Math 6462, W22. Assignment # 4 solutions

In all questions, H stands for the separable infinite-dimensional Hilbert space.

(1) (10pts) Suppose that (X,μ) is a probability measure space and that T is a Hilbert–Schmidt operator on $L^2(X,\mu)$. Prove that there is $k\in L^2(X\times X,\mu\times\mu)$ such that T is the operator associated to the kernel k, so that for every $\xi\in L^2(X,\mu)$ we have

$$T\xi(x)=\int_X k(x,y)\xi(y)\,d\mu(y).$$

(This is the converse of the theorem stating that every operator associated with such kernel is Hilbert–Schmidt, proved in class.)

Solution. The easiest way to go is to use the orthonormal basis for the space of Hilbert–Schmidt operators given in (2) below (originally I wanted to use a corollary of the spectral theorem for compact operators that gives the structure of an arbitrary compact operator, modified for the Hilbert–Schmidt operators, but the approach given here is simpler).

Let (f_j) be an orthonormal basis for $L^2(X,\mu)$. Then $(f_j\odot f_k)_{j,k}$ is an orthonormal basis for the Hilbert–Schmidt operators. Thus every T can be written as $T=\sum\lambda_{jk}f_j\odot f_k$ (the convergence is in the Hilbert–Schmidt norm), with $\sum_{jk}|\lambda_{jk}|^2<\infty$.

Let's see what $f_j\odot f_k$ looks like for some j and k. For $g\in L^2(X,\mu)$ we have

$$(f_j\odot f_k)(\xi)(x)=\int\overline{f_j(y)}f_k(x)\xi(y)\,d\mu(y)$$

for almost all $x\in X$. That is $f_j\odot f_k$ corresponds to the operator with the kernel $(x,y)\mapsto\overline{f_j(y)}f_k(x)$ as in Example 2.8.5 in Arveson. These functions form an orthonormal basis in $L^2(X\times X,\mu^2)$, and since $\sum|\lambda_{jk}|^2<\infty$, the sum $\sum_{jk}\lambda_{jk}\overline{f_j(y)}f_k(x)$ converges in $L^2(X\times X,\mu^2)$. The limit is the desired kernel k. (This can be proven e.g., by the Dominated Convergence Theorem.)

(2) (10pts) Suppose that H is a Hilbert space with orthonormal basis (e_n) (the basis is not necessarily countable; separability of H makes no difference in this question).

(a) (3pts) Prove that the Hilbert–Schmidt operators on H form a Hilbert space, with respect to the inner product $(S|T)=\mathrm{Tr}(T^*S)$. (Most of the work has been done in class, and you can freely cite it.)

Solution. This was straightforward. $(S|T)$ is a sesquilinear form with $(S|S)=\sum_\xi\|S\xi\|_2\geq 0$ (here ξ ranges over a fixed orthonormal basis). Also, $S\neq 0$ implies $S\xi\neq 0$ for some ξ in this basis, and $\mathrm{Tr}(S^*S)\geq(S^*S\xi|\xi)=\|S\xi\|_2>0$, so the form is strictly positive.

(b) (7pts) Find a nice orthonormal basis for the Hilbert space of Hilbert–Schmidt operators on H, and prove that it is indeed an orthonormal basis.

Solution. Suppose that $(\xi_j)_{j\in J}$ is an orthonormal basis for H (finite, countable, uncountable, does not matter). Define $\xi\odot\eta$ to be the rank one operator $(\xi\odot\eta)(\zeta)=(\zeta|\eta)\xi$. We claim that the operators $\xi_j\odot\xi_k$, for j,k in J, form an orthonormal basis for the space of Hilbert–Schmidt operators.

First check that they form an orthonormal system. A computation gives $(\xi_j\odot\xi_k)^*=\xi_k\odot\xi_j$ and also $(\xi_j\odot\xi_k)(\xi_l\odot\xi_m)=(\xi_j\odot\xi_m)$ if $k=l$ and 0 otherwise. Also, $\mathrm{Tr}(\xi_j\odot\xi_k)=1$ if $j=k$ and 0 otherwise.

Thus $(\xi_j\odot\xi_k|\xi_l\odot\xi_m)=1$ if $j=l$ and $k=m$ and 0 otherwise, as required.

It remains to check that this set is a basis for the space of Hilbert–Schmidt operators. The easiest way to do so is to compute its orthogonal. Suppose T is such that $(T|\xi_j\odot\xi_k)=0$ for all j and k. But

$$(T|\xi_j\odot\xi_k)=\mathrm{Tr}((\xi_k\odot\xi_j)T)=\mathrm{Tr}(\xi_k\odot T^*\xi_j)=(T\xi_k|\xi_j).$$

This means that $T\xi_k=0$ for all k, and therefore $T=0$, as required.

