Math 6462, W22. Assignment # 2 solutions

(1) With $\bar{\mathbb{D}} = \{z \in \mathbb{C} | |z| \leq 1\}$ let $\mathcal{A}$ be the subalgebra (not a Banach subalgebra!) of $C(\bar{\mathbb{D}})$ consisting of those functions that can be represented by the convergent series $\sum_{n=0}^{\infty} a_n z^n$ such that $\sum_n |a_n| < \infty$. Here z is the identity function on $\bar{\mathbb{D}}$.[1]

(a) (4pts) Verify that $\mathcal{A}$ is (algebraically) isomorphic to the Banach subalgebra $\ell^1(\mathbb{N})$ of $\ell^1(\mathbb{Z})$ defined as

$$\ell^1(\mathbb{N}) = \{x \in \ell^1(\mathbb{Z}) | x_n = 0 \text{ for all } n < 0\}.$$

(Recall that for me $0 \in \mathbb{N}$.)

Solution. Define $\Phi\colon \mathcal{A} \to \ell^1(\mathbb{N})$ by $\Phi(\sum_n a_n z^n) = (a_n)_{n\in\mathbb{N}}$. One verifies that Φ is a homomorphism by straightforward calculations. It is surjective because if $a \in \ell^1(\mathbb{N})$ then $\sum_n a_n z^n$ is absolutely convergent on $\bar{\mathbb{D}}$, and therefore in $\mathcal{A}$. This is an algebraic isomorphism between $\mathcal{A}$ and $\ell^1(\mathbb{N})$.

(b) (6pts) Prove that $f \in \mathcal{A}$ satisfies $f(z) \neq 0$ for all $z \in \bar{\mathbb{D}}$ if and only if $f \in \mathrm{GL}(\mathcal{A})$.

Solution. The salient point is that $\mathrm{Sp}(\ell^1(\mathbb{N}))$ is homeomorphic to $\bar{\mathbb{D}}$. If $\lambda \in \bar{\mathbb{D}}$, define $\varphi_\lambda\colon \ell^1(\mathbb{N}) \to \mathbb{C}$ by

$$\varphi_\lambda(\textstyle\sum_{n=0}^{\infty} a_n z^n) = \sum_{n=0}^{\infty} a_n \lambda^n.$$

This is a convergent series, and direct computation shows that φ_λ is a character on $\ell^1(\mathbb{N})$. If $\varphi \in \mathrm{Sp}(\ell^1(\mathbb{N}))$, let $\lambda_\varphi = \varphi(z)$. Then $|\lambda_\varphi| \leq \|z\| = 1$, thus $\lambda_\varphi \in \bar{\mathbb{D}}$. A direct computation shows that for $\varphi \in \mathrm{Sp}(\ell^1(\mathbb{N}))$, $\varphi_{\lambda_\varphi}$ and φ agree on finitely-supported sums in $\ell^1(\mathbb{N})$. This is a dense subset of $\ell^1(\mathbb{N})$, hence $\varphi = \varphi_{\lambda_\varphi}$. On the other hand, for $\lambda \in \bar{\mathbb{D}}$ we obviously have $\lambda_{\varphi_\lambda} = \lambda$.

Therefore $\varphi \mapsto \lambda_\varphi$ is a bijection between $\mathrm{Sp}(\ell^1(\mathbb{N}))$ and $\bar{\mathbb{D}}$. Being evaluation at a fixed element of $\ell^1(\mathbb{N})$, this map is continuous. It is therefore, as a continuous bijection between compact Hausdorff spaces, a homeomorphism.

Thus $f \in \mathcal{A}$ is invertible iff $\Phi(f) \in \ell^1(\mathbb{N})$ is invertible if and only if $\sum_{n=0}^{\infty} \Phi(f)_n \lambda^n \neq 0$ for all $\lambda \in \bar{\mathbb{D}}$. But $\sum_{n=0}^{\infty} \Phi(f)_n \lambda^n = f(\lambda)$, proving the claim.

(2) (10pts) (Continuing (1).) Let S be the isometric shift on $\ell^1(\mathbb{N})$, defined by $S(x)_{n+1} = x_n$ for all $n \in \mathbb{N}$, $S(x)_0 = 0$. Prove that for every $x \in \ell^1(\mathbb{N})$, the set of translates $\{S^n x | n \in \mathbb{N}\}$ spans a dense subset of $\ell^1(\mathbb{N})$ if and only if the power series $f(z) = \sum_{n=0}^{\infty} x_n z^n$ has no zeros on $\bar{\mathbb{D}}$.

Solution. Let $\mathcal{S}$ denote the linear span of $\{S^n x | n \in \mathbb{N}\}$.

