Math 6462, W22. Assignment # 3 solutions

In all questions, H stands for the separable infinite-dimensional Hilbert space.

(1) (a) (4pts) Find two self-adjoint operators on H with spectrum $[0,1]$ and empty point-spectrum that are not unitarily equivalent.
The solution is somewhat more detailed than necessary.

Solution. Consider strictly positive Borel measures μ and ν on $[0,1]$. Let S be M_{id} on $L^2([0,1],\mu)$ and let T be M_{id} on $L^2([0,1],\nu)$. Since μ is strictly positive, $\sigma(S)=[0,1]$, and similarly $\sigma(T)=[0,1]$. We'll see when is there a unitary $U\colon L^2([0,1],\mu)\to L^2([0,1],\nu)$ such that $S=UTU^*$.

Claim 1. *Suppose f_n is a bounded sequence in $L^\infty(X,\mu)$ such that $f_n\geq f_{n+1}\geq 0$ for all n. Then f_n converges to 0 almost everywhere if and only if* $\text{SOT-}\lim_n M_{f_n}=0$.

Proof. The forward implication uses the Dominated Convergence Theorem, as we did in class when proving Theorem 2.6.3 (and Remark 2.6.2) in Arveson, that a nondegenerate representation of $C(X)$ extends to a σ-representation of $\mathcal{B}(X)$.
For the converse use the assumption that $f_n\geq f_{n+1}\geq 0$ for all n. If f_n do not converge to 0 almost everywhere, then there exists $\varepsilon>0$ such that the sets $A_{\varepsilon,n}=\{x|f_n(x)\geq\varepsilon\}$ satisfy $\limsup_n\mu(A_{\varepsilon,n})=\delta>0$. But $A_{\varepsilon,n}\supseteq A_{\varepsilon,n+1}$, and therefore $A=\bigcap_n A_{\varepsilon,n}$ satisfies $\mu(A)\geq\delta$. The characteristic function of A gives a vector ξ such that $\|M_{f_n}\xi\|\not\to 0$. □

By linearity, the claim implies that if $f_n\geq f_{n+1}\geq g$ are real in $L^\infty(X,\mu)$, then f_n converges to g almost everywhere if and only if M_{f_n} SOT-converges to M_g.
The conjugation by U, $R\mapsto URU^*$, is norm-norm continuous, but it is also SOT-SOT continuous (proof is straightforward and a good exercise). It is also WOT-WOT continuous (another little exercise) but we don't need this.
For a Borel $A\subseteq[0,1]$, consider the projection $P(A)=M_{1_A}$ in $\mathcal{B}(L^2([0,1],\mu))$ and the projection $Q(A)=M_{1_A}$ in $\mathcal{B}(L^2([0,1],\nu))$.

Claim 2. *For every Borel $A\subseteq[0,1]$ we have $UP(A)U^*=Q(A)$.*

Proof. Since $UTU^*=S$ (i.e., $UM_{\mathrm{id}}U^*=M_{\mathrm{id}}$), for every polynomial and every $f\in C([0,1])$ we have $UM_fU^*=M_f$.
If $A\subseteq[0,1]$, is closed, then 1_A is the pointwise infimum of a decreasing sequence f_n of functions in $C([0,1])$. Then $\text{SOT-}\lim_n M_{f_n}=P(A)$ and by the first claim $UP(A)U^*=Q(A)$. By looking at the complements, the same holds for the open sets.
If $A\subseteq[0,1]$ is G_δ, then it is the intersection of open sets, and using the first claim again we have $UP(A)U^*=Q(A)$.
Every measurable $A\subseteq[0,1]$ is equal to a G_δ set A' modulo a null set, thus $P(A)=P(A')$ and the conclusion follows for all Borel sets. □

