COMBINATORIAL SET THEORY FOR C*-ALGEBRA ERRATA

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I am indebted to Sina Adineh, Bruce Blackadar, Takeshi Katsura, Luciano Salvetti, and Fanxin Wu for pointing out to some mistakes that mysteriously crept into my book while I wasn't looking. The proofs that were included in their entirety were not entirely wrong. Even if there were a few minor changes in a proof, I found it that writing out the complete (and completely correct) proof appeared to be the right way to go. This is the list of all errors in [5] (and their corrections) known to me. It includes clarifications of some of the obscurities found in the text, as well as more elegant rewritings of some of the (correct) proofs. The list will be updated as needed.

In the last line of the proof of Lemma 1.1.2. $\|c\| = 1$ should be $\|c\| \le 1$.

P. 6, line 6: 'if $x \ge \varepsilon$' should be 'for almost all x.'

P. 6, line -10: $\kappa \times \kappa$ should be $\mathbb{J} \times \mathbb{J}$.

Lemma 1.2.8. x should be ξ.

Lemma 1.3.4. The statement should be "Every homomorphism $\Phi \colon A \to C$ between abelian C*-algebras is necessarily a *-homomorphism."'

A stronger version of Lemma 1.3.4, asserting that every homomorphism from an abelian C*-algebra into an arbitrary C*-algebra is a *-homomorphism, is false as the following example communicated to me by Takeshi Katsura shows. Define $\Phi \colon \mathbb{C}^2 \to M_2(\mathbb{C})$ by $\Phi((x,y)) = \begin{pmatrix} x & x-y \\ 0 & y \end{pmatrix}$. This is a homomorphism, but not a *-homomorphism.

Lemma 1.4.8. Here *is a version of this lemma that is both correct and useful.*

Lemma 1.4.8

(1) For every n and *-polynomial $f(\bar{x})$ in non-commuting variables $x_0, \ldots, x_{n-1}$ there is a universal constant K_f such that every C*-algebra A, every n-tuple of contractions $\bar{a}$ in A, and every $b \in A$ satisfy

$$\|[f(\bar{a}),b]\| \le K_f \sum_{j<n}(\|[a_j,b]\| + \|[a_j^*,b]\|).$$

(2) For every compact $S \subseteq \mathbb{C}$ and every $f \in C(S)$ there is a constant $K_f < \infty$ such that every unital C*-algebra A, every normal $a \in A$ with $\mathrm{sp}(a) \subseteq S$, and every $b \in A$ satisfy $\|[f(a),b]\| \le K_f(\|[a,b]\| + \|[a^*,b]\|)$.

Proof. A telescoping calculation shows that $[a^k, b] = \sum_{j=0}^{k-1} a^{k-1-j}[a,b]a^j$, hence $\|[a^k, b]\| \leq k\|[a,b]\|$ (using $\|a\| \leq 1$). A similar calculation shows that if c_i, for $i < m$, are contractions then $\|[\prod_{i<m} c_i, b]\| \leq \sum_{i<m} \|[c_i, b]\|$. Since $f(\bar{x})$ is the sum of monomials in x_j and x_j^*, for $j < n$, taking K_f to be the sum of all $k|\alpha|$ such that α is a coefficient of a monomial of the k-th degree in f suffices.

For the second part, if $L = \max\{|z| | z \in S\} \leq 1$ then use the Stone–Weierstrass theorem to approximate f by a polynomial, and then use the first part of this lemma with $n = 1$. The assumption that the algebra A be unital is used to assure that $C^*(a, 1) \cong C(\mathrm{sp}(a))$ is included in A. If $L > 1$, then renormalize prior to using this argument.

Lemma 1.5.7 (4). e should be ε.

The proof of Corollary 1.5.8, second line. $\|p_i - p_{i+1}\| < 2$ should be $\|p_i - p_{i+1}\| < 1$.

In the paragraph following Definition 1.6.7. ... countable directed set without a maximal element has a cofinal subset isomorphic to $(\mathbb{N}, \leq)$.

The proof of Corollary 1.6.12. x_n should be $a^{1/2}((b_1^* b_1) + 1/n)^{-1} b_1^* b_1$, and the remainder of the proof should be changed by replacing a with $a^{1/2}$. This also makes the last sentence unnecessary.

The proof of Corollary 1.6.13. In first and second lines of the proof, $vf(|b|)$ should be $vg(|b|)$. Also, a period is missing at the end of this line.

The proof of Corollary 1.8.1., line 2: $\|a(1 - e_\lambda)\|$ should be $\|a(1 - e_\lambda)\|^2$.

Definition 1.9.1. In the second line, delete 'for $n \in \mathbb{N}$'.
At the end, $\lambda \leq \mu$ should be $\lambda < \mu$.

Lemma 1.9.2. $\mathscr{B}(H)^n$ should be $\mathscr{B}(H)$.

Exercise 1.11.6. The hint is misleading; $\mathrm{sp}_B(b) \supseteq \mathrm{sp}_A(b)$ is trivial. A better hint would suggest to first prove that $\mathrm{GL}(B)$ is a relatively clopen subset of $\mathrm{GL}(A)$.

Exercise 1.11.33: In 1, it is not necessary to assume that the C*-algebra be abelian.
In 2, 'scalar projections' should be 'nonzero scalar projections'.
The solution suggested by the hint is ok, but it is not the most straightforward one; just ignore the hint.

Exercise 1.11.39. In the hint, add 'identify H with $K \oplus K$'

Add Exercise 1.11.39$\frac{1}{2}$. If a is a contraction, prove that

$$a(1 - a^* a)^{1/2} = (1 - aa^*)^{1/2} a.$$

Exercise 1.11.40. In the hint, 'identify H with $K \oplus K$ and try $\begin{pmatrix} a & \sqrt{1 - aa^*} \\ \sqrt{1 - a^* a} & -a^* \end{pmatrix}$'.

Exercise 1.11.41 (5). "If $a_i a_j = 0$" should be "If $a_i^* a_j = 0$".
Also, there is a square missing in the last formula: $\sum_{j<n} \|a_j^* b_j\|^2$.

p. 48, line 8: $\bigoplus_{j\in\mathbb{J}}$ should be $\bigoplus_{j\in\mathbb{J}} A_j$.

§2.1.3, line 2. 'with is a' should be 'with a'

§2.1.5, line 13: Right parenthesis missing in $\mathscr{K}(\ell_2(\mathbb{N}))$

The proof of Lemma 2.2.4, line $-4.$: '$M_n(\mathbb{C})\odot M_{m/n}(\mathbb{C})$' should be '$M_m(\mathbb{C})\odot M_{n/m}(\mathbb{C})$'.

Lemma 2.3.1. (*) should be (C).

Lemma 2.4.4. 'minimal projection in $M_n(\mathbb{C})$'

Example 2.4.11 (5). $a \in A$ is identified with $\sum_{i<n}\alpha_{g(i)}^{-1}(a)\otimes e_{i,i}$, where $g(i)$, for $i < n$, is an enumeration of Γ.

page 72, line 7. implies $(1-d)r \approx_\varepsilon 0$.

Proposition 3.2.11. (Starting with the paragraph preceding it.)
One can recover a simple C*-algebra from its image under a c.p.c. order zero map with a trivial kernel.

Proposition 3.2.11 *Suppose A is a unital* C*-*algebra, B is a* C*-*algebra, and* $\varphi: A \to B$ *is a c.p.c. map of order zero. Then there exist a* C*-*algebra C and a completely positive map* $\psi: \varphi[A] \to C$ *such that* $\psi \circ \varphi: A \to C$ *is a* *-*homomorphism whose kernel is equal to* $\ker(\varphi)$. *In particular, if* $\varphi[A] \neq \{0\}$ *and A is simple then* $\psi \circ \varphi[A]$ *is isomorphic to A.*

Proof. We may assume that $B = C^*(\varphi[A])$. Theorem 3.2.9 implies that there are a positive contraction $h \in \mathscr{M}(B)$ and a *-homomorphism $\pi: A \to \{h\}' \cap \mathscr{M}(B)$ such that $\varphi(a) = h\pi(a)$. We will have $C := \pi[A]$.

