Math 6462, W22,
Homework 1 solutions.

(1) $sp(f) = range(f)$.

$\supseteq$: If $\lambda \notin sp(f)$, let $(f-\lambda)^{-1} = g$,
then $g(x)(f(x)-\lambda) = 1$ for all $x$, hence
$f(x) - \lambda \neq 0$, and $\lambda \notin range(f)$.

$\subseteq$ If $\lambda \notin range(f)$ then since $X$ is
cpct and $f$ is cts, there is $\varepsilon > 0$
such that $B_\varepsilon(\lambda)$ (= the $\varepsilon$-ball centered at $\lambda$)
is disjoint from $f[X]$. Thus $g(x) = 1/(f(x)-\lambda)$, $x \in X$
is $(f-\lambda)^{-1}$.

(2a) $X = \{0\} \cup [1, \infty)$ (Some may insist on $X = [1,\infty)$, see below)

(b) $1 \in X$ - take any normalized unital
Banach algebra $A$ e.g., $C([0,1])$,
For $r \geq 1$, on $A$ define $|||a||| = r\|a\|$.
Then $|||\cdot|||$ is a Banach space norm
equivalent to $\|\cdot\|$ because $\frac{1}{r}|||\cdot||| \leq \|\cdot\| \leq r|||\cdot|||$.
It is also submultiplicative, since $r^2 \geq r$.
So $r \in X$, and $[r, \infty) \subseteq X$.
clearly $X \subseteq [0, \infty)$. Fix $0 < r < 1$, and
assume $\|1_A\| = r$ for some $A$.

Then $r = \|1_A\| \leq \|1_A\| \cdot \|1_A\| = r^2$ - Contradiction.
Finally, $\|1_A\| = 0 \Rightarrow 1_A = 0$, so $1_A = 0$.
and i $A = \{0\}$. (Some people allow this, and I don't care.)

ⓒ No. Any unital Banach algebra can be renormalized so that $\|1_A\| = 1$ (proved in class) and the proof of ⓑ shows that it can be further renormalized to have $\|1_A\| = r$ for any $r \in X$.
(If you answered $X = [1, \infty)$ to ⓐ, then the correct answer to ⓒ is "no".)

③ ⓐ Fix $\xi \in l_2(\mathbb{N})$, $\xi \neq 0$. We claim $\xi$ is not an eigenvector for $S$. Otherwise, $S\xi = \lambda \xi$ for some $\lambda$. ~~Replacing $\xi$ with $\xi/\|\xi\|_2$, we may assume $\|\xi\|_2 = 1$.~~ Since $\ker(S) = \{0\}$, $\lambda \neq 0$. Let $n$ be minimal such that $\xi_n \neq 0$.
But $(S^n \xi)_n = 0$, and $(S^n \xi)_n = \lambda^n \xi_n \neq 0$ - contradiction.

ⓑ Fix $\lambda$, $|\lambda| \leq 1$. Fix $\varepsilon > 0$, and $n > 2/\varepsilon^2$.
Let $\xi = \left(\frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, \frac{n}{n}, \frac{n-1}{n}, \dots, \frac{1}{n}, 0, 0, \dots\right)$
(I.e., $\xi_k = \frac{k+1}{n}$ for $0 \leq k \leq n-1$,
$\xi_k = \frac{2n-k}{n}$, for $n \leq k \leq 2n$
$\xi_k = 0$, for $k > 2n$.)

Then $S\xi - \xi = (\underbrace{-\tfrac{1}{n}, -\tfrac{1}{n}, \dots, -\tfrac{1}{n}}_{n \text{ times}}, \underbrace{\tfrac{1}{n}, \dots, \tfrac{1}{n}}_{n \text{ times}})$

So $\|S\xi - \xi\|_2 = \sqrt{\frac{2n}{n^2}} < \varepsilon$.

Since $\varepsilon > 0$ was arbitrary, this implies $(S-1)$ is not invertible! If $T(S-1)=1$ then $\|\xi\| = \|T(S-1)\xi\| > \varepsilon \|(S-1)\xi\|$, contradicting $\|T\| < \infty$.

ⓒ $\|S\| = 1$, hence $Sp(S) \subseteq \{\lambda \mid |\lambda| \leq 1\}$. To prove the converse inclusion, fix $\lambda$, $|\lambda| \leq 1$. Fix $\varepsilon > 0$, and $n > 2/\sqrt{\varepsilon}$. Let

$$\xi = \left(\tfrac{1}{n}, \tfrac{2}{n}\bar\lambda, \tfrac{3}{n}\bar\lambda^2, \dots, \tfrac{n}{n}\bar\lambda^{n-1}, \tfrac{n-1}{n}\bar\lambda^{n}, \dots, \tfrac{1}{n}\bar\lambda^{2n}, 0, 0, \dots\right)$$

As in ⓑ, $\|S\xi - \lambda\xi\| = \sqrt{\frac{2}{n}} < \varepsilon$, and $\lambda \in Sp(S)$.

④ ⓐ Fix $x$, $\|x\| < 1$. Let $f : [0,1] \to GL(A)$
$f_1(r) = 1 - rx$
be $f_0(r) = (1 - rx)^{-1}$. Since $\|rx\| = |r|\|x\| < 1$, $f_0, f_1$ is well-defined. Since $(\cdot)^{-1}$ is cts, and
$f_1(0) = 1_A$, $f_1(r) = f_0(r)^{-1}$, $f_1(1) = (1-x)$,
$f_0(0) = 1_A$, $f_0(1) = (1-x)^{-1}$, ⓐ follows.

ⓑ we'll prove that the path component of $x \in GL(A)$ is open in $A$.
As we proved in class, $x \in GL(A)$ and $\|h\| < \|x^{-1}\|^{-1}$ implies $x + h \in GL(A)$. Thus
$Y = \{y \in A \mid \|x - y\| < \|x^{-1}\|^{-1}\} \subseteq GL(A)$, and for $y \in Y$
$f : [0,1] \to GL(A)$ defined by $f(r) = x - r(x - y)$
is a cts map from $[0,1]$ into $GL(A)$ (proved as in ⓐ), $f(0) = x$, and $f(1) = y$.

(c) Let $\Gamma = \{ \prod_{j=1}^{n} (1-x_j)^{-1} \mid n \geq 1, \ \|x_j\| < 1, \text{ for } j \leq n \}$.

By (b), $\Gamma \subseteq G_0$.

To prove $G_0 \subseteq \Gamma$, fix $x \in G_0$ and $f : [0,1] \to GL(A)$ that is ctns, $f(0) = 1$, and $f(1) = x$.

Let $g : [0,1] \to GL(A)$ be $g(r) = f(r)^{-1}$; then $g$ is ctns.

Let $M = \max_{r \in [0,1]} \|g(r)\|$. Fix $n > M$. For $0 \leq k \leq n$
such that $|r-s| \leq \frac{1}{n} \Rightarrow \|f(r) - f(s)\| < 1/M$

let $x_k = f(k/n)$. Then $x_0 = 1$, $x_n = f(1)$, and

$\|x_k - x_{k+1}\| < \frac{1}{M} \leq \|x_k^{-1}\|^{-1}$. Thus (with $h_k = x_k - x_{k+1}$)

$x_{k+1} = x_k - h_k = x_k(1 - x_k^{-1} h_k)$, with $\|x_k^{-1} h\| < 1$.
since $\|x_k^{-1}\|^{-1} \leq M$

This implies that (with $y_k = x_k^{-1} h_k$, $0 \leq k \leq n-1$,
we have $f(1) = (1-y_0)(1-y_1) \cdots (1-y_{n-1})$
as required.