(Digression: The claim implies that for every $f\in L^\infty(X,\mu)$ we have $UM_fU^*=M_f$. By linearity, for all step functions f we have $UM_fU^*=M_f$. Since every $f\in L^\infty(X,\mu)$ is a uniform limit of step functions, the conclusion follows. We don't need this fact.)
Claim implies that $\mu(A)=0$ if and only if $\nu(A)=0$. In other words, μ is absolutely continuous with respect to ν and vice versa.
Finally note that the pure point spectrum of T is empty if and only if μ has no atoms (i.e., every singleton has measure zero).
The question therefore reduces to finding two strictly positive measures on $[0,1]$, neither one of which has atoms, such that one of them is not absolutely continuous with respect to the other.
Take μ to be the Lebesgue measure. Let $P\subseteq[0,1]$ be a perfect set such that $\mu(P)=0$. There is a strictly positive Borel probability measure ν_P on P with no atoms. (E.g.,

identify P with the Cantor set. Then identify the Cantor set with the product $\{0,1\}^{\mathbb{N}}$, and on this space consider the product of uniform probability measures on $\{0,1\}$.)
Then μ and $\frac{1}{2}\mu + \frac{1}{2}\mu_P$ are as required.

(b) (6pts) Find infinitely many self-adjoint operators on H with spectrum $[0,1]$ and empty point-spectrum that are pairwise unitarily inequivalent.

Solution. Continuing the argument of part (a), pick infinitely many disjoint perfect μ-null subsets P_n of $[0,1]$. (There are infinitely many ways to find such P_n; see e.g., the solution to part (c).) On each P_n choose a measure ν_n as in part (a). Then for example $\mu_n = \frac{1}{2}\mu + \frac{1}{2}\nu_n$ are as required.

(c) (Bonus 5pts) Find uncountably many self-adjoint operators on H with spectrum $[0,1]$ and empty point-spectrum that are pairwise unitarily inequivalent.

Solution. Continuing the argument of part (b), one only needs uncountably many perfect μ-null subsets of $[0,1]$. For this, note that the Cantor space $\{0,1\}^{\mathbb{N}}$ is homeomorphic to its square $(\{0,1\}^{\mathbb{N}})^2$. Therefore the vertical sections of this square give continuum many disjoint perfect subsets of the Cantor space, and therefore continuum many disjoint perfect subsets of a perfect μ-null subset P of $[0,1]$.
Suppose that T is a normal operator on H with spectrum $X = \sigma(T)$. By the Spectral Theorem, there are a σ-finite Borel measure[1] μ on X and a unitary $U\colon L^2(X,\mu) \to H$ such that $T = UM_fU^*$, where f is the identity function on X. By the Bounded Borel Functional calculus, there is a σ-representation π of $B(X)$ on H.
On the other hand, by Theorem 2.1.3 in Arveson, the map $\Phi\colon L^\infty(X,\mu) \ni f \mapsto M_f \in \mathcal{B}(L^2(X,\mu))$ is an isometric isomorphism.
Let $\iota\colon C(X) \to L^\infty(X,\mu)$ be the identity map. Note that it is injective, since μ is strictly positive on X.

$$
\begin{array}{ccc}
C(X) & \longrightarrow & L^\infty(X,\mu) \\
\downarrow & \swarrow & \\
\mathcal{B}(H) & &
\end{array}
$$

(2) (a) (5pts) Suppose that T is a self-adjoint operator on H such that $\|T^2 - T\| < 1/4$. Prove that there is a projection in $\mathrm{C}^*(T)$ (the C^*-algebra generated by T).

