Math 6462, W22. Assignment # 2. Due date: February 13, 11:59pm.

Choose any five questions. Clearly indicate which five, and *submit only their solutions.* If you submit solutions to more than five questions, I will grade only five of them.

(1) With $\bar{\mathbb{D}} = \{z \in \mathbb{C} | |z| \leq 1\}$ let $\mathcal{A}$ be the Banach subalgebra of $C(\bar{\mathbb{D}})$ consisting of those functions that can be represented by the convergent series $\sum_{n=0}^{\infty} a_n z^n$ such that $\sum_n |a_n| < \infty$. Here z is the identity function on $\bar{\mathbb{D}}$.

   (a) (4pts) Verify that $\mathcal{A}$ is isomorphic to the Banach subalgebra $\ell^1(\mathbb{N})$ of $\ell^1(\mathbb{Z})$ defined as

$$\ell^1(\mathbb{N}) = \{x \in \ell^1(\mathbb{Z}) | x_n = 0 \text{ for all } n < 0\}.$$

   (Recall that for me $0 \in \mathbb{N}$.)

   (b) (6pts) Prove that $f \in \mathcal{A}$ satisfies $f(z) \neq 0$ for all $z \in \bar{\mathbb{D}}$ if and only if $f \in \mathrm{GL}(\mathcal{A})$.

(2) (10pts) (Continuing (1).) Let S be the isometric shift on $\ell^1(\mathbb{N})$, defined by $S(x)_{n+1} = x_n$ for all $n \in \mathbb{N}$, $S(x)_0 = 0$. Prove that for every $x \in \ell^1(\mathbb{N})$, the set of translates $\{S^n x | n \in \mathbb{N}\}$ spans a dense subset of $\ell^1(\mathbb{N})$ if and only if the power series $f(z) = \sum_{n=0}^{\infty} x_n z^n$ has no zeros on $\bar{\mathbb{D}}$.

(3) (10pts) Let $M_n(\mathbb{C})$ denote the Banach algebra of $n \times n$ complex matrices, with respect to the operator norm. Suppose that A is a unital Banach algebra and $n \geq 2$. Prove that every homomorphism $\varphi \colon A \to M_n(\mathbb{C})$ is continuous.

(4) (10pts) Let H be $\ell^2(\mathbb{N})$. Prove that the adjoint operation $A \mapsto A^*$ on $\mathcal{B}(H)$ is continuous in the weak operator topology (WOT), but not in the strong operator topology(SOT).

(5) (10pts) Suppose that A is a commutative, unital, Banach algebra. Prove that the Gelfand map is an isometry if and only if $\|x\|^2 = \|x^2\|$ for all $x \in A$.

(6) (10pts) Suppose that A is a unital Banach algebra generated by 1 and x. Prove that $\mathbb{C} \setminus \sigma_A(x)$ is connected.

(7) Let H be $\ell^2(\mathbb{N})$.

   (a) (5pts) Suppose that $(U_\lambda)_\lambda$ is a SOT-convergent net of unitaries, with limit A. Prove that A is an isometry.

   (b) (5pts) Show that a SOT-limit of unitaries is not necessarily a unitary. (Hint: Consider the unilateral shift on $\ell^2(\mathbb{N})$, defined by $A(\xi)_{n+1} = \xi_n$ for $n \in \mathbb{N}$ and $A(\xi)_0 = 0$. Find a sequence of unitaries U_n that SOT-converges to A.