(3) (10pts) Show that a multiplication operator M_f on $L^2(X,\mu)$ is self-adjoint and $\sigma(M_f) \subseteq [0,\infty)$ if and only if $(M_f\xi|\xi) \geq 0$ for all $\xi \in L^2(X,\mu)$.

Solution. Suppose $\sigma(M_f) \subseteq [0,\infty)$. Then for every $\xi \in L^2(X,\mu)$ we have $(M_f\xi|\xi) = \int f(x)|\xi(x)|^2\,d\mu(x) \geq 0$ (because both $f(x)$ and $f(x)|\xi(x)|^2$ are then ≥ 0 almost everywhere). Conversely, assume that $\sigma(M_f) \not\subseteq [0,\infty)$. Fix $\lambda\imath\sigma(M_f)$ and $\varepsilon > 0$ such that $B_\varepsilon(\lambda) \cap [0,\infty) = \emptyset$. Let $E \subseteq X$ be a measurable, μ-positive set such that $0 < \mu(E) < \infty$ and let $\xi = 1_E$. Then $|(M_f\xi|\xi) - \lambda\mu(E)| = |\int_E (f(x)-\lambda)|\xi(x)|^2\,d\mu(x)| \leq \int_E |f(x)-\lambda|\,d\mu(x) \leq \varepsilon$. Therefore $(M_f\xi|\xi) \in B_\varepsilon(\lambda)$, and in particular it is not in $[0,\infty)$.

(4) (10pts) Suppose that X is a compact Hausdorff space and F is a proper closed subset of X. Let $J_F = \{f \in C(X) | f(x) = 0 \text{ for all } x \in F\}$. This is a norm-closed, self-adjoint ideal of $C(X)$ (we know that 'self-adjoint' is redundant but never mind).

(a) (3pts) What property of F is equivalent to the assertion that the ideal J_F is unital? Prove your claim.

Solution. J_F is unital if and only if F is open (in addition to being closed). If F is open, then 1_F is in $C(X)$, and it is the unit for J_F. Conversely, if J_F has a unit then it has to be 1_F; but $1_F \in C(X)$ if and only if F is clopen (both closed and open).

(b) (3pts) In case J_F is not unital, describe the compact Hausdorff space Y such that $C(X)/J_F$ is isomorphic to $C(Y)$.

Solution. This is the case when F is closed, but not open. Let Y be the quotient space of X obtained by identifying all points in F to a single point. The quotient $C(X)/J_F$ is isomorphic to $C(Y)$.

(c) (4pts) In case J_F is not unital, describe the compact Hausdorff space Y such that the unitization J_F^e is isomorphic to $C(Y)$.

Solution. (Note that J_F is isomorphic to $C_0(F)$.) This is the one-point compactification of F–the space $Y = F \cup \{\infty\}$, where the copy of F is homeomorphic to F and the open neighbourhoods of ∞ are co-compact subsets of F. Then $f \in C(Y)$ if and only if the restriction of $f - f(\infty)$ to F vanishes at infinity (no pun intended), that is belongs to $C_0(F)$.

(5) Suppose that X is a locally compact, but not compact, Hausdorff space.

(a) (0pts) Check that $C_b(X) = \{f\colon X \to \mathbb{C} | f \text{ is continuous and bounded}\}$ is a C*-algebra, and conclude that there is a compact Hausdorff space Y such that $C_b(X) \cong C(Y)$ (no need to submit a proof of this).

(b) (5pts) Recall that the points of a compact Hausdorff space Y are in bijective correspondence with the characters of $C(Y)$. Use this to define a natural homeomorphism from X onto a dense open subspace $\tilde{X}$ of Y.

Solution. Since X is locally compact, by for example Tietze extension theorem continuous functions from X into $[0,1]$ separate the points of X (i.e., X is completely regular).
Send $x \in X$ to the evaluation character α_x, $\alpha_x(f) = f(x)$, on $C_b(X)$. By the first paragraph, a net (x_λ) in X converges to y in X if and only if for every $f \in C_b(X)$ we have $\lim_\lambda f(x_\lambda) = f(y)$. But this is equivalent to asserting that $\lim_\lambda \alpha_{x_\lambda} = \alpha_x$. Thus $x \mapsto \alpha_x$ is a homeomorphism.
To check that the image of X in Y is dense, note that $C_b(X)$ is a C*-algebra, and therefore isomorphic to $C(Y)$. Every $f \in C(Y)$ is uniquely determined by its restriction of $\tilde{X}$ (because $C(Y)$ is isomorphic to $C_b(X)$). Since Y is compact and Hausdorff, this implies $\tilde{X}$ is dense in Y.