Since h is a positive contraction, the sequence $h^{1/n}$, for $n \in \mathbb{N}$, is an approximate unit for $\mathscr{M}(B)$ and it strictly converges to 1. This sequence belongs to the commutant of C. For a and b in A and $n \geq 1$ we have $h\pi(a) = h\pi(b)$ if and only if $h^{1/n}\pi(a) = h^{1/n}\pi(b)$. Therefore the map $\psi_n(ha) := h^{1/n}a$ is well-defined. It is clearly completely positive. The sequence $\psi_n(ha)$ strictly converges to a, and therefore the map $\psi(ha) = a$ is also completely positive and well-defined. Since the sequence $h^{1/n}$ is increasing, we have $\ker\psi = \ker\psi_n = \ker\varphi$ for all n.

For $a \in A$ we have $\psi \circ \varphi(a) = \psi(h\pi(a)) = \pi(a)$ and $\psi \circ \varphi$ is a unital *-homomorphism on A whose kernel is $\ker(\varphi)$.

The proof of Lemma 3.4.3, line 3. $\|BB^*\xi\| \leq \|(1-AA^*)\xi\|$ should be $(BB^*\xi|\xi) \leq ((1-AA^*)\xi|\xi)$.

The proof of Proposition 3.6.5, (1) implies (2): *No errors here, but an additional line of explanation could be helpful. By Lemma 1.7.4. (3), a positive functional* θ *satisfies* $\|\theta\| = \sup\{\theta(a)|0 \leq a \leq 1\}$. *Because of this, using the notation from the proof of 3.6.5 (1)* $\Rightarrow$ *(2),* $\varphi = \psi + (\varphi - \psi)$ *implies* $\|\varphi\| = \|\psi\| + \|\varphi - \psi\|$, *and therefore* φ *is a convex combination of the states* $\frac{1}{\|\psi\|}\psi$ *and* $\frac{1}{\|\varphi-\psi\|}(\varphi - \psi)$.

Theorem 3.7.2. *The theorem as stated is not due to Glimm. What's worse, unlike Glimm's theorem it is false.*[1] *Here is the actual statement of Glimm's theorem.*

Theorem 3.7.2 (Glimm). *Every non-type-I* C^**-algebra has a* C^**-subalgebra B that has a quotient isomorphic to the CAR algebra.*

   The proof of this theorem is contained in the (correct part of) the proof given in the book.[2] *The required corrections are listed below.*

p. 108, line 2. 'a partial isometry' should be 'an isometry'.

Lemma 3.7.3 (1). The first Ad should be Ad w

The proof of Theorem 3.7.2 on p. 109. *Replace the last three lines (i.e., the text following 'Proposition 3.7.5') with* 'implies the desired conclusion.'

Proposition 3.8.5 (4). $\|a\| \leq \pi$ should be $\|a\| \leq 2\pi$, or even $a \leq 2\pi$.

Proposition 3.9.4. *It is true that (3)* $\Rightarrow$ *(1)* $\Leftrightarrow$ *(2), but (1)* $\not\Rightarrow$ *(3). Here is what is missing.*
   (2) $\Rightarrow$ (1): The salient point is that for $a \geq 0$ and a projection p we have $a = pap$ if and only if $a \leq \|a\|p$: Since $pap \leq \|a\|p$, $a = pap$ implies $a \leq \|a\|p$. Conversely, if $a \leq pap$ then $\|a(1-p)\| \leq \|pap(1-p)\| = 0$, hence $a(1-p) = (1-p)a = 0$ and $a = pap$.
   Fix a projection $p \in A^{**}$. Then the set $\{a \in A \mid pap = a\}$ is clearly a C^*-subalgebra of A, and it is hereditary by the first paragraph.
   (2) $\not\Rightarrow$ (3): Take $B = \mathscr{K}(H)$ (or any other nonunital C^*-algebra) and let A be its unitization, $B^\dagger$. Clearly B is a hereditary C^*-subalgebra of A. However, $B^\perp = \{0\}$ and therefore $(B^\perp)^\perp \neq B$.

Exercise 3.10.12. *This is Proposition 3.2.11.*

Lemma 4.1.8. Add "with $\mathsf{T}(A_n) \neq \emptyset$".

The proof of Proposition 4.1.10. In the second line, $O_a \subseteq \{\tau(a)\}$ should be $O_a \cap \mathbb{C} \subseteq \{\tau(a)\}$.

The proof of Lemma 4.3.1. *In some of the formulas, u_0 should be u_0^* and u_0^* should be u_0 for most of the time. More helpfully, the first two displayed formulas and the sentence between them should read as follows:*

$$\|\textstyle\sum_{j=0}^n u_j bu_j^*\|^2 = \|u_0(b + \sum_{j=1}^n u_0^* u_j bu_j^* u_0)u_0^*\|^2 = \|b + \sum_{j=1}^n u_0^* u_j bu_j^* u_0\|^2.$$

Let $a := \sum_{j=1}^n u_0^* u_j bu_j^* u_0$. Since $(1-p)b = b$, we have

$$(1-p)a = \textstyle\sum_{j=1}^n (1-p)u_0^* u_j (1-p)bu_j^* u_0 = 0.$$

Example at the bottom of page 128: Add the requirement '$k(j) \neq k(j+1)$ for $j < n-1$'.

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[1] It seems that my subconsciousness was trying to tell me something when it chose the Kopatchinskaya quote that opens this section.
[2] The proof does give more, but this is not a good place to dwell on this.

Page 129, l. 4. 'powers of g' should be 'powers of h'.

The proof of Lemma 5.2.3. In the displayed line, $\pi_\varphi(\xi_\varphi)$ should be $\pi_\varphi(a)\xi_\varphi$.

Example 5.2.4 (2). 'of and only if' should be 'if and only if'.

Proposition 5.2.8. $0 \le a \le \pi_\varphi$ should be $0 \le a \le 2\pi$.

Proposition 5.2.9. $0 \le a \le \pi$ should be $0 \le a \le 2\pi$.

Lemma 5.2.13. *This lemma and its proof are equally silly, and they are not needed in the text anymore; let's just forget them.*

p. 155, 3–4 lines above Theorem 5.5.1. 'all automorphisms are inner' should be 'all inner automorphisms are trivial'.

Theorem 5.6.1. *This theorem is stated for n-tuples of pure states. Its application on p. 304 requires a stronger result, where one is given two infinite sequences of pure states. This is Exercise 5.7.27 (2). Its proof is an extension of the proof of Theorem 5.6.1.*

Exercise 5.7.3. *(1) and (2) are not equivalent. The exercise should read as:*
    Prove that (1) implies (2).
    ((2) is equivalent to A being prime; therefore we have the equivalence if A is separable, but not in general by a result of Nik Weaver.)

Exercise 5.7.7. Prove that every nonsingular pure state on $\mathscr{B}(H)$ is a vector state.

Exercise 5.7.18. *The first sentence should read as follows:*
    Assuming B is a unital C*-subalgebra of a unital C*-algebra A, prove the following.

Exercise 5.7.27 (2). *Add a footnote with the following text:* Used in the proof of Proposition 11.2.1.

p. 173, line -8. *A word is missing:* Hence factorial representations of type I are exactly the multiples of irreducible representations

The line above Proposition 6.2.9. $f\colon [X]^{\aleph_0} \to X$ should be $f\colon [X]^{<\aleph_0} \to X$.

The proof of Proposition 6.3.4. In the third line (from above) 'in $\mathbb{P}$' should be 'in $\triangle_{x\in X}C_x$'.
    In the third line from below, '$\subseteq \triangle_{x\in X}C_x$' should be 'and hence is in $\triangle_{x\in X}C_x$'.

§6.4. *Definitions of winning strategies for players I and II were somewhat garbled (at some places $2n$ should be $2n+1$ and vice versa), and in addition I made the game unnecessarily difficult for player II by assuming they cannot remember the order in which the elemebts of X have been played;[3] here is how this paragraph should read.*

I wins if $\mathsf{Y} := \{x_j : j \in \mathbb{N}\} \notin \mathsf{C}$, and II wins if $\mathsf{Y} \in \mathsf{C}$. A *winning strategy* for II is a function $f\colon \bigcup_n \mathsf{X}^{2n+1} \to \mathsf{X}$ such that for every choice of $x_{2n} \in \mathsf{X}$, for $n \in \mathbb{N}$, with $x_{2n+1} := f(\langle x_j : j < 2n \rangle)$ the set $\{x_n : n \in \mathbb{N}\}$ belongs to C. A winning strategy for I is $f\colon \bigcup_n \mathsf{X}^{2n} \to \mathsf{X}$ such that for every choice of $x_{2n+1} \in \mathsf{X}$, for $n \in \mathbb{N}$, with $x_{2n} := f(\langle x_j : j < 2n+1 \rangle)$ the set $\{x_n : n \in \mathbb{N}\}$ does not belong to C.

