, \|x^* - x\|_2^2, |\tau^+(x-y)|, |\tau^+(y-x)|\}$$

is $\exists$-max. If $\tau \in \partial_e T(A)$, then by the assumptions on f there exists a projection p in $\pi_\tau[A]''$ such that $\tau(p) = f(\tau)$, hence $\tau(p) = \tau(a)$. Since τ^+ is interpreted as τ in $\pi_\tau[A]''$, we have $\varphi^\tau(\pi_\tau(a)) = 0$. Since τ was an arbitrary extremal tracial state, Theorem 7.1 implies that $\varphi^{\mathsf{A}}(a) = 0$, for $\mathsf{A} = (\overline{A}^{T(A)}, T(A))$. Therefore for every $n \geq 1$ there exists $b_n \in A^{\mathcal{U}}$ such that $\varphi^{(A^{\mathcal{U}},(T(A))^{\mathcal{U}})}(b_n) < 1/n$. Since $A^{\mathcal{U}}$ is countably saturated ([14, Theorem 16.4.1]), this type is realized and the conclusion follows.

For the moreover part, modify the type used in the previous paragraph by adding the conditions $\|[c_n, x]\|_2^2$ where c_n ranges over a dense subset of S. $\square$

8.4. **C*-Dynamics.** A recent application of CPoU in the equivariant setting which is in line with the framework of this paper is contained in [19]. This paper deals with dynamical systems of amenable groups on simple, separable, unital, $\mathcal{Z}$-stable C*-algebras which, under suitable regularity assumptions, are shown to have α-CPoU.

thm:equiCPoU

Theorem 8.13. *Let A be a separable, nuclear, $\mathcal{Z}$-stable, unital C*-algebra with non-empty tracial state space, and let $\alpha : G \to \mathrm{Aut}(A)$ be an action of a discrete, countable, amenable group which is cocycle conjugate to $\alpha \otimes \mathrm{id}_{\mathcal{Z}}$ and such that the induced action on $T(A)$ factors through a finite group action. Then $(\overline{A}^{T(A)}, T(A), \alpha)$ has α-CPoU.*[13]

Assuming that an action α as in the statement is suitably outer, the presence of α-CPoU is used in [19] to prove that the dynamical system satisfies a certain Rokhlin-type condition. From the perspective given by the present paper, the argument used to show this fact consists of a particular instance of the Tracial Transfer Property. Let us give a quick sketch of this argument, after pointing out that model-theoretic study of C*-dynamics has been used in [20].

In the context of actions of amenable groups on von Neumann algebras, the Rokhlin Property is a noncommutative analogue of the classical result in ergodic theory known as Rokhlin's Lemma. For actions of $\mathbb{Z}$ on tracial von Neumann algebras, the exact definition is given below. We refer to [28, §6.1] for a definition suitable to a more general setting.

rokhlinprop

Definition 8.14 (Rokhlin Property for actions of $\mathbb{Z}$)**.** Let $(\mathcal{M}, \tau)$ be a tracial von Neumann algebra. The automorphism $\alpha \in \mathrm{Aut}(\mathcal{M})$ has the *Rokhlin Property* if for every $n \in \mathbb{N}$ and every $\|\cdot\|_{2,\tau}$-separable subset S of the tracial von Neumann ultrapower $\mathcal{M}^{\mathcal{U}}$ (which in the notation introduced in §**??** would be $\mathcal{M}^{\mathcal{U}}_{\{\tau\}}$) there are orthogonal projections $p_1, \ldots, p_n \in \mathcal{M}^{\mathcal{U}} \cap S'$ such that

[13]The definition of α-CPoU given in [19] is slightly different from Definition **??**, since the inequality $\tau(a_i p_i) \leq M\delta\tau(p_i)$ is required to hold for some $M \in \mathbb{N}$ possibly larger than 1. This is just an apparent difference, since the two definitions can be proved to be equivalent using an argument analogous to the one in the proof of Lemma 3.2.

(1) $\sum_{i=1}^n p_i = 1$,
(2) $\alpha(p_i) = p_{i+1 \mod (n)}$.