By (1), $f(z) = \sum_{n=0}^{\infty} x_n z^n$ has no zeros on $\bar{\mathbb{D}}$ if and only if $f(z) \in \mathrm{GL}(\mathcal{A})$ if and only if $x \in \mathrm{GL}(\ell^1(\mathbb{N}))$. It therefore suffices to prove that the linear span of $\mathcal{S}$ is dense in $\ell^1(\mathbb{N})$ if and only if $x \in \mathrm{GL}(\ell^1(\mathbb{N}))$. Here is a proof.

If the linear span of $\mathcal{S}$ is dense in $\ell^1(\mathbb{N})$, then there is a finite linear combination $y = \sum_n \lambda_n S^n x$ such that $\|y - 1\| < 1$. Therefore y is invertible, and (since $\ell^1(\mathbb{N})$ is abelian) so is x.

Conversely, if x is invertible, then $x^{-1} \in \overline{\mathcal{S}}$. Let $y_m \in \mathcal{S}$ be such that $\|y_m - x^{-1}\| < 1/m$. Every finitely-supported element of $\ell^1(\mathbb{N})$ is of the form $w = \sum_n \lambda_n S^n = \sum_n \lambda_n S^n x^{-1} x$. For every m, $w_m = (\sum_n S^n) y_m \in \mathcal{S}$, and $\lim_m w_m x = w$. Since w was an arbitrary finitely supported element and such elements are dense in $\ell^1(\mathbb{N})$, the conclusion follows.

(3) (10pts) Let $M_n(\mathbb{C})$ denote the Banach algebra of $n \times n$ complex matrices, with respect to the operator norm. Suppose that A is a unital Banach algebra and $n \geq 2$. Prove that every surjective homomorphism $\varphi\colon A \to M_n(\mathbb{C})$ is continuous.[2]

[1] There was a misleading typo in this question, stating that $\mathcal{A}$ was a Banach subalgebra. It is not; it is a proper dense subalgebra. Please let me know if this affected you.

[2] The special bonus offer is still valid: Is the assertion true without the assumption that φ is onto?

Solution. Let $J = \ker(\varphi)$. Then (by an isomorphism theorem from algebra) A/J is isomorphic to $M_n(\mathbb{C})$. Since $M_n(\mathbb{C})$ is simple, so is A/J and therefore J is a maximal ideal.

[This was mentioned in class in passing; in case I did not provide a proof, here it is: Suppose J is not a maximal ideal. Since A is unital, there is a maximal proper ideal $I \supseteq J$. Then $\varphi[I]$ is a proper ideal in $M_n(\mathbb{C})$; contradiction.]

Since A is unital, every maximal ideal is closed. Therefore the quotient map $\pi\colon A \to A/J$ is continuous (a basic Banach space theory fact). Let $\tilde{\varphi}$ denote the isomorphism between A/J and $M_n(\mathbb{C})$ given by $\tilde{\varphi}(a+J) = \varphi(a)$. Then $\varphi = \tilde{\varphi} \circ \pi$, and it suffices to check that $\tilde{\varphi}$ is continuous. But it is an isomorphism between finite-dimensional (more precisely, n^2-dimensional) Banach spaces. Since all norms on a finite dimensional space are equivalent, $\tilde{\varphi}$ is continuous.

(4) (10pts) Let H be $\ell^2(\mathbb{N})$. Prove that the adjoint operation $A \mapsto A^*$ on $\mathcal{B}(H)$ is continuous in the weak operator topology (WOT), but not in the strong operator topology(SOT).

Solution. It suffices to prove the continuity at 0. WOT-subbasic neighbourhoods of 0 in $\mathcal{B}(H)$ are of the form $U_{\xi,\eta} = \{a | |(a\xi, \eta)| < \varepsilon\}$ for $\varepsilon > 0$ and ξ and η in H. The preimage of $U_{\xi,\eta}$ under the adjoint map is $\{a | |(a^*\xi, \eta)| < \varepsilon\} = \{a | |(\xi, a\eta)| < \varepsilon\} = \{a | |(a\eta, \xi)| < \varepsilon\} = U_{\eta,\xi}$.

Since ε, ξ and η were arbitrary, this proves that the preimage of every WOT-subbasic open neighbourhood of 0 is WOT-open, hence the map is continuous at 0, Since it is conjugate linear, it is WOT-continuous.

For the second part, let S denote the shift with respect to the standard orthonormal basis (ξ_j) of H, and let $a_n = (S^*)^n$ for $n \in \mathbb{N}$. Then $a_n(\xi_j) = \xi_{j-n}$ if $n \leq j$ and $a_n(\xi_j) = 0$ if $n > j$. Thus for every vector $\eta \in H$ we have $\lim_n \|a_n(\eta)\| = 0$. (If η has finite support, then $a_n(\eta) = 0$ for a large enough n. Every η can be approximated arbitrarily well by a vector with a finite support, and therefore $a_n(\eta) \to 0$ as $n \to \infty$.)