The proof of Theorem 6.4.1. *The text following (1) $\Rightarrow$ (2) is a proof of (1) $\Rightarrow$ (3) (this is really most of the proof of the Theorem), and this 'change of direction' makes the given (correct and easy) proofs of (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1) useless. Here is how to complete the proof.*

(3) $\Rightarrow$ (2): Fix f such that every $p \in [\mathsf{X}]^{\aleph_0}$ closed under f belongs to C. Enumerate all finite nonempty subsets of $\mathbb{N}$ as s_n, for $n \in \mathbb{N}$, so that $s_n \subseteq 2n+1$ for all n. Let σ be the strategy for player II in $G(\mathsf{C})$ according to which in the n-th move, they play $f(\{x_j | j \in s_n\})$. Since $s_n \subseteq 2n+1$, σ is well-defined. If $p = \{x_n | n \in \mathbb{N}\}$ is the result of a game in which II played according to σ, then p is closed under f and therefore belongs to C. This is therefore a winning strategy for II.

(2) $\Rightarrow$ (1): Suppose that II has a winning strategy, call it σ, in $G(\mathsf{C})$. Consider (X, σ) as a structure in the language that has a symbol for an n-ary function, interpreted as $\sigma \restriction \mathsf{X}^n$, for each $n \geq 1$. By the Downwards Löwenheim–Skolem theorem (Theorem 7.1.3), countable elementary submodels M of (X, σ) form a club. It therefore suffices to show that that the universe of every countable $M \preceq (\mathsf{X}, \sigma)$ is in C. Fix such M and enumerate its universe as y_n, for $n \in \mathbb{N}$. Now let Player I manipulate Player II in $G(\mathsf{C})$ by playing y_m in the m-th move. If in this game II follows σ, then since M is closed under σ, the outcome of the game is equal to (the universe of) M and therefore belongs to C, as required.

Proposition 8.6.5. 'increasing' should be 'decreasing'. *The result as stated is correct. The given proof produces a decreasing function, but a minor modification to the definition of L_0 results in a proof of 8.6.5 as stated.*

Definition 9.1.1 (1). 'has the finite intersection property' should be 'is closed under finite intersections' (this is a subtle, but nontrivial, difference).

Example 9.1.4 (2). $\mathrm{Fin} \times \emptyset = \{\mathsf{X} \subseteq \mathbb{N}^2 : (\exists m)(\forall m' \geq m)(\forall n)(m', n) \notin \mathsf{X}\}$.

Definition 9.3.1. In line 6, 'of both of' should be 'if both of'.

Lemma 9.3.3. The formula at the end of the second line of the proof should read as $\mathsf{A}_n \setminus \mathsf{C} \subseteq \mathsf{A}_n \cap \bigcup_{j<n} \mathsf{B}_j$.

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[3]Theorem 6.4.1 is still true with this change, by a finer bookkeeping argument in the proof of (3) $\Rightarrow$ (2).

p. 253, lines -7 and -6. $g \colon \mathsf{Y} \to \mathsf{X}$ should be $g \colon \mathsf{X} \to \mathsf{Y}$ and $\{\mathsf{A} \subseteq \mathsf{X} : g^{-1} \in \mathscr{F}\}$ should be $\{\mathsf{A} \subseteq \mathsf{Y} : g^{-1}[\mathsf{A}] \in \mathscr{F}\}$.

The proof of Proposition 9.4.4. In line 1, add 'and $g \colon \mathbb{N} \to \mathbb{N}$' and in line 3 $h = f \circ g$ should be $h = g \circ f$.

Example 9.5.4 (3). Should be $\mathfrak{d}_{(\kappa, \leq)} = \mathrm{cof}(\kappa)$. My brain was out to lunch when I was writing this.

§9.7, second line: cofinal should be cofinite

The proof of Theorem 9.7.8. *The function g as defined belongs to $\mathbb{N}^{\mathbb{N}}$ but not $\mathbb{N}^{\uparrow \mathbb{N}}$. This does not affect the proof, but one could modify Φ by choosing $\Phi(\mathbf{E})$ to be any element of $\mathbb{N}^{\uparrow \mathbb{N}}$ which is $\geq^* g$; for example, take g' defined by*
$$g'(m) = \max(g(m), g(m-1)+1).$$

Exercise 9.10.28. Delete this.

Exercise 9.10.31. In (2), 'countable ideal' should be 'countably generated ideal'.

Lemma 10.2.6 (1). As the anonymous referee of [6] pointed out, the proof that the unitaries u_{s} form a Schauder basis for $\mathrm{CCR}(G)$ is problematic. Therefore the first part of this lemma should be deleted (the second part is trivially true). I therefore owe a verification that each of the uses of Lemma 10.2.6 (1) can be replaced by a correct argument.

(a) p. 282, in the proof of Lemma 10.2.7: The linear span of u_{s} is dense in $\mathrm{CCR}(G)$, and this is all that is needed.

(b) p. 283, in the proof of Lemma 10.2.10. One way to fix the argument is to prove the analogue of [6, Corollary 2.7], asserting that if $X \subseteq [V]^{< \aleph_0}$ is a subgroup (with $G = (V, E)$), then $a = \sum_{\mathsf{s}} \lambda_{\mathsf{s}} u_{\mathsf{s}}$ belongs to $\mathrm{CCR}(G, X)$ if and only if $\lambda_{\mathsf{s}} = 0$ for all $\mathsf{s} \notin X$. (This is really a special case of [6, Corollary 2.7].)

(c) p. 285, the end of the proof of Proposition 10.2.13: again it suffices to know that the linear span of u_{s} is dense in $\mathrm{CCR}(G)$.

(d) p. 286 ditto.

(e) The assertion that $\mathrm{CCR}(G)$ has a Schauder basis, repeated on p. 275 and p. 279, can be deleted with no harm to anything but my ego.

(I don't know whether u_{s} do form a Schauder basis or not, but Pavlos Motakis provided an example of a pair of Banach spaces showing that the argument used in the proof of Lemma 10.2.6 is faulty.)

Lemma 10.2.11. One should explicitly add the assumptions $\{(k, 0), (l, 0)\} \notin E$ for all $k \neq l$ and $\{(k, 1), (l, 1)\} \notin E$ for all $k \neq l$, and state that they imply $[u_k, u_l] = [v_k, v_l] = 0$ for all k and l.

Claim in the proof of Lemma 10.2.11. $u_l v_j = -v_j u_l$ should be $u_l v_j = v_j u_l$.

p. 285. *A left curly bracket is missing in the definition of F^+; it should read as*
$$F^+ := \{\{(\xi, 0), (\xi, 1)\} : \xi \in F \setminus \{\xi_{n-1}\}\} \cup \{(\xi_{n-1}, 0), *\}$$

Claim on p. 290. In the second line of the proof, 'nonstationary' should be 'stationary'.

p. 304, line 1. *Theorem 5.6.1 should be Exercise 5.7.27 (2).*

Proposition 12.3.4. add 'If $\lambda \leq \kappa$ are cardinals and...'

Proposition 12.3.5. add 'If $\lambda \leq \kappa$ are cardinals and...'