The set $\{p_1, \ldots, p_n\}$ is referred to as a *Rokhlin tower*.

In [9] Connes proved that an automorphism of a tracial von Neumann algebra $(\mathcal{M}, \tau)$ has the Rokhlin property if and only if all its powers are properly outer (this was later expanded to outer actions of discrete countable amenable groups on II_1 factors in [23] and [28]). Instead of defining proper outerness, we will define and use a related condition.

Sticking to the case $G = \mathbb{Z}$, an action $\alpha\colon \mathbb{Z} \to \mathrm{Aut}(A)$ on a simple C*-algebra A is *strongly outer* if α^τ is outer on $\pi_\tau[A]''$ for every τ in $T(A)^\alpha$. Strong outerness of α implies that α^τ is properly outer for every $\tau \in T(A)^\alpha$ (see e.g. [33, Proposition 5.7]), hence by Connes' result this allows one to verify the Rokhlin Property fiberwise. In case the action induced by α on $T(A)$ factors through a finite group action, α-CPoU can be used to glue together these local Rokhlin towers from the fibers to obtain a global Rokhlin-type condition, known as *uniform tracial Rokhlin Property*, which states the existence of families of projections as in Definition 8.14 in the tracial ultrapower $A^{\mathcal{U}}$.

What we just exposed is a simplified version of what happens in [19], where the framework is much wider and it includes actions of general amenable, countable, discrete groups. On the other hand, the argument in the previous paragraph is an immediate consequence of Theorem 8.13 and Theorem 7.2. Indeed, the existence of Rokhlin towers can be expressed by $\forall\exists$-max sentences indexed over $\mathbb{N} \times \mathbb{N}$ as follows, in an n-tuple of variables $\bar{x}$ and an m-tuple of variables $\bar{y}$ ($\varphi_p(x)$ is the formula axiomatizing projections introduced in Example 5.4(2))

$$\rho_{n,m} = \sup_{\|\bar{x}\|\leq 1} \inf_{\|\bar{y}\|\leq 1} \max_{i\leq n, j\leq m} \{\varphi_p(y_j), \|\sum_{j=1}^n y_j - 1\|_2^2, \\ \|\alpha(y_j) - y_{j+1}\|_2^2, \|[x_i, y_j]\|_2^2\}. \tag{8.1}$$

Another property that can be easily transferred from the fibers of a tracially complete C*-algebra to the algebra itself is approximate innerness of automorphisms. Let $(\mathcal{M}, X)$ be a tracially complete C*-algebra and let $\alpha \in \mathrm{Aut}(\mathcal{M})$. We say that α is *approximately inner* if for every finite set $F \subset \mathcal{M}$ and every $\epsilon > 0$ there exists a unitary $u \in \mathcal{M}$ such that

$$\|\alpha(a) - uau^*\|_{2,X} < \epsilon, \ \forall a \in F.$$

P.approximately-inner

Proposition 8.15. *Let $(\mathcal{M}, X, \alpha)$ be a factorial $\mathbb{Z}$-tracially complete C*-algebras such that the action induced by α on X factors through a finite group action. Suppose that $(\mathcal{M}, X, \alpha)$ has α-CPoU. Then α is approximately inner if and only if α^τ is approximately inner on $(\pi_\tau[A]'', \tau)$ for every $\tau \in T(A)^\alpha$.*

Proof. If α is approximately inner, it is immediate to check that α^τ is approximately inner for every $\tau \in T(A)^\alpha$.

For the other direction, consider the collection of the following formulas in an n-tuple of variables $\bar{x}$

$$\psi_n = \sup_{\|\bar{x}\|\leq 1} \inf_{\|y\|\leq 1} \max_{j\leq n} \{\|\alpha(x_j) - \exp(i2\pi y^* y) x_j \exp(-i2\pi y^* y)\|_2^2\}.$$

An automorphism of a tracial von Neumann algebra is approximately inner if and only if all ψ_n's are evaluated as zero, since all unitaries admit a logarithm on von Neumann algebras. The conclusion then follows by Theorem 7.2. $\square$