Solution. (This one was trivial as one could take the projection to be 0; I should have e.g., added the assumption that $\{0,1\} \subseteq \sigma(T)$ and required that the projection is nontrivial, i.e., neither 0 nor 1. This is what the proof shows.) The Spectral Theorem easily implies that $\mathrm{C}^*(T,1) \cong C(\sigma(T))$. Since $T = T^*$, the Continuous Functional Calculus implies $\sigma(T) \subseteq \mathbb{R}$.[2]
Since $\|T - T^2\| < 1/4$, and (again by the Continuous Functional Calculus, CFC) $\|T - T^2\| = \sup\{|t - t^2| : t \in \sigma(T)\}$, we have $1/2 \notin \sigma(T)$. Let $g\colon \sigma(T) \to \{0,1\}$ be defined by $g(t) = 0$ if $t < 1/2$ and $g(t) = 1$ if $t > 1/2$. Since $1/2 \notin \sigma(T)$, g is continuous on $\sigma(T)$, and therefore (CFC again) $P = f(T) \in \mathrm{C}^*(T,1)$. By CFC, $g^2 = g$ implies $P^* = P$ and $g = \overline{g}$ implies $P = P^*$, hence P is a projection.

(b) (5pts) Prove that there is a decreasing $f\colon (0,1/2) \to (0,1]$ that satisfies $\lim_{t\to 0} f(t) = 0$ and such that if T is a self-adjoint operator on H and $\|T^2 - T\| = \varepsilon < 1/2$, then there is a projection P in $\mathrm{C}^*(T)$ such that $\|P - T\| \leq f(\varepsilon)$.

[1] There is a probability measure with the same property; this is an exercise.
[2] We are shooting a fly with a cannon; proving this from the scratch is a nice exercise.

Solution. This is a refinement of the solution to (a). Fix $0 < \varepsilon < 1/2$ and that $\|T - T^2\| = \varepsilon$. The solution to the inequality $|t - t^2| \leq \varepsilon$ is

$$\left(-\frac{1-\sqrt{1+4\varepsilon}}{2}, \frac{1-\sqrt{1-4\varepsilon}}{2}\right) \cup \left(\frac{1+\sqrt{1-4\varepsilon}}{2}, \frac{1+\sqrt{1+4\varepsilon}}{2}\right).$$

Let

$$f(\varepsilon) = \min\left|\frac{1 \pm \sqrt{1 \pm 4\varepsilon}}{2}\right|.$$

By CFC, with g as in the first part, $\|T - g(T)\| = f(\varepsilon)$. Clearly $\lim_{\varepsilon \to 0^+} f(\varepsilon) = 0$, as required. (One could work out a nicer-looking bound, but this was not the point of the exercise.)

(3) (Continuity of the continuous functional calculus.) Fix a continuous function $f\colon \mathbb{C} \to \mathbb{C}$. For a normal operator T on H, $f(T)$ is defined by the (obvious extension of) the continuous functional calculus. Prove that if T_n and T are normal operators such that $\lim_n \|T_n - T\| = 0$ then $\lim_n \|f(T_n) - f(T)\| = 0$.

Solution. First we prove that this is true when $f(x) = x^n$ for some $n \geq 1$. Fix n. We have (writing $T^0 = 1$)

$$T^n - S^n = \sum_{j=0}^{n-1} S^j(T - S)T^{n-1-j}$$

and therefore $\|T^n - S^n\| \leq \sum_{j=0}^{n-1} \|S - T\|\|S^j\|\|T^{n-1-j}\|$. If T is normal, then $\|T^j\| = \|T\|^j$, and therefore with $M = \max(\|S\|, \|T\|)$ we have $\|T^n - S^n\| \leq nM^{n-1}\|T - S\|$.

Now suppose $\|T_m - T\| \to 0$ as $m \to \infty$. Then $\|T_m\| \to \|T\|$, and in particular $M = \sup_m \max_{j<n} \|T_m\| < \infty$. By the above calculations $\|T_m^n - T^n\| < nM^{n-1}\|T_m - T\| \to 0$ as $n \to \infty$.

Also, $\|T_m^* - T^*\| = \|T_m - T\|$.

Let $p(x)$ be a complex *-polynomial such that x and x^* commute, hence $p(x)$ is a complex linear combination of terms of the form $x^m(x^*)^k$ for $m \geq 0$ and $k \geq 0$. By the continuity of the addition and scalar multiplication, $\|T_n - T\| \to 0$ implies $\|p(T_n) - p(T)\| \to 0$.