Exercise 12.6.13. *The short proof of Lemma 12.4.4 suggested here is circular... and there are even shorter circular proofs of virtually anything. Let's stick to the medium-length proof given in the text.*

Lemma 13.1.5. *In the sketch of the proof that the multiplier algebra $\mathscr{M}(A)$ is the strict completion of A I assumed that the relevant nets have the same index set. Not only that there is no reason why this should be true, but since all Cauchy nets in A don't form a set (one has to consider arbitrary index sets), one cannot proceed to define a completion in this way. Instead, one has to consider the space of all strictly Cauchy filters over A. A proof of how this can be done for an arbitrary weak topology induced by a family of seminorms can be found in Gabriel Nagy's lecture notes (https://www.math.ksu.edu/ nagy/func-an-F07-S08.html, lecture TVS IV.). In our setting there is an extra hurdle, since we need to assure that the norm on $\mathscr{M}(A)$ is well-defined. The details are included below. Once the completion is defined, one can revert back to nets and the proof works as indicated.*

The idea for the following proof was adapted from [10].

Lemma 13.1.4$\frac{1}{3}$ *Suppose that $\mathscr{F}$ is a family of seminorms on a vector space X. Then there is a vector space $\tilde{X}$ with the following properties.*

(1) It contains X as a linear subspace.
(2) Every $\rho \in \mathscr{F}$ extends to a seminorm $\tilde{\rho}$ on $\tilde{X}$.
(3) Every Cauchy net in $\tilde{X}$ converges.
(4) The subspace X is dense in $\tilde{X}$ with respect to the weak topology induced by $\tilde{\mathscr{F}} := \{\tilde{\rho}|\rho \in \mathscr{F}\}$.

If X is an algebra with involution, then $\tilde{X}$ has the structure of an algebra with involution. The space $\tilde{X}$ is called the completion *of X.*

Given the correct setup, the proof is analogous with the construction of the completion of a metric space. Since all Cauchy nets on a set do not form a set, we will construct $\tilde{X}$ as the space of (equivalence classes of) Cauchy filters on X.

Definition 13.1.4$\frac{2}{3}$ *Suppose that $\mathbb{F}$ is a filter on a space X equipped with a family of seminorms $\mathscr{F}$.*

(1) The filter $\mathbb{F}$ converges to $x \in X$ if for every $\rho \in \mathscr{N}$ and every $\varepsilon > 0$ we have $\{y \in X|\rho(x-y) < \varepsilon\} \in \mathbb{F}$.
(2) The filter $\mathbb{F}$ is Cauchy *if for every $\rho \in \mathscr{N}$ and every $\varepsilon > 0$ there exists $Y \in \mathbb{F}$ such that $|\rho(x-y)| < \varepsilon$ for all x and y in Y.*

Proof of Lemma $1.13.1.4\frac{1}{3}$. Let $\mathbb{CF}(X)$ be the set of all Cauchy filters on X. On this set define the algebraic operations in the natural fashion (with the sum $Y + Z :=$ $\{y + z | y \in Y, z \in Z\}$, and similarly defined λY for a scalar λ):

$$\mathbb{F} + \mathbb{G} := \{F + G | F \in \mathbb{F}, G \in \mathbb{G}\}$$
$$\lambda\mathbb{F} := \{\lambda F | F \in \mathbb{F}\}.$$

Standard ε, δ arguments show that if $\mathbb{F}$ and $\mathbb{G}$ are Cauchy filters, then so are $\mathbb{F} + \mathbb{G}$ and $\lambda\mathbb{F}$.

For $\rho \in \mathscr{F}$ we define $\tilde{\rho} \colon \mathbb{CF}(X) \to \mathbb{C}$ as follows. Fix $\mathbb{F} \in \mathbb{CF}(X)$. Since it is Cauchy, there exists $\lambda \in \mathbb{C}$ such that for every $\varepsilon > 0$ the set

$$U_{\rho,\lambda,\varepsilon} := \{x \in X | |\rho(x) - \lambda| < \varepsilon\}$$

belongs to $\mathbb{F}$. If this applies, we write $\lim_{x \to \mathbb{F}} \rho(x)$ for λ, and define

$$\tilde{\rho}(\mathbb{F}) := \lim_{x \to \mathbb{F}} \rho(x).$$

It is straightforward to check that $\tilde{\rho}$ is a seminorm on $\mathbb{CF}(X)$.

On $\mathbb{CF}(X)$ define the relation

$$\mathbb{F} \approx \mathbb{G} \Leftrightarrow \mathbb{F} + (-1)\mathbb{G} \text{ converges to } 0.$$

A calculation shows that this is an equivalence relation which is a congruence with respect to addition and multiplication by scalars, as well as for each $\tilde{\rho} \in \mathscr{F}$. Therefore we can identify $\tilde{\rho}$ with a seminorm on $\mathbb{CF}(X)/\approx$. Finally, embed X into the quotient $\mathbb{CF}(X)/\approx$ by sending x to the principal filter

$$\mathbb{F}_x := \{Y \subseteq X | x \in Y\}.$$

A calculation shows that $\tilde{X} := \mathbb{CF}(X)/\approx$ is a linear space, that each $\tilde{\rho}$ is a seminorm on $\tilde{X}$, that $\tilde{X}$ is complete with respect to $\tilde{\mathscr{F}}$, and that the copy of X inside $\tilde{X}$ is dense.

We'll now prove that $\tilde{X}$ is complete with respect to the weak topology induced by $\tilde{\mathscr{F}}$. Fix a Cauchy net in $\tilde{X}$. Lift it to a net in $\mathbb{CF}(X)$, $\mathbb{F}_\mu$, for μ ranging over some directed set M. Since the net is Cauchy, for every $\rho \in \mathscr{N}$ the limit

$$\Phi(\rho) := \lim_\mu \tilde{\rho}(\mathbb{F}_\mu)$$

exists. For $F \Subset \mathscr{N}$ and $\varepsilon > 0$ fix $\mu = \mu_{F,\varepsilon}$ such that for $\nu > \mu$ and $\rho \in F$ we have $\tilde{\rho}(\mathbb{F}_\mu + (-1)\mathbb{F}_\nu) < \varepsilon/2$. Fix $Y \in \mathbb{F}_\mu$ such that $\max_{\rho \in F} \sup_{x,y \in Y} |\rho(x - y)| < \varepsilon/2$ and fix $y \in Y$. Then the set

$$Z_{F,\varepsilon} := \{z \in X | \max_{\rho \in F} |\rho(z - y)| < \varepsilon\}$$

belongs to $\mathbb{F}_\nu$ for all $\nu \geq \mu$. The family $\{Z_{F,\varepsilon} | F \Subset \mathscr{N}, \varepsilon > 0\}$ has the finite intersection property, since each of its finite subsets belongs to $\mathbb{F}_\nu$ for some $\nu \in M$. Let $\mathbb{G}$ be the filter generated by this family. By the choice of the generators, it is a Cauchy filter on X. Also, for every $\rho \in \mathscr{N}$ and $\varepsilon > 0$, if $\mu \geq \mu_{\{\rho\},\varepsilon}$ then $\tilde{\rho}(\mathbb{F}_\mu + (-1)\mathbb{G}) < \varepsilon$, and therefore $\mathbb{G}/\approx$ is the limit of the Cauchy net $(\mathbb{F}_\mu)$.

In the case when X is an algebra with an involution *, on $\mathbb{CF}(X)$ we can define (writing $FG := \{xy | x \in F, y \in G\}$ and $F^* := \{x^* | x \in F\}$):

$$\mathbb{FG} := \{FG | F \in \mathbb{F}, G \in \mathbb{G}\},$$
$$\mathbb{F}^* := \{F^* | F \in \mathbb{F}\}.$$

Standard ε, δ arguments show that if $\mathbb{F}$ and $\mathbb{G}$ are Cauchy filters, then so are $\mathbb{FG}$ and $\mathbb{F}^*$. Arguments analogous to those in the case when X was a vector space show that in this case $\tilde{X}$ is an algebra.

Lemma 13.1.5 *The completion $\mathscr{M}(A)$ of A in the strict topology is equipped with a unital C^*-algebra structure such that A is an essential ideal in $\mathscr{M}(A)$.*

Proof. Applying Lemma $13.1.4\frac{1}{3}$ to A and $\mathscr{F} := \{\rho_h, \lambda_h | h \in A\}$, we obtain the algebra $\mathscr{M}(A) := \tilde{A}$.

In order to define a norm on $\mathscr{M}(A)$, fix an approximate unit $\mathscr{E}$ for A.