9. Concluding remarks

The Tracial Transfer Property (and therefore CPoU) is strictly weaker than $\mathcal{Z}$-stability, on separable simple nuclear C*-algebras. For example, the C*-algebra constructed in [30, Theorem 1.4] (see also [16, Proposition 3.5.6]) is nuclear, simple, separable and unital, but not $\mathcal{Z}$-stable, and its tracial completion is isomorphic to the hyperfinite II$_1$-factor $\mathcal{R}$ and therefore has the Tracial Transfer Property. Examples of separable C*-algebra without the Tracial Transfer Property include $C([0,1], \mathrm{C}^*_r(F_2))$ and $C(K)$ for K compact, Hausdorff, connected and having more than one point (see [6, Proposition 3.31]). Presently, no example of a nuclear, simple, and infinite-dimensional C*-algebra A and a closed face $X \subseteq T(A)$ such that the tracial completion of $(\overline{A}^X, \|\cdot\|_{2,X})$ fails the Tracial Transfer Property is known.

S.Beyond

9.1. Beyond $\forall\exists$-max formulas.

It is not too difficult to show that if a tracially complete C*-algebra satisfies the Tracial Transfer Property, then the class of formulas satisfying equality (5.9) (or (5.10) of Definition 5.5 is larger than the family of $\forall\exists$-max formulas. Indeed, suppose that $\psi(\bar{x})$ is an $\mathcal{L}_{\|\cdot\|_2}$-formula which, for every tracially complete C*-algebra $\mathsf{M} := (\mathcal{M}, X)$ and every tuple $\bar{a}$ in $\mathcal{M}$ of the appropriate sort, satisfies

eq:TTP-

$$\psi^{\mathsf{M}}(\bar{a}) = \sup_{\tau\in X} \psi^\tau(\pi_\tau(\bar{a})). \tag{9.1}$$

Then $\varphi(\bar{x}) := h(\psi(\bar{x}))$ also satisfies the equality in (9.1), whenever $h\colon \mathbb{R} \to \mathbb{R}$ is a continuous increasing function. The reason is the following simple computation, exploiting the fact that the sup commutes with continuous increasing functions on $\mathbb{R}$

$$\begin{aligned} \varphi^{\mathsf{M}}(\bar{a}) &= h(\psi^{\mathsf{M}}(\bar{a})) \\ &\overset{(9.1)}{=} h(\sup_{\tau\in X} \psi^\tau(\pi_\tau(\bar{a}))) \\ &= \sup_{\tau\in X} h(\psi^\tau(\pi_\tau(\bar{a}))) \\ &= \sup_{\tau\in X} \varphi^\tau(\pi_\tau(\bar{a})). \end{aligned}$$

Analogous computations show that if $\psi_1(\bar{x}), \dots \psi_k(\bar{x})$ satisfy (9.1), then also the formula $\max\{\psi_1(\bar{x}), \dots \psi_k(\bar{x})\}$ does. In this paper we content ourselves with showing that CPoU entail the tracial transfer for $\forall\exists$-max formulas, as these proved to be the most useful in our applications in §8. Nevertheless, it would be interesting to investigate what is the largest class of formulas to which this transfer property applies (keeping in mind that some inevitable limits are necessary, as discussed in Remark 5.8), and whether it is possible to characterize it abstractly.

These considerations may be a part of a bigger picture.

Ghasemi's Feferman–Vaught theorem for continuous logic ([21], see also [14, §16.3]) is indispensable in study of reduced products of C*-algebras (see for example the proof that the reduced products are countably saturated in [14, §16.5] or the proof in [15] that the exact sequence $0 \mapsto c_{\mathcal{U}}(B) \mapsto B^\infty \to B^{\mathcal{U}} \to 0$ splits). Our results in §6 may be an initial step towards proving a theorem of this type. The following is a consequence of the proof of Theorem 6.2 (for simplicity we do not state the dynamic version).