Thus $\text{SOT-}\lim_n a_n = 0$.

On the other hand, $a_n(\xi_0) = \xi_n$, and therefore $\lim_n a_n(\xi_0)$ does not exist (as a matter of fact, $\lim_n a_n(\eta)$ exists iff $\eta = 0$). But $\text{SOT-}\lim_n a_n = b$ is equivalent to $\lim_n a_n(\eta) = b(\eta)$ for all η. In particular, $\text{SOT-}\lim_n (a_n)^{**} \neq 0^*$ and the adjoint is not SOT-continuous.

(5) (10pts) Suppose that A is a commutative, unital, Banach algebra. Prove that the Gelfand map is an isometry if and only if $\|x\|^2 = \|x^2\|$ for all $x \in A$.

Solution. Since $\|x^2\| = \|x\|^2$ is clearly true in $C(X)$, if Γ_A is an isometry then $\|x\|^2 = \|x^2\|$ for all $x \in A$.

For the converse, assume $\|x^2\| = \|x\|^2$ for all $x \in A$. By induction this implies that $\|x^{2^n}\| = \|x\|^{2^n}$. Therefore $r(x) = \lim_n \|x^n\|^{1/n} = \lim_n \|x^{2^n}\|^{2^{-n}} = \|x\|$. Since $r(\Gamma_A(x)) = \|\Gamma_A(x)\|$, this implies $\Gamma_A(x)$ is an isometry.

(6) (10pts) Suppose that A is a unital Banach algebra generated by 1 and x. Prove that $\mathbb{C} \setminus \sigma_A(x)$ is connected.

Solution. Suppose otherwise. Let U be a bounded connected component of $\mathbb{C} \setminus \sigma(x)$. Fix $\lambda \in U$. Since A is generated by x and 1, there are complex polynomials $P_n(z)$ such that $\lim_n \|P_n(x)(x - \lambda)\| = 1$.

The Gelfand transform Γ sends A into a subalgebra of $C(\sigma(x))$. It sends x to the identify function on $\sigma(x)$ and a polynomial $\sum_n \lambda_n x^n$ to the restriction of the complex polynomial $\sum_n \lambda_n z^n$ to $\sigma(x)$. Thus $Q_n(z) = \Gamma(P_n(x)(x - \lambda))$ are complex polynomials that uniformly converge to 1 on $\sigma(x)$. Also $Q_n(\lambda) = 0$ for all n. By compactness of $\sigma(x)$, for a large n we have $1 - Q_n(\nu) < 1/2$ for all $\nu \in \sigma(x)$ but $1 - Q_n(\lambda) = 0$. In particular, Q_n attains maximum on $\sigma(x) \cup U$ in the interior of this set. This contradicts the maximum modulus principle.

(7) Let H be $\ell^2(\mathbb{N})$.

(a) (5pts) Suppose that $(U_\lambda)_\lambda$ is a SOT-convergent net of unitaries, with limit A. Prove that A is an isometry.

Solution. We have SOT-$\lim_n U_n = A$ if and only if $\lim_n U_n(\eta) = A(\eta)$ (this is the norm limit) for all $\eta \in H$. But $\|U_n(\eta)\| = \|\eta\|$ for all n and η, and therefore $\|A(\eta)\| = \eta$ for all η—but this means that A is an isometry.

(b) (5pts) Show that a SOT-limit of unitaries is not necessarily a unitary. (Hint: Consider the unilateral shift on $\ell^2(\mathbb{N})$, defined by $A(\xi)_{n+1} = \xi_n$ for $n \in \mathbb{N}$ and $A(\xi)_0 = 0$. Find a sequence of unitaries U_n that SOT-converges to A.

Solution. We can choose U_n to be a permutation unitary, i.e., a unitary that permutes the basis vectors. Let $U_n(\xi_j) = \xi_{j+1}$ if $j < n$, $U_n(\xi_n) = \xi_0$, and $U_n(\xi_j) = \xi_j$ if $j > n$. This is a unitary. For every basis vector ξ_j, we have $U_n(\xi_j) = \xi_{j+1}$ for all large enough n. Therefore $\lim_n U_n(\xi_j) = \xi_{j+1}$. By linearity, one verifies that $\lim_n U_n$ and A agree on the finite linear combinations of basis vectors. Since this is a dense set, we conclude that $\lim_n U_n(\eta) = A(\eta)$ for all η, and therefore SOT-$\lim_n U_n = A$ as required.