Claim. *If $\mathbb{F}$ is a Cauchy filter on A, then both $\|\mathbb{F}\|_\lambda := \sup_{e \in \mathscr{E}} \lim_{x \to \mathbb{F}} \lambda_e(x)$ and $\|\mathbb{F}\|_\rho := \sup_{e \in \mathscr{E}} \lim_{x \to \mathbb{F}} \rho_e(x)$ are finite and equal to one another.*

Also, $\|\mathbb{F}\| := \|\mathbb{F}\|_\lambda$ defines a norm on $\mathscr{M}(A)$ that satisfies the C^*-equality.

Proof. Suppose that $\sup_{e \in \mathscr{E}} \lim_{x \to \mathbb{F}} \lambda_e(x) = \infty$. For $m \in \mathbb{N}$ fix $e_m \in \mathscr{E}$ such that $\lim_{x \to \mathbb{F}} \lambda_{e_m}(x) > 2^{2m}$. Let $e := \sum_{m=0}^\infty 2^{-m-1} e_m$. Fix $K > \lim_{x \to \mathbb{F}} \lambda_e(x)$ and m such that $2^m > K$.

Let $Y \in \mathbb{F}$ be such that every $y \in Y$ satisfies $\lambda_e(y) < K$ and $\lambda_{e_m}(y) > 2^{2m}$. Fix $y \in Y$. Since $e \geq 2^{-m} e_m$, we have $K > \|ey\| > 2^{-m} \|e_m y\| > 2^m$; contradiction.

This proves that $\|\mathbb{F}\|_\lambda$ is well-defined. The mirror image of this argument shows that $\|\mathbb{F}\|_\rho$ is also well-defined.

Standard ε, δ arguments, used with the C^*-equality, show that $\|\mathbb{F}\|_\lambda^2 = \|\mathbb{FF}^*\|_\lambda$ and $\|\mathbb{F}\|_\rho^2 = \|\mathbb{FF}^*\|_\rho$. Since clearly $\|\mathbb{FF}^*\|_\lambda = \|\mathbb{FF}^*\|_\rho$, this implies $\|\mathbb{F}\|_\lambda = \|\mathbb{F}\|_\rho$ and that $\|\mathbb{F}\| = \|\mathbb{F}\|_\lambda$ satisfies the C^*-equality.

Given that the proof of the Claim has had an added bonus—the norm $\|\mathbb{F}\|$ satisfies the C^*-equality—the reader may still have energy and interest to go over a proof that the algebraic operations are continuous with respect to this norm. For this, note that (using the fact that $\mathscr{E}$ is a net) the norm defined in Claim satisfies

$$\|\mathbb{F}\| = \lim_{e \to \mathscr{E}} \lambda_e(\mathbb{F}) = \lim_{e \to \mathscr{E}} \rho_e(\mathbb{F}).$$

Using this ε, δ arguments, a relentless reader can easily check that the operations are norm-continuous. This remark applies to the verification of the completeness of the norm, and completes the proof.

Lemma 13.1.8. *Part (1) is false, and it was never used. The proof of (1) given in the text is a proof of the (useful) part (2) of the lemma.*

Proposition 13.2.1. *. . . unique faithful representation. . .*

Exercise 13.4.15. *The C^*-algebras A and B have to be assumed to be separable.*

The second claim in the proof of Theorem 14.2.1, p. 353. In the second line of the proof, $\|(p_{\mathsf{X}}+p_{\mathsf{X}})c\|$ should be $\|(p_{\mathsf{X}}+p_{\mathsf{Y}})c\|$.

p. 369, the paragraph following the proof of the Claim. *The use of Lemma 1.4.8 (and the lemma itself, see above) should be made more precise. Here is the correct text*

By a proof similar to the proof of Lemma 1.4.8, for every *-polynomial $P(\bar{x})$ with coefficients in C there is a universal constant $K < \infty$ depending only on $P(\bar{x})$ such that

$$\|[P(\bar{b}),a]\| \leq K \max_{c} \|[c,a]\|,$$

where c ranges over the coefficients of P and the entries of $\bar{b}$. By applying the second part of Lemma 1.4.8, there is a continuous function $g\colon (0,1] \to (0,1]$ such that…

The proof of part (3) of the Claim on page 370. *The assertion that "This implies $\|\xi_n - a_n f_n^2 \xi_n\| < 2\varepsilon$" on p. 371 in this proof is not quite clear. Here is the proof, complete with the much needed details.*

(3) If $\limsup_n \|a_n f_n^2\| = 0$ there is nothing to prove. We may therefore assume $r := \limsup_n \|a_n f_n^2\|$ is nonzero. By replacing a_m with a_m/r, we may assume that $\limsup_m \|a_m\| = 1$. In addition, $|\|a_n\| - \|a_n f_n^2\|| \to 0$ implies $\limsup_m \|a_m f_m^2\| = 1$.

Fix $\varepsilon > 0$ and n such that $1 - \varepsilon < \|a_n\| < 1 + \varepsilon$, $1 - \varepsilon < \|a_n f_n^2\| < 1 + \varepsilon$, and $\|a_{n+1}\| < 1 + \varepsilon$. Let ξ_n^0 be a unit vector that satisfies $\|a_n f_n^2 \xi_n^0\| > \|a_n f_n^2\| - \varepsilon$. If p_n is the support projection of f_n, then $\xi_n := \|p_n \xi_n^0\|^{-1} \xi_n^0$ still has the property of ξ_n^0 and in addition $f_m \xi_n = 0$ for all $m > n + 1$.

We have $\|a_n f_n^2 \xi_n\| > \|a_n f_n^2\| - \varepsilon > 1 - 3\varepsilon$, and therefore

$$(f_n^4 \xi_n | \xi_n) = \|f_n^2 \xi_n\| > (1 - 2\varepsilon)\|a_n\|^{-1} > 1 - 2\varepsilon/(1+\varepsilon) > 1 - 3\varepsilon.$$

Since $0 \leq f_n \leq 1$, we also have $f_n^4 \leq f_n^2$, and

$$\|\xi_n - f_n^2 \xi_n\| = \|\xi_n\|^2 - 2(f_n^2 \xi_n | \xi_n) + (f_n^4 \xi_n | \xi_n) < 1 - (f_n^2 \xi_n | \xi_n) < 3\varepsilon.$$

Also, $f_{n+1}^2 \leq 1 - f_n^2$ hence $\|f_{n+1}^2 \xi_n\| < 3\varepsilon$. Together with $\sum_{m \geq n+2} f_m^2 \xi_n = 0$ and $\|a_{n+1}\| < 1 + \varepsilon$, this gives

$$\|\textstyle\sum_{m \geq n} a_m f_m^2\| \geq \|a_n f_n^2 \xi_n\| - \|a_{n+1} f_{n+1}^2 \xi_n\| > \|a_n f_n^2\| - 5\varepsilon.$$

Since this holds for arbitrarily large n and the sequence $\sum_{m \leq k} f_m^2$, for $k \in \mathbb{N}$, is an approximate unit for A, we have $\|\pi(\sum_m a_m f_m^2)\| \geq 1 - 3\varepsilon\|a_m\|$. Since $\varepsilon > 0$ was arbitrary, $\|\pi(\sum_m a_m f_m^2)\| \geq 1 = \limsup_n \|a_n f_n^2\|$, as required.

Exercise 15.6.11. *The second sentence in the question should read as*

If $c \in C$ satisfies $Jc = \{0\}$ then f can be chosen to satisfy $fc = 0$.

Exercise 15.6.12. Add the assumption $cd = 0$. (This is necessary. unless A is unital: Otherwise one could take $c = d$ self-adjoint and nonzero in the annihilator of A.)

Exercise 15.6.17. *C should be $C(\beta\mathbb{N} \setminus \mathbb{N})$.*

Exercise 15.6.18. *$C(\beta\mathbb{N} \setminus \mathbb{N})$ should be C.*

Example 16.1.3 (3). In the fourth line, $\{\|x^*x - x\| = 0, \|x^x - x\| = 0\}$. should be $\{\|x^* - x\| = 0, \|x^2 - x\| = 0\}$..

p. 396, line 2. '$\chi(M)$-*saturated*' should be 'κ-*saturated, where κ is the density character of M.*'

p. 407, in (2). $j \in \mathsf{Y}_j$ *should be* $j \in \bigcup_{i \leq j} \mathsf{Y}_i$.

p. 407, in the definition of Y_2. $m(3)$ *should be m.*

p. 407, in the definition of Y_{k+1}. $\bar{m}(k+2)$ should be *m*.