Finally, let X denote the closed disk of radius M (as above) centered at 0. Then $\sigma(T_n) \subseteq X$ for all n. By the Stone-Weierstrass theorem, the algebra of complex *-polynomials is norm-dense in $C(X)$. Fix $f \in C(X)$ and let $p_m(x)$ be a sequence of polynomials that converges to f uniformly on X. Then $\lim_n \|p_m(T_n) - p_m(T)\| = 0$. By the Continuous Functional Calculus, $\lim_m \|p_m(T) - f(T)\| = 0$, and $\lim_m \|p_m(T_n) - f(T_n)\| = 0$ for all n. Choosing $m(n) \to \infty$ such that both $\|p_{m(n)}(T_n) - f(T_n)\| < 1/n$ and $\|p_{m(n)}(T) - f(T)\| < 1/n$ for all n, $\|f(T_n) - f(T)\| \to 0$ follows.

(4) Suppose that X and Y are compact Hausdorff spaces and that $\Phi\colon C(X) \to C(Y)$ is a unital homomorphism (Φ is not assumed to be continuous).

(a) Prove that there is a function $\tilde{\Phi}\colon Y \to X$ such that $(\Phi(f))(y) = f(\tilde{\Phi}(y))$ for all $y \in Y$.

Solution. Since $\Phi\colon C(X) \to C(Y)$ is a bounded linear operator, we can consider its transpose $\Phi^*\colon C(Y)^* \to C(X)^*$ such that for all $f \in C(X)$ and $\varphi \in C(Y)^*$ we have

$$(\Phi(f), \varphi) = (f, \Phi^*(\varphi)).$$

Let $\tilde{\Phi}$ be the restriction of Φ to $\mathrm{Sp}(C(Y))$ (identified with Y). Then (using the definitions) for $y \in Y$ and $f \in C(X)$ we have

$$\Phi(f)(y) = (\Phi(f), y) = (f, \tilde{\Phi}(y)) = f(\tilde{\Phi}(y))$$

as required.

(b) Prove that $\tilde{\Phi}$ is continuous.

Solution. The transpose of a bounded linear operator is continuous in the weak operator topology.

(c) Prove that $\Phi(f^*) = \Phi(f)^*$ for all $f \in C(X)$.

Solution. For every $y \in Y$ we have $\Phi(f^*)(y) = f^*(\tilde\Phi(y)) = \overline{\tilde\Phi(f)(y))} = \overline{\Phi(f)(y)}$. Since this is true for all $y \in Y$, the identify follows.

(5) This is a continuation of Question (4), with Φ, X, Y, and $\tilde\Phi$ as before. In each of the following, first complete the sentence by inserting the right word (a property of a continuous map) and prove the assertion obtained in this way.

(a) Prove that Φ is an isomorphism if and only if $\tilde\Phi$ is....

Solution. The magic word is 'homeomorphism'. Since Φ is an isomorphism iff it is injective and surjective, and $\tilde\Phi$ is a homeomorphism iff it is injective and surjective, this will follow from the other two parts of the question.

(b) Prove that $\ker(\Phi) = \{0\}$ if and only if $\tilde\Phi$ is....

Solution. The magic word is 'surjective'. Suppose $\tilde\Phi$ is surjective. If $f \in C(X)$ is nonzero, fix $y \in Y$ such that $f(\tilde\Phi(y)) \neq 0$. Then $\tilde\Phi(f)(y) \neq 0$. Since f was arbitrary, $\ker(\tilde\Phi) = \{0\}$. Conversely, if $\tilde\Phi$ is not surjective, then $X \setminus \tilde\Phi[Y]$ is a nonempty open set. (Since Y is compact, its image is closed.) Fix $x \in X \setminus \tilde\Phi[Y]$. By the Tietze Extension Theorem find $f \in C(X)$ that vanishes on $\tilde\Phi[Y]$ such that $f(x) = 1$. Then $\Phi(f) = 0$, hence $\ker(\Phi)$ is nontrivial.