(c) (5pts) Identify X with $\tilde{X}$. Prove that every continuous function from X into $[0,1]$ has a unique extension to a continuous function from Y into $[0,1]$.

Solution. Identify X with $\tilde{X}$. The Gelfand map sends continuous $f\colon X \to [0,1]$ to $\Gamma(f) \in C(Y)$ that is a continuous extension of f. It is unique because X is dense in Y.

(d) (0pts) The property (5c) determines Y uniquely, in the sense that if X is homeomorphic to a dense subspace of a compact Hausdorff space Z with the same property, then there is a homeomorphism $f\colon Y \to Z$ that is equal to the identity on X. (Again, no need to submit a proof of this.)

(6) Let X and Y be as in Question 5.

(a) (10pts) Prove that $C_b(X)/C_0(X) \cong C(Y \setminus X)$.

Solution. $C_b(X)/C_0(X)$ is, by the previous question, isomorphic to $C(Y)/C_0(X)$ (with X identified with $\tilde{X}$). Since X is locally compact, every $x \in X$ has an open neighbourhood U whose closure is included in X. By Tietze, X is an open subset of Y. We already know it is dense. The function that sends $f \in C(Y)$ to $f \restriction (Y \setminus X) \in C(Y \setminus X)$ is a *-homomorphism onto (Tietze) whose kernel is $C_0(X)$.

(b) (0pts) Think about the cases $X = \mathbb{N}$ and $X = [0, \infty)$.

(7) (10pts) Use the GNS representation to prove that every separable C*-algebra A has a faithful representation on a separable Hilbert space. (A representation π is faithful if $\ker(\pi) = \{0\}$. For C*-algebras, a representation is faithful if and only if it is isometric.)

Solution. For example, fix a countable dense subset D of A. For each $d \in D$ fix a state $\varphi_d \in S(A)$ such that the associated GNS representation π_d satisfies $\|\pi_d(d)\| = \|d\|$.

Then $\pi = \bigoplus_d \pi_d$ is a faithful representation. This is because $\|\pi(d)\| = \|d\|$ for all $d \in D$, and every $a \in A$ is a limit of a convergent sequence in D. It remains to check that the associated Hilbert space is separable.

For each $d \in D$, the Hilbert space H_d associated with the GNS representation π_d is separable. This is because $\varphi_d(a^*a) \leq \|a^*a\| = \|a\|^2$ for all $a \in A$, hence $\|a\|_{2,\varphi_d} \leq \|a\|$. Thus (for example) the image of D is dense in H_d.

Therefore the space associated with π is, being a countable ℓ_2-sum of separable Hilbert spaces, separable.

(8) (10pts) Prove that the Calkin algebra does not have a nonzero representation on a separable Hilbert space. (Bonus 5pts: Prove that the Calkin algebra does not have a faithful representation on $\ell_2(\kappa)$ unless $\kappa \geq 2^{\aleph_0}$.)

Solution. First we need a family of infinite subsets X_r of $\mathbb{N}$ that is uncountable but the intersection of any two of them is finite. (For example: Enumerate the rationals as q_j, for $j \in \mathbb{N}$. For each $r \in \mathbb{R} \setminus \mathbb{Q}$ let $X_r \subseteq \mathbb{N}$ be such that q_j, for $j \in X_r$, converges to r.)

To each X_r let p_r be the projection in ℓ_∞ associated to 1_{X_r}. Then in ℓ_∞/c_0 the projections (using $\dot{a}$ for the image of a under the quotient map) $\dot{p}_r$, for $r \in \mathbb{R} \setminus \mathbb{Q}$, are nonzero and since for $r \neq s$ the projection $p_r p_s$ has finite rank, $\dot{p}_r \dot{p}_s = 0$. Now embed ℓ_∞/c_0 into the Calkin algebra $\mathcal{Q}(H) = \mathcal{B}(H)/\mathcal{K}(H)$ diagonally. The image of $\dot{p}_r$, for $r \in \mathbb{R} \setminus \mathbb{Q}$, is an uncountable orthogonal family of nonzero projections. If $\pi\colon \mathcal{Q}(H) \to \mathcal{B}(K)$ for some Hilbert space K is a faithful representation[1], then the image of this family is an uncountable family of orthogonal projections that gives uncountably many orthogonal nonzero subspaces of K. Thus K cannot be separable. (The proof shows more that the orthonormal basis for K cannot have fewer than $2^{\aleph_0}$ elements. Since this is the cardinality of $\mathcal{Q}(H)$, this suffices and $\mathcal{Q}(H)$ has a faithful representation on $\mathcal{B}(\ell^2(2^{\aleph_0}))$.)

[1] Or a nonzero representation—every representation of $\mathcal{Q}(H)$ is faithful, this is a simple C*-algebra.