In Exercise 16.8.32 once and Exercise 16.8.33 twice. incomplete *should be* complete.

§17.1. *In the fourth line of 17.1,* … certain inverse system. . .

The proof of Lemma 17.1.9. *The second half of the last sentence of the proof should be replaced with the following, somewhat more informative, text.*

By Lemma 9.7.6, every $a \in \mathscr{F}[\mathbf{E}]$ can be written as a sum of an element of $\mathscr{D}[\mathbf{E}^{\text{even}}]$, an element of $\mathscr{D}[\mathbf{E}^{\text{odd}}]$, and a compact element. Also, both $\mathscr{D}[\mathbf{E}^{\text{even}}]$ and $\mathscr{D}[\mathbf{E}^{\text{odd}}]$ are included in $\mathscr{F}[\mathbf{E}]$. Therefore $\lim_n \operatorname{diam}(\operatorname{sp}(w_n)) = 0$ is equivalent to $u \sim_{\mathbf{E}} v$, completing the chain of equivalences.

Claim in the proof of Proposition 17.1.11. If α is a countable limit ordinal. . .

Theorem 17.1.12. … at least $2^{\aleph_1}$…

Corollary 17.1.13. … at least $2^{\aleph_1}$…

Theorem 17.1.15. … at least $2^{\aleph_1}$…

Theorem 17.2.3 and its proof.
  (1) In the statement of Theorem 17.2.3, 'von Neumann algebras' should be replaced with 'C*-algebras' and 'Borel-measurable function' should be replaced with 'uniformly bounded Borel-measurable function'.
  (2) In the proof, Θ' is defined by using Bochner integral. (This is the analog of Lebesgue integral for Banach space-valued functions; see e.g., [2, Appendix E]).
  (3) It should be pointed out that Θ' itself is Borel-measurable and uniformly bounded. (This was proved in [8, p. 7].)

p. 429, line 9. $\Theta'(g)$ should be $|\Theta'(g)|$.

p. 429, line 10. $\Theta'(g)^{-1}$ should be $|\Theta'(g)|^{-1}$.

The proof of Lemma 17.4.5. All three instances of $\mathsf{D}_{\tilde{\mathsf{X}}}$ should be replaced with D_{X}.

Theorem 17.2.6. Replace 'von Neumann algebra' with 'C*-algebra'.

The proof of Theorem 17.2.6. *Eager to use the full power of the Burger–Ozawa–Thom Theorem 17.2.3, I turned a correct proof given in [8] into a seemingly more elegant, albeit incorrect, one (needless to say, this was entirely my fault). The proof of Theorem 17.2.6 should be replaced by the following hybrid of the two proofs.*

Proof. By Theorem 17.2.3, there is a Borel-measurable $\Lambda\colon \mathsf{U}(M_m(\mathbb{C})) \to A$ such that $\Lambda(uv) = \Lambda(u)\Lambda(v)$ for all u and v, and $\|\Lambda - \Theta\| \leq 4\varepsilon$. By replacing A with a C*-algebra generated by the range of Λ, we may assume that A is separable. Therefore Λ is a Borel-measurable homomorphism between Polish groups. By Pettis's Theorem ([7, Theorem 9.10]), Λ is continuous.

The continuity of Λ implies that if $a \in M_m(\mathbb{C})$ is self-adjoint, then the one-parameter group of untiaries $\Lambda(\exp(ira))$, for $r \in \mathbb{R}$, is norm-continuous in r. By Stone's Theorem ([11, Theorem 5.3.15], and the observation that the norm-continuity implies S is bounded) there exists $\tilde{\Lambda}(a) \in A_{sa}$ such that

$$\Lambda(\exp(ira)) = \exp(ir\tilde{\Lambda}(a))$$

for all $r \in \mathbb{R}$.

Claim.   *(1)* $\tilde{\Lambda}(1) = 1$.
    (2) If p is a projection, then $\tilde{\Lambda}(p)$ is a projection.

Proof. The proofs of both parts use the same idea. Assume that $b := \tilde{\Lambda}(1)$ is not 1, and therefore $\mathrm{sp}(b) \neq \{1\}$. Fix $\lambda \in \mathrm{sp}(b)$ and $s \in \mathbb{R}$ such that $s(1 - \lambda) = \pi$. Then $\exp(is) = \Theta(\exp(is)) \approx_{\varepsilon} \exp(isb)$ but Lemma 1.4.7 implies

$$\|\exp(is) - \exp(isb)\| \geq |\exp(is) - \exp(-i\lambda s)| = |\exp(i\pi)| = 2;$$

contradiction. Now suppose $\tilde{\Lambda}(p)$ is not a projection. Fix $\lambda \in \mathrm{sp}(\tilde{\Lambda}(p) \setminus \{0, 1\})$. Then $\{\exp(i2k\pi\lambda) | k \in \mathbb{Z}\}$ is a nontrivial subgroup of $\mathbb{T}$, and some $k \in \mathbb{Z}$ satisfies $|\exp(i2k\pi\lambda) - 1| \geq \sqrt{2}$. But $\exp(ik\pi p) = 1$, hence $\exp(ik\pi\tilde{\Lambda}(p)) \approx_{\varepsilon} 1$, while Lemma 1.4.7 implies $\|\exp(ik\pi\tilde{\Lambda}(p)) - 1\| \geq \sqrt{2}$; contradiction.

For a projection p let

$$u_p := \exp(i\pi p)$$

(equivalently, $u_p = 1 - 2p$).

Claim. *Suppose p and q are projections.*
    (1) We have $\tilde{\Lambda}(p) = \frac{1}{2}(1 - \Lambda(u_p))$.
    (2) If p and q are Murray–von Neumann equivalent, then so are $\tilde{\Lambda}(p)$ and $\tilde{\Lambda}(q)$.
    (3) If p and q commute, then so do $\tilde{\Lambda}(p)$ and $\tilde{\Lambda}(q)$.
    (4) If $pq = 0$, then $\tilde{\Lambda}(p)\tilde{\Lambda}(q) = 0$ and $\tilde{\Lambda}(p + q) = \tilde{\Lambda}(p) + \tilde{\Lambda}(q)$.
    (5) If $\sum_{j<m} p_j = 1$ for projections p_j, for $j < m$, then $\sum_{j<m} \tilde{\Lambda}(p_j) = 1$.

Proof. (1) The equivalent formula, $\Lambda(u_p) = 1 - 2\tilde{\Lambda}(p)$, follows from the definitions.

(2) By the first claim, $\tilde{\Lambda}$ sends p and q to projections. Since p and q belong to $M_m(\mathbb{C})$, Murray–von Neumann equivalence coincides with the unitary equivalence. If w is a unitary such that $wpw^* = q$, then $wu_pw^* = u_q$ and a simple computation

using (1) shows that $\Lambda(w)$ implements Murray-von Neumann equivalence of $\tilde{\Lambda}(p)$ and $\tilde{\Lambda}(p)$.

(3) Immediate from (1) and the equivalence of $[p,q]=0$ and $[u_p,u_q]=0$.

(4) We have $pq=0$ if and only if $p+q$ is a projection, if and only if $u_p u_q = u_{p+q}$; therefore this follows from (1).

(5) This follows immediately from the first claim and (4).

Let τ be the unique tracial state on $M_m(\mathbb{C})$ and fix a faithful tracial state σ on A. (Since A is finite-dimensional, it has a faithful tracial state.) Our next task is to prove $\tau(u) = \sigma(\Lambda(u))$ for every unitary u in $M_m(\mathbb{C})$. The spectral theorem implies

$$u = \sum_{j<m} \exp(i\lambda_j) p_j = \prod_{j<m} \exp(i\lambda_j p_j),$$

where p_j are rank-1 projections and λ_j are the real numbers such that $\exp(i\lambda_j)$ are the eigenvalues of u (the eigenvalues of multiplicity n are repeated n times). Since the p_j are Murray–von Neumann equivalent, the second Claim implies that all $\tilde{\Lambda}(p_j)$ are Murray–von Neumann equivalent and $\sum_{j<m} \tilde{\Lambda}(p_j) = 1$. Therefore $\sigma(\tilde{\Lambda}(p_j)) = 1/m$ for all j.