T.II1.convex.transfer **Theorem 9.1.** *Suppose that (T,μ) is a Radon probability measure space, $\{\mathcal{N}_\sigma\}_{\sigma\in T}$ is a measurable field of type II_1 von Neumann algebras with separable predual, and $\mathcal{N} = \int_T^\oplus \mathcal{N}_\sigma \, d\mu(\sigma)$. If $\varphi(\bar{x})$ is an $\forall\exists$-convex $\mathcal{L}_{\|\cdot\|_2}$-formula or an $\forall\exists$-convex definable predicate, then every tuple $\bar{a} = (\bar{a}_\sigma)_{\sigma\in T}$ in $\mathcal{N}$ of the appropriate sort satisfies $\varphi^{\mathcal{N}}(\bar{a}) \leq \|\sigma \mapsto \varphi^{\mathcal{N}_\sigma}(\bar{a}_\sigma)\|_\infty$.* □

We conjecture that in the situation of Theorem 9.1 it is possible to effectively compute the theory of $\mathcal{N}$ from the $L^\infty(T,\mu)$ functions $\psi \mapsto \psi^{\mathcal{N}_\sigma}(\bar{a}_\sigma)$ (more precisely, that a measurable analog of [14, Definition 16.3.2] applies to every φ). The following question is more challenging.

Question 9.2. *Is there a Feferman-Vaught-type theorem that computes the theory of a tracially complete C*-algebra $\mathsf{M} = (\mathcal{M}, X)$ with CPoU from the information on the sets*

$$Z_{\psi,r} = \{\sigma \in \partial_e X : \psi^\sigma \leq r\}$$

for a sentence ψ and $r \in \mathbb{R}$? What about the analogous question for G-tracially complete C-algebras?*

The answer is probably negative in the case when M does not satisfy CPoU.

Appendix A. Axiomatizability

In this appendix we prove the axiomatizability in $\mathcal{L}_{\|\cdot\|_2}$ and $\mathcal{L}_{\|\cdot\|_2,G}$ of both the tracially complete C*-algebras (Theorem A.1) and tracial von Neumann algebras and factors (Theorem A.2; the difference with the known result from [17] is that we prove axiomatizability in the slightly more restrictive language of tracially complete C*-algebras).

It is currently not known whether the class of factorial tracially complete C*-algebras is axiomatizable (however factorial tracially complete C*-algebras with CPoU are axiomatizable, [18]). In particular it is open whether the ultraproduct of a family of factorial tracially complete C*-algebras is still factorial.

A.1. **Tracially complete C*-algebras.** We introduce two equivalent categories - the category of tracially complete C*-algebras with embeddings, and the category of tracially complete C*-algebras thought of as metric $\mathcal{L}_{\|\cdot\|_2}$-structures, again with embeddings. We then show that the latter class of metric structures is elementary.

TC is the category of tracially complete C*-algebras $(\mathcal{M}, X)$. Since we really only retain the uniform 2-norm for a given $(\mathcal{M}, X)$ there is some ambiguity here in that different X's can give the same tracially complete algebra. We will assume that X is maximal when we talk about tracially complete algebras in this appendix. This will mean that an additional assumption on X is that for $\tau \in T(M)$, if $\|\cdot\|_\tau \leq \|\cdot\|_{2,X}$ then $\tau \in X$. This clearly implies that X is convex and weak*-closed. We are now able to describe the morphisms in TC as follows:

$$\varphi : (\mathcal{N}, Y) \to (\mathcal{M}, X)$$

is a morphism if φ is an injective *-homomorphism from $\mathcal{N}$ to $\mathcal{M}$ such that $\varphi^*(X) = Y$. An application of the Hahn-Banach theorem shows that Y satisfies the maximality condition we are assuming.

The category of metric structures MTC has as objects tracially complete C*-algebras in the language $\mathcal{L}_{\|\cdot\|_2}$ defined in §**??**. Since the sorts D_n are still given by the operator norm balls of radius n, these balls are complete with respect to the uniform 2-norm. The morphisms in this category are injective *-homomorphisms which preserve the uniform 2-norm.

