Math 6462, W22. Assignment # 4. Due date: March 28, 11:59pm.

Choose any five questions. Clearly indicate which five, and *submit only their solutions.* If you submit solutions to more than five questions, I will grade only five of them.

In all questions, H stands for the separable infinite-dimensional Hilbert space.

(1) (10pts) Suppose that (X, μ) is a probability measure space and that T is a Hilbert–Schmidt operator on $L^2(X, \mu)$. Prove that there is $k \in L^2(X \times X, \mu \times \mu)$ such that T is the operator associated to the kernel k, so that for every $\xi \in L^2(X, \mu)$ we have

$$T\xi(x) = \int_X k(x, y)\xi(y)\, d\mu(y).$$

(This is the converse of the theorem stating that every operator associated with such kernel is Hilbert–Schmidt, proved in class.)

(2) (10pts) Suppose that H is a Hilbert space with orthonormal basis (e_n) (the basis is not necessarily countable; separability of H makes no difference in this question).

  (a) (3pts) Prove that the Hilbert–Schmidt operators on H form a Hilbert space, with respect to the inner product $(S|T) = \operatorname{Tr}(T^*S)$. (Most of the work has been done in class, and you can freely cite it.)

  (b) (7pts) Find a nice orthonormal basis for the Hilbert space of Hilbert–Schmidt operators on H, and prove that it is indeed an orthonormal basis.

(3) (10pts) Show that a multiplication operator M_f on $L^2(X, \mu)$ is self-adjoint and $\sigma(M_f) \subseteq [0, \infty)$ if and only if $(M_f\xi|\xi) \geq 0$ for all $\xi \in L^2(X, \mu)$.

(4) (10pts) Suppose that X is a compact Hausdorff space and F is a proper closed subset of X. Let $J_F = \{f \in C(X) | f(x) = 0 \text{ for all } x \in F\}$. This is a norm-closed, self-adjoint ideal of $C(X)$ (we know that 'self-adjoint' is redundant but never mind).

  (a) (3pts) What property of F is equivalent to the assertion that the ideal J_F is unital? Prove your claim.

  (b) (3pts) In case J_F not unital, describe the compact Hausdorff space Y such that J_F is isomorphic to $C(Y)$.

  (c) (4pts) In case J_F is not unital, describe the compact Hausdorff space Y such that the unitization J_F^e is isomorphic to $C(Y)$.

(5) Suppose that X is a locally compact, but not compact, Hausdorff space.

  (a) (0pts) Check that $C_b(X) = \{f : X \to \mathbb{C} | f \text{ is continuous and bounded}\}$ is a C*-algebra, and conclude that there is a compact Hausdorff space Y such that $C_b(X) \cong C(Y)$ (no need to submit a proof of this).

  (b) (5pts) Recall that the points of a compact Hausdorff space Y are in bijective correspondence with the characters of $C(Y)$. Use this to define a natural homeomorphism from X onto a dense open subspace $\tilde{X}$ of Y.

  (c) (5pts) Identify X with $\tilde{X}$. Prove that every continuous function from X into $[0, 1]$ has a unique extension to a continuous function from Y into $[0, 1]$.

  (d) (0pts) The property (5c) determines Y uniquely, in the sense that if X is homeomorphic to a dense subspace of a compact Hausdorff space Z with the same property, then there is a homeomorphism $f : Y \to Z$ that is equal to the identity on X. (Again, no need to submit a proof of this.)

(6) Let X and Y be as in Question 5.

  (a) (10pts) Prove that $C_b(X)/C_0(X) \cong C(Y \setminus X)$.

  (b) (0pts) Think about the cases $X = \mathbb{N}$ and $X = [0, \infty)$.

(7) (10pts) Use the GNS representation to prove that every separable C*-algebra A has a faithful representation on a separable Hilbert space. (A representation π is faithful if $\ker(\pi) = \{0\}$. For C*-algebras, a representation is faithful if and only if it is isometric.)

(8) (10pts) Prove that the Calkin algebra does not have a nonzero representation on a separable Hilbert space. (Bonus 5pts: Prove that the Calkin algebra does not have a faithful representation on $\ell_2(\kappa)$ unless $\kappa \geq 2^{\aleph_0}$.)