Let $a := \sum_{j<m} \lambda_j p_j$, so that $u = \exp(ia)$. Then for all $r \in \mathbb{R}$ we have (using the orthogonality of p_j's and the orthogonality of $\tilde{\Lambda}(p_j)$'s as needed)

$$\exp(ir\tilde{\Lambda}(a)) = \Lambda(\exp(ira)) = \Lambda(\prod_{j<m} \exp(ir\lambda_j p_j))$$
$$= \prod_{j<m} \Lambda(\exp(ir\lambda_j p_j)) = \prod_{j<m} \exp(ir\lambda_j \tilde{\Lambda}(p_j))$$
$$= \exp(ir\sum_{j<m} \lambda_j \tilde{\Lambda}(p_j)).$$

By taking logarithms, $\tilde{\Lambda}(a) = \sum_{j<m} \lambda_j \tilde{\Lambda}_j p_j$ and $\Lambda(u) = \sum_{j<m} \exp(i\lambda_j) \tilde{\Lambda}(p_j)$. Since $\sigma(\tilde{\Lambda}(p_j)) = 1/m = \tau(p_j)$ for all j, we have $\tau(u) = \frac{1}{m} \sum_{j<m} \exp(i\lambda_j) = \sigma(\tilde{\Lambda}(u))$. By Lemma 17.2.4, $\Phi(\sum_{j<4} \lambda_j u_j) := \sum_{j<4} \lambda_j \tilde{\Lambda}(u_j)$ is a well-defined *-homomorphism.

Also, $\|\Phi(a) - \Theta(a)\| \le 4 \sup_{x \in \mathsf{U}(M_m(\mathbb{C}))} \|\Phi(u) - \Lambda(u)\| \le 8\varepsilon$, as required.

Definition 17.3.3. ... if its restriction to the unit ball has a lifting which is Borel-measurable with respect to the strict topology (this is a Polish topology).

The statement of Lemma 17.4.8. It should be emphasized that Θ is assumed to be continuous when the codomain, $\mathscr{B}(H)_1$, is considered with respect to the strong operator topology (although the proof works with the weak operator topology).

The proof of Lemma 17.4.8; this is a simplified proof (using the easy Exercise 1 below), posted March 15, 2021. *Missing details and typos added up to an impasse. The proof of Lemma 17.4.8 should be replaced with the following.*

We recursively find an increasing sequence $(n(j))_j$, $s(j) \in \mathsf{D}_{(n(j),n(j+1))}$ (with $n(0) := 0$), and an increasing sequence of finite-rank projections (r_j), for $j \in \mathbb{N}$, so that the following holds for all j, and all x and y in $\mathsf{D}_{\mathbb{N}\setminus(n(j),n(j+1))}$, with $z := x \restriction (n(j)+1) + y \restriction [n(j+1),\infty)$:

   (1) $\|(\Theta(y+s(j)) - \Theta(z+s(j)))(1-r_j)\| \le 2^{-j}$,
   (2) $\|(1-r_j)(\Theta(y+s(j)) - \Theta(z+s(j)))\| \le 2^{-j}$,

(3) $\|(\Theta(x+s(j)) - \Theta(z+s(j)))r_j\| \leq 2^{-j}$,
(4) $\|r_j(\Theta(x+s(j)) - \Theta(y+s(j)))\| \leq 2^{-j}$.

Set $n(0) := 0$. Suppose that $k \geq 0$ and that $n(j)$, $s(j)$, and r_j, for $j < k$, have been chosen to satisfy (1)–(4).

Towards a contradiction, assume that $n(k)$ and $s(k)$ that satisfy (1) cannot be found. We will find an increasing sequence $m(i) \in \mathbb{N}$ and $t(i) \in \mathsf{D}_{[m(i),m(i+1))}$, so that $m(0) := n(k-1)+1$, and for every i there are $a(i)$ and $b(i)$ in $\mathsf{D}_{[0,n(k-1))}$ and c in $\mathsf{D}_{(m(i+1),\infty)}$ such that $p_i := \mathrm{proj}_{\overline{\mathrm{span}}\{\xi_l : l \leq i\}}$ (with ξ_l being the orthonormal basis for H fixed earlier) satisfies

$$\|(\Theta(a(i) + \textstyle\sum_{l \leq i} t(l) + c) - \Theta(b(i) + \textstyle\sum_{l \leq i} t(l) + c))(1 - p_i)\| > 2^{-k+1}.$$

Suppose that $a(i)$, $b(i)$, $t(l)$, and $m(l')$, as required have been chosen for $l \leq i$ and $l' \leq i+1$. By the assumed failure of (1) for $s(i) := \sum_{l \leq i} t(l)$, there exists c in $\mathsf{D}_{[m(i+1),\infty)}$ such that

$$\|(\Theta(a + s(i) + c) - \Theta(b + s(i) + c))(1 - p_i)\| > 2^{-k+1}$$

By the continuity of Θ, there is a large enough $m(i+2)$ such that with

$$t(i+1) := c p_{[m(i+1),m(i+2))},$$

for all d in $\mathsf{D}_{(m(i+2),\infty)}$ we have

$$\|(\Theta(a(i) + \textstyle\sum_{l \leq i+1} t(l) + d) - \Theta(b(i) + \textstyle\sum_{l \leq i+1} t(l+1) + d))(1 - p_i)\| > 2^{-k+1}.$$

This describes the recursive construction of $a(i)$, $b(i)$, and $t(i)$ as required.

Since there are only finitely many choices for $a(i)$ and $b(i)$, some pair $\tilde{a}, \tilde{b}$ appears as $a(i), b(i)$ infinitely often. Let $t := \sum_l t(l)$ (the partial sums strongly converge to an element of D). Then

$$\|(\Theta(\tilde{a} + t) - \Theta(\tilde{b} + t))(1 - p_i)\| > 2^{-k+1}$$

for all i, and $\Theta(\tilde{a} + t) - \Theta(\tilde{b} + t)$ is not compact. But $(\tilde{a} + t) - (\tilde{b} + t)$ has finite rank. This contradicts the assumption that Θ lifts Φ. We can therefore choose $n^0(k)$, $s^0(k) \in \mathsf{D}_{(n(k),n^0(k)]}$, and r_k^0 such that (1) holds. Note that this condition holds when $s^0(k)$ is end-extended and r_k is increased.

An argument analogous to the one used to secure (1) shows that we can find $n^1(k) > n^0(k)$, $s^1(k) \in \mathsf{D}_{(n(k),n^1(k)]}$ extending $s^0(k)$, and r_k such that (2) holds. This extension does not affect the condition (1).

Enumerate $\mathsf{D}_{n(k-1)+1}$ as $a(i)$, for $i < m$. Since r_k has finite rank, each of the functions $x \mapsto \Theta(x)r_k$ and $x \mapsto r_k\Theta(x)$ is continuous with respect to the norm topology in the range. As in the proof of Theorem 9.9.1, recursively, in m steps, choose $n(k) \geq n^1(k)$ and $s(k) \supseteq s^1(k)$ to produce a basic open set $[(n(k-1),n(k)),s(k)]$ such that both (3) and (4) hold for all x and y that satisfy $x \restriction (n(k-1)+1) = y \restriction (n(k-1)+1) = a(i)$ for some $i < m$. This describes the recursive construction of $n(i)$, $s(i)$, and r_i as required.