T.utc.axiomatizable **Theorem A.1.** *The category of tracially complete C*-algebras* TC *is categorically equivalent to the category* MTC*. Furthermore,* MTC *is an elementary class.*

Proof. Define a functor $F : \mathrm{TC} \to \mathrm{MTC}$ sending $(\mathcal{M}, X) \in \mathrm{TC}$ to the metric structure associated with $\mathcal{M}$ and $\|\cdot\|_{2,X}$. On morphisms, F sends an injective *-homomorphism to essentially the same map only partitioned by operator norm. We leave it as an exercise to see that this is an equivalence. One small point is that if one starts with an object $(M, \|\cdot\|_{2,X})$ in MTC then by the convention that we choose a maximal $X \subseteq T(M)$ compatible with the uniform 2-norm, there is no ambiguity about which tracially complete C*-algebra $(\mathcal{M}, \|\cdot\|_{2,X})$ is sent to it by F. Instead of exhibiting axioms for MTC, we take an indirect route and use [16, Theorem 2.4.1]. By this result, a class of metric structures is elementary if (and only if) it is closed under isomorphisms, ultraproducts, and elementary submodels (or simply ultraroots). MTC is clearly closed under isomorphisms.

Suppose that $(\mathcal{M}_j, \|\cdot\|_{2,X_j})$, for $j \in \mathbb{J}$, is a family of tracially complete C*-algebras in MTC and $\mathcal{U}$ is an ultrafilter on $\mathbb{J}$. Note that we are assuming that X_j is maximal in the sense mentioned before the Theorem and so X_j is both convex and closed for all j. The tracial ultraproduct $\mathcal{M} := \prod^{\mathcal{U}}(\mathcal{M}_j, \|\cdot\|_{2,X_j})$ is the quotient of the direct product $\prod_{j\in\mathbb{J}} \mathcal{M}_j$ by the ideal $\{(a_j)_j \,|\, \lim_{j\to\mathcal{U}} \|a_j\|_{2,X_j} = 0\}$.

Let Y_0 be $\prod^{\mathcal{U}} X_j$, the set of all limit tracial states on $\mathcal{M}$ (as defined in §**??**). It is easy to see that Y_0 is convex and sequentially closed, but as pointed out in §**??** it is not necessarily weak*-closed. Let $Y := \sum^{\mathcal{U}} X_j$ denote the weak*-closure of Y_0 and note that the norms $\|\cdot\|_{2,Y}$ and $\|\cdot\|_{2,Y_0}$ coincide. We claim that the ultraproduct 2-norm on $\mathcal{M}$ is equal to $\|\cdot\|_{2,Y}$; once proven, this will imply that $\mathcal{M}$ is isometrically isomorphic to $(\prod^{\mathcal{U}} \mathcal{M}_j, Y)$ and therefore tracially complete. For a representing sequence (a_j) of an element of $\mathcal{M}$ we have

$$\|(a_j)\|_{2,Y} = \sup_{(\tau_j)_j} \lim_{j\to\mathcal{U}} \|a_j\|_{2,\tau_j}$$

where the supremum is taken over all limit tracial states $(\tau_j)_{j\in\mathbb{J}}$ in $\prod^{\mathcal{U}} X_j$. For every $j \in \mathbb{J}$, since the evaluation of tracial states at $a_j^* a_j$ is weak*-continuous and X_j is compact, this supremum is attained at some tracial state denoted $\tau_j \in X_j$. Thus we have $\|(a_j)\|_{2,Y} = \lim_{j\to\mathcal{U}} \|a_j\|_{2,\tau_j}$. The right-hand side is not greater than $\lim_{j\to\mathcal{U}} \|a_j\|_{2,X_j}$. On the other hand, $\lim_{j\to\mathcal{U}} \|a_j\|_{2,X_j}$ is (again by weak*-continuity) equal to $\lim_{j\to\mathcal{U}} \|a_j\|_{2,\tau_j}$ for a limit tracial state (τ_j), and therefore not greater than $\|(a_j)\|_{2,Y}$.

This proves the equality, and therefore the ultraproduct is the tracially complete C*-algebra $(\prod^{\mathcal{U}} \mathcal{M}_j, \sum^{\mathcal{U}} X_j)$.

