Math 6462, W22. Assignment # 3. Due date: March 10, 11:59pm.

Choose any five questions. Clearly indicate which five, and *submit only their solutions.* If you submit solutions to more than five questions, I will grade only five of them.

Note that only Question 1 has some bonus points associated to it. If you submit solutions to parts (a) and (b) only, this counts as complete solution. For a correct solution to (c) you get bonus points.

In all questions, H stands for the separable infinite-dimensional Hilbert space.

(1) (a) (4pts) Find two self-adjoint operators on H with spectrum $[0,1]$ and empty point-spectrum that are not unitarily equivalent.
(b) (6pts) Find infinitely many self-adjoint operators on H with spectrum $[0,1]$ and empty point-spectrum that are pairwise unitarily inequivalent.
(c) (Bonus 5pts) Find uncountably many self-adjoint operators on H with spectrum $[0,1]$ and empty point-spectrum that are pairwise unitarily inequivalent.

(2) (a) (5pts) Suppose that T is a self-adjoint operator on H such that $\|T^2 - T\| < 1/4$. Prove that there is a projection in $\mathrm{C}^*(T)$ (the C^*-algebra generated by T).
(b) (5pts) Prove that there is a decreasing $f\colon (0,1/2) \to (0,1]$ that satisfies $\lim_{t\to 0} f(t) = 0$ and such that if T is a self-adjoint operator on H and $\|T^2 - T\| = \varepsilon < 1/2$, then there is a projection P in $\mathrm{C}^*(T)$ such that $\|P - T\| \leq f(\varepsilon)$.

(3) (Continuity of the continuous functional calculus.) Fix a continuous function $f\colon \mathbb{C} \to \mathbb{C}$. For a normal operator T on H, $f(T)$ is defined by the (obvious extension of) the continuous functional calculus. Prove that if T_n and T are normal operators such that $\lim_n \|T_n - T\| = 0$ then $\lim_n \|f(T_n) - f(T)\| = 0$.

(4) Suppose that X and Y are compact Hausdorff spaces and that $\Phi\colon C(X) \to C(Y)$ is a unital homomorphism (Φ is not assumed to be continuous).
(a) Prove that there is a function $\tilde{\Phi}\colon Y \to X$ such that $(\Phi(f))(y) = f(\tilde{\Phi}(y))$ for all $y \in Y$.
(b) Prove that $\tilde{\Phi}$ is continuous.
(c) Prove that $\Phi(f^*) = \Phi(f)^*$ for all $f \in C(X)$.

(5) This is a continuation of Question (4), with Φ, X, Y, and $\tilde{\Phi}$ as before. In each of the following, first complete the sentence by inserting the right word (a property of a continuous map) and prove the assertion obtained in this way.
(a) Prove that Φ is an isomorphism if and only if $\tilde{\Phi}$ is....
(b) Prove that $\ker(\Phi) = \{0\}$ if and only if $\tilde{\Phi}$ is....
(c) Prove that Φ is surjective if and only if $\tilde{\Phi}$ is....

(6) Suppose that π is a representation of a Banach *-algebra A on H. Prove that the following are equivalent.
(a) Every nonzero $\xi \in H$ is cyclic for π.
(b) The only closed subspaces of H that are invariant[1] for $\pi(a)$ for every $a \in A$ are $\{0\}$ and H.
(c) If $T \in \mathcal{B}(H)$ commutes with $\pi(a)$ for all $a \in A$, then T is a scalar multiple of the identity.

(7) Suppose that $T \in \mathcal{B}(H)$ is a normal operator. Prove that the range of $\mathcal{B}(\sigma(T))$ under the Bounded Borel Functional Calculus associated to T is a von Neumann algebra. (Hint: This is easy; you just need to recall the only result that we proved about von Neumann algebras so far.)

[1]A subspace K of H is invariant for T if $T[K] \subseteq K$.