(c) Prove that Φ is surjective if and only if $\tilde\Phi$ is....

Solution. The magic word is 'injective'. Suppose $\tilde\Phi$ is injective. By the compactness of Y and of $X_0 = \tilde\Phi[Y]$, $\tilde\Phi$ is a homeomorphism between X_0 and Y. For $h \in C(Y)$ define $f_0\colon X_0 \to \mathbb{C}$ by $f_0 = h \circ \tilde\Phi^{-1}$. Then f_0 is continuous, and by the Tietze Extension Theorem it has a continuous extension to $f \in C(X)$. Then $\Phi(f) = f \circ \tilde\Phi = h$.

Conversely, suppose $\tilde\Phi$ is not injective and fix $y \neq y'$ in Y such that $\tilde\Phi(y) = \tilde\Phi(y')$. Then for every $f \in C(X)$, $\Phi(f)(y) = f \circ \tilde\Phi(y) = f \circ \tilde\Phi(y') = \Phi(f)(y')$. Since $C(Y)$ separates the points of Y and $\Phi[C(X)]$ does not, Φ is not surjective.

(6) Suppose that π is a representation of a Banach *-algebra A on H. Prove that the following are equivalent.

(a) Every nonzero $\xi \in H$ is cyclic for π.

(b) The only closed subspaces of H that are invariant[3] for $\pi(a)$ for every $a \in A$ are $\{0\}$ and H.

(c) If $T \in \mathcal{B}(H)$ commutes with $\pi(a)$ for all $a \in A$, then T is a scalar multiple of the identity.

Solution. A nontrivial invariant subspace cannot contain a cyclic vector (and it contains nonzero vectors). This easily implies that (6a) and (6b) are equivalent.

(6c) implies (6b): If K is a nontrivial closed invariant subspace, then the ortohogonal projection to K is an invariant nonscalar operator.

(6b) implies (6c): This is the only nontrivial implication. Suppose that (6c) fails. Let A be the subalgebra of $\mathcal{B}(H)$ consisting of all operators that commute with $\pi(a)$ for all $a \in A$.[4] A routine calculation shows that A is WOT-closed, and therefore a von Neumann algebra.

If all self-adjoint operators in A were scalar, then every operator in A would be scalar. THis is because $T = T_0 + iT_1$, for self-adjoint T_0 and T_1. We can therefore fix a self-adjoint, nonscalar, $T \in \pi[A]'$. Thus $\sigma(T) \subseteq \mathbb{R}$ and it has more than one point. Let U be a bounded open interval in $\mathbb{R}$ such that both $U \cap \sigma(T)$ and $\sigma(T) \setminus U$ are nonempty. Let $X = \sigma(T) \cap U$. Let $f_n\colon \mathbb{R} \to [0,1]$ be a sequence of continuous functions on converging to χ_U pointwise (Tietze).

By the Bounded Borel Functional Calculus, let $S = \chi_X(T)$.

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[3]A subspace K of H is invariant for T if $T[K] \subseteq K$.

[4]This is the *relative commutant* of $\pi[A]$, denoted by $\pi[A]'$; this is not used in the proof.

(7) Suppose that $T \in \mathcal{B}(H)$ is a normal operator. Prove that the range of $\mathcal{B}(\sigma(T))$ under the Bounded Borel Functional Calculus associated to T is a von Neumann algebra.

There is some overlap with the solution to Question (1).