It remains to prove that if $(\mathcal{M}, X)$ is tracially complete and N is an elementary submodel of $(\mathcal{M}, X)$ with domain $\mathcal{N}$, then the restriction of $\|\cdot\|_{2,X}$ to $\mathcal{N}$ is equal to $\|\cdot\|_{2,X(\mathcal{N})}$, where $X(\mathcal{N}) := \{\tau \restriction \mathcal{N} : \tau \in X\}$. As N is elementary submodel of $(\mathcal{M}, X)$, for every $a \in \mathcal{N}$ we have that $\|a\|_2^{\mathsf{N}} = \|a\|_{2,X} \leq \|a\|_{2,X(\mathcal{N})}$. Moreover, as $\|\cdot\|_{2,X} \leq \|\cdot\|$, $\mathcal{N}$ is closed with respect to the operator norm, and therefore a C*-algebra. Finally, an application of the Hahn-Banach theorem shows that $X(\mathcal{N})$ is maximal in the necessary sense, i.e. every $\tau \in T(\mathcal{N})$ which satisfies $\|\cdot\|_\tau \leq \|\cdot\|_{2,X(\mathcal{N})}$ can be extended to a tracial state on $\mathcal{M}$ which is in X. Hence $\|a\|_{2,X(\mathcal{N})} \leq \|a\|_2^{\mathsf{N}}$ and we are done. $\square$

S.vNA

A.2. **Tracial von Neumann Algebras and Factors.**

In what follows, by tracial von Neumann algebra we mean a pair $(\mathcal{M}, \tau)$ where τ is a faithful tracial state on a von Neumann algebra $\mathcal{M}$.

In [17, §3.2] it is proved that both classes of tracial von Neumann algebras and of II_1 factors are elementary in the language of tracial von Neumann algebras defined in [17, §2.4]. The latter language is richer than our $\mathcal{L}_{\|\cdot\|_2}$ from §**??**, since it also contains two unary predicates, tr^r and tr^i, which are interpreted on tracial von Neumann algebras as the real and the imaginary part of the tracial state.

In this section we prove that the classes of tracial von Neumann algebras and of II_1 factors are also elementary in the context considered in this paper, that is as $\mathcal{L}_{\|\cdot\|_2}$-structures. To do this, we rely on the predicate τ^+ introduced in Definition 5.9, and on the fact that it is definable on tracially complete C*-algebras (Lemma 5.10).

thm:vNas **Theorem A.2.** *The classes of tracial von Neumann algebras and of II_1 factors, seen as structures in $\mathcal{L}_{\|\cdot\|_2}$, are elementary.*

Proof. We start by showing that the class of tracial von Neumann algebras $(\mathcal{M}, \tau)$ is elementary. Note that the domain $\mathcal{M}$ of a tracially complete C*-algebra $(\mathcal{M}, X)$ such that X is a singleton τ is automatically a von Neumann algebra. Indeed, in this case the $\|\cdot\|_{2,\tau}$-norm induces the strong topology on $\mathcal{M}$ in the GNS-representation associated with τ, hence by the Kaplansky density theorem and completeness of the unit ball of $\mathcal{M}$ with respect of the $\|\cdot\|_{2,\tau}$-norm, we can conclude that $\mathcal{M} \cong \pi_\tau[\mathcal{M}] = \pi_\tau[\mathcal{M}]''$.

In view of this, it suffices to axiomatize the class of those tracially complete C*-algebras $\mathsf{M} = (\mathcal{M}, X)$ such that $X = \{\tau\}$. By Theorem A.1 we already know that the class of tracially complete C*-algebras is elementary. A standard argument combining the Hahn-Banach Theorem with Theorem 8.11 shows that $X = \{\tau\}$ if and only if the definable predicate τ^+ is additive on self-adjoint elements of $\mathcal{M}$. By Lemma 5.10, the latter property is expressible via the following definable predicate

$$\sup_{\|x\|,\|y\|\leq 1} \Big(\tau^+\Big(\frac{x+x^*}{2} + \frac{y+y^*}{2}\Big) - \tau^+\Big(\frac{x+x^*}{2}\Big) - \tau^+\Big(\frac{y+y^*}{2}\Big)\Big). \tag{A.1}$$

Axiomatizability of the class of II$_1$ factors can be obtained using axioms (16) and (17) from [17, §3.2] (see [17, Proposition 3.4], also §6.3.2). □

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