Math 6462, W22. Assignment # 1. Due date: January 31, 11:59pm.

(1) (10pts) Consider the unital Banach algebra $C([0,1])$. Prove that the spectrum of $f \in C([0,1])$ is equal to the range of f. (Note that this is not about $[0,1]$—the analogous result holds when $[0,1]$ is replaced with any compact Hausdorff space.)

(2) (a) (3pts) Describe the set $X = \{\|1_A\| : A \text{ is a unital Banach algebra}\}$. (The answer should be a concrete description of a set of reals, such as $(-3,-2) \cup \{0\}$.[1]

(b) (3pts) Prove that your description of the set X is correct. More precisely, prove that there is a Banach algebra A such that $\|1_A\| = r$ if and only if $r \in X$, for every $r \in \mathbb{R}$.

(c) (4pts) For a unital Banach algebra A, consider the set $X_A = \{|||1_A||| : |||\cdot||| \text{ is a norm on } A \text{ equivalent to the original one}\}$. Clearly $X_A \subseteq X$ for every A. Is there A such that $X_A \neq X$? Prove your claim.

(3) In $\ell_2(\mathbb{N})$, for a vector ξ let ξ_n denote its n-th coordinate (in this course $0 \in \mathbb{N}$). Define $S \in B(\ell_2(\mathbb{N}))$ by $S(\xi)_{n+1} = \xi_n$ for all n and $S(\xi)_0 = 0$. This is a bounded linear operator (convince yourself that this is so, although it is not a part of the question).

(a) (3pts) Prove that $\sigma_p(S) = \emptyset$ (i.e., S has no eigenvalues).

(b) (3pts) Prove that $1 \in \sigma(S)$.

(c) (4pts) Prove that $\lambda \in \sigma(S)$ if and only if $|\lambda| \leq 1$.

(4) Suppose that A is a unital Banach algebra. Let G_0 be the path-connected component of 1_A in $\mathrm{GL}(A)$. (That is, G_0 is the set of all $a \in \mathrm{GL}(A)$ such that there is a continuous function $f\colon [0,1] \to \mathrm{GL}(A)$ such that $f(0) = 1_A$ and $f(1) = a$.)

(a) (3pts) Prove that if $\|x\| < 1$ then $(1-x)^{-1} \in G_0$.

(b) (3pts) Prove that G_0 is open in $\mathrm{GL}(A)$.

(c) (4pts) Prove that G_0 is equal to the subgroup of all products of the form $\prod_{j=1}^{n}(1-x_j)^{-1}$, for $n \geq 1$ and $x_1, \dots, x_n$ of norm < 1.

[1] Clearly this is not the correct answer