Solution. Let $\pi\colon \mathcal{B}(\sigma(T)) \to \mathcal{B}(H)$ be the σ-representation given by the Bounded Borel Functional Calculus. Denote its range by N. Meanwhile, by the Spectral Theorem we may assume that $H = L^2(X, \mu)$ and $T \in L^\infty(X, \mu)$ for some probability measure space X. We will write

$$\mathcal{A} = L^\infty(X, \mu).$$

Let P be the spectral measure associated with π and let Borel(X) denote the σ-algebra of Borel subsets of $\sigma(T)$. Then for every $A \in \mathcal{B}$ we have $P(A) \in \mathcal{A}$.

This fact requires a proof. We will prove that the family $\mathcal{F} = \{A \in \text{Borel}(\sigma(T)) | P(A) \in \mathcal{A}\}$ includes all open sets and is closed under complements and countable unions. We will use the fact that if (S_n) is a sequence in $\mathcal{A}$ and $S = \text{SOT-}\lim S_n$, then $S \in \mathcal{A}$.

If A is open, then the characteristic function $1_A \in \mathcal{B}(X)$ is a pointwise limit of continuous functions. Since π is a σ-representation, $P(A) \in \mathcal{A}$. If $A = \bigcup_n A_n$, and $P(A_n) \in \mathcal{A}$ then again because π is a σ-representation $P(A) \in \mathcal{A}$. Clearly if $P(A) \in \mathcal{A}$ then $P(\mathbb{C} \setminus A) = 1 - P(A)$ is in $\mathcal{A}$.

Fix $\pi(f) \in N$. Then f is a pointwise limit of step functions f_n. Every step function f_n is a linear combination of characteristic functions 1_A, and therefore by the previous paragraph $\pi(f_n) \in \mathcal{A}$. Since π is a σ-representation, $\pi(f) \in \mathcal{A}$. Thus $\pi[\mathcal{B}(X)] \subseteq \mathcal{A}$.

Almost there. Fix S in the SOT-closure of N. We need to prove that $S \in N$.

We claim that S commutes with every element of $\mathcal{A}$. This is 'obvious' but let's prove it. For $T \in \mathcal{B}(H)$ the function $F_T(X) \mapsto TX - XT$ is SOT-continuous. Then F_T vanishes on $\mathcal{A}$ if and only if T commutes with $\mathcal{A}$.[5] Let (T_λ) be a net in N such that $S = \text{SOT-}\lim T_\lambda$. For every $T \in N$, F_T vanishes on $\mathcal{A}$ because $N \subseteq \mathcal{A}$ and $\mathcal{A}$ is abelian. Thus for every $R \in \mathcal{A}$ we have $F_S(R) = \text{SOT-}\lim F_{T_\lambda}(R) = 0$, and F_S vanishes on $\mathcal{A}$.

For a measurable $A \subseteq X$ write ξ_A for the characteristic function of A considered as a vector in $L^2(X, \mu)$[6] and write Q_A for the same function considered as a projection in $\mathcal{A}$.

Fix measurable $A \subseteq B \subseteq X$. Then

(1) $$Q_A S \xi_B = S Q_A \xi_B = S\xi_A.$$

We claim that $\eta = S\xi_X$ is in $L^\infty(X, \mu)$. (It is clearly in $L^2(X, \mu)$.) Otherwise, for every $m \in \mathbb{N}$ the set

$$A(m) = \{x \in X | |\eta(x)| \geq K\}$$

is not μ-null. But (1) implies $S\xi_{A(m)} = P_{A(m)}\eta$, thus $\|S\xi_{A(m)}\|_2 \geq m\|\xi_{A(m)}\|_2$ and $\|S\| \geq m$ for all m; contradiction.

Thus $\eta \in \mathcal{A}$. We claim that $S = M_f$ (the multiplication operator). For measurable $A \subseteq X$ we have $SQ_A = M_\eta Q_A$ by (1). By linearity, this equality also holds for linear combinations of projections in $\mathcal{A}$.

[5]This is also equivalent to $T \in \mathcal{A}$, but we don't need this.

[6]Then $\xi_A \in L^2(X, \mu)$ since we are assuming that μ is a probability measure, which is not a loss of generality.