Let $\mathsf{X} := \{n(j) : j \in \mathbb{N}\}$. The sum $s := \sum_j s(j)$ strongly converges to an element of $\mathsf{D}_{\mathbb{N} \setminus \mathsf{X}}$. We claim that for every $x \in \mathsf{D}_\mathsf{X}$ and every $j \in \mathbb{N}$ we have

$$(1) \qquad \|[\Theta(x+s) - \Theta(s), r_j]\| \leq 2^{-j+2}.$$

To prove this, using (1)–(4) and writing $x_- := x \upharpoonright (n(j)+1)$, $x_+ := x \upharpoonright [n(j+1), \infty)$, and $\varepsilon := 2^{-j}$),

$$(\Theta(x+s) - \Theta(s))r_j \approx^\varepsilon (\Theta(x_- + s) - \Theta(s))r_j \approx^\varepsilon (\Theta(x_- + s) - \Theta(s))$$
$$\approx^\varepsilon r_j(\Theta(x_- + s) - \Theta(s)) \approx^\varepsilon r_j(\Theta(x+s) - \Theta(s)),$$

as required. Define $\Xi_j \colon \mathsf{D}_{n(j)} \to (r_{j+1} - r_j)\mathscr{B}(H)_{\leq 1}(r_{j+1} - r_j)$ by

$$\Xi_j(x) := (r_{j+1} - r_j)(\Theta(s+x) - \Theta(s))(r_{j+1} - r_j).$$

Fix $x \in \mathsf{D}_\mathsf{X}$. By (1), we have $\sum_j \|[\Theta(x+s) - \Theta(s), r_j]\| \leq 4\varepsilon$. By Exercise 1, the function Ξ of product type determined by (Ξ_j) satisfies $\Xi(x) \approx^{\mathscr{K}} \Theta(x+s) - \Theta(s)$, and is therefore a required lifting of Φ on D_X.

Exercise 1.   *(1) Suppose that r is a projection. Then every operator a satisfies*

$$\|[a,r]\| = \|a - rar - (1-r)a(1-r)\|.$$

(2) Suppose that $a \in \mathscr{B}(H)$, (r_j) is an approximate unit for $\mathscr{K}(H)$ consisting of projections, and $\sum_j \|[a, r_j]\| < \infty$.
  Use (1) to prove that $a - \sum_j (r_{j+1} - r_j)a(r_{j+1} - r_j)$ is compact.

Lemma 17.5.8. *The following lemma should be added at the end of §17.5, on p. 441.*

Lemma 17.5.8. *Suppose that $\mathbf{E} \in \mathrm{Part}_{\mathbb{N}}$, u_e and u_o are in $\mathsf{U}(\ell_\infty(\mathbb{N}))$, and $\mathrm{Ad}\,u_e$ and $\mathrm{Ad}\,u_o$ agree on $\mathscr{D}[\mathbf{E}]$ modulo the compacts. Then there is $w \in \mathsf{U}(\ell_\infty(\mathbb{N}))$ such that $\mathrm{Ad}\,w$ agrees with $\mathrm{Ad}\,u_e$ on $\mathscr{D}[\mathbf{E}^{\mathrm{even}}]$ modulo the compacts and with $\mathrm{Ad}\,u_o$ on $\mathscr{D}[\mathbf{E}^{\mathrm{odd}}]$ modulo the compacts.*

Proof. The assumption implies that $u_e u_o^*$ belongs to $\pi[\mathscr{D}[\mathbf{E}]]' \cap \mathscr{Q}(H)$. By [12] $\pi[\mathscr{D}[\mathbf{E}]]' \cap \mathscr{Q}(H) = \pi[\mathscr{D}[\mathbf{E}]']$, hence there is $a \in \mathscr{D}[\mathbf{E}]'$ such that $a - u_e u_o^*$ is compact. But (writing p_i for $p_i^{\mathbf{E}}$) $\mathscr{D}[\mathbf{E}]' = \mathrm{W}^*(p_i : i \in \mathbb{N})$ (see §9.7.1).

Fix $\lambda_i \in \mathbb{T}$, for $i \in \mathbb{N}$, such that $a = \sum_i \lambda_i p_i$ (all infinite sums in this proof are SOT-convergent). Define $\eta_i \in \mathbb{T}$, for $i \in \mathbb{N}$, recursively by setting $\eta_0 := 1$, $\eta_{2i+1} := \eta_{2i}$, and $\eta_{2i+2} := \lambda_{2i+1}\lambda_{2i}^{-1}\eta_{2i+1}$ for $i \geq 1$.

Let $w := \sum_i \eta_i p_i$. It clearly belongs to $\mathsf{U}(\ell_\infty(\mathbb{N}))$.

Then $wu_e^* = \sum_i \eta_{2i}(p_{2i} + p_{2i+1})$, hence it belongs to $\mathscr{D}[\mathbf{E}^{\mathrm{even}}]'$, and $\mathrm{Ad}\,w$ and $\mathrm{Ad}\,u_e$ agree on $\mathscr{D}[\mathbf{E}^{\mathrm{even}}]$. Also, $wu_o^* - \sum_i \eta_{2i+1}(p_{2i+1} + p_{2i+2})$ is compact, hence $\pi(wu_o^*)$ belongs to $\pi[\mathscr{D}[\mathbf{E}^{\mathrm{odd}}]]' \cap \mathscr{Q}(H)$. Therefore $\mathrm{Ad}\,w$ and $\mathrm{Ad}\,u_o$ agree on $\mathscr{D}[\mathbf{E}^{\mathrm{odd}}]$ modulo the compacts.

Definition 17.6.1. (a) It should be specified that 'Borel' refers to the weak operator topology on $\mathscr{B}(H)_{\leq 1}$.

(b) *The last sentence should state the following.*

A σ-narrow lifting on $\mathscr{D}[\mathbf{E}]$ or $\mathsf{D}[\mathbf{E}]$ and a σ-narrow ε-approximation on $\mathscr{D}[\mathbf{E}]$ or $\mathsf{D}[\mathbf{E}]$ are defined analogously.

Example 17.6.2. *In the first sentence 'ℓ_∞/c_0' should be $\mathscr{P}(\mathbb{N})/\mathrm{Fin}$.*

The second and third sentences should read as follows:

Let $\mathscr{U}$ be a nonprincipal ultrafilter on $\mathbb{N}$. Define $\Upsilon\colon \mathscr{P}(\mathbb{N}) \to \mathscr{P}(\mathbb{N})$ by $\Upsilon(\mathsf{A}) := \mathbb{N}$ if $\mathsf{A} \in \mathscr{U}$ and $\Upsilon(\mathsf{A}) = \emptyset$ otherwise.

The proof of Lemma 17.6.3. *This proof was wrong. Here is what it should have been.*

Proof. Throughout this proof $\mathscr{B}(H)_1$ is considered with respect to the weak operator topology, and 'Borel' refers to Borel sets with respect to this topology. We will use the notation from Definition 17.4.1. It will also be convenient to use the notation $\mathscr{D}_\mathsf{X}[\mathbf{E}]$ and $\mathsf{D}_\mathsf{X}[\mathbf{E}]$ with the intervals in $\mathbf{E}$ indexed by $\{0,1\}^{<\mathbb{N}}$ and $\mathsf{X} \subseteq \{0,1\}^{<\mathbb{N}}$. Fix $\mathbf{E}$ so that $\lim_n \min_{|s|=n} |E_s| = \infty$. If $\mathsf{X} \subseteq \{0,1\}^{<\mathbb{N}}$ is infinite and included in a branch of $\{0,1\}^{<\mathbb{N}}$, then this branch is denoted $\mathsf{B}(\mathsf{X})$. Fix a discretization $\mathsf{D}[\mathbf{E}]$ of $\mathscr{D}[\mathbf{E}]$. The definition of a discretization depends on the indexing of the intervals of $\mathbf{E}$ by $\mathbb{N}$, but all that we need is that for every n and every $s \in \{0,1\}^n$ the set D_s is 2^{-n}-dense in the unit ball of $\mathscr{D}_{\{n\}}[\mathbf{E}]$. Also fix $d \geq (2\varepsilon)^{-1}$ and $n \geq 1$. Let (omitting $[\mathbf{E}]$ when clear from the context)

$$\mathscr{X} := \{(\mathsf{X},a) : \mathsf{B}(\mathsf{X}) \text{ is defined and } a \in \mathsf{D}_\mathsf{X}\}.$$

and let $\{(\mathsf{X},a),(\mathsf{Y},b)\} \in M_0^{d,n}$ if the following conditions are satisfied (writing p_X for $p_\mathsf{X}^\mathbf{E}$):

($M_0^d 1$) $\mathsf{B}(\mathsf{X}) \neq \mathsf{B}(\mathsf{Y})$,

($M_0^d 2$) $p_\mathsf{X} b = p_\mathsf{Y} a$, and

($M_0^{d,n} 3$) $\max(\|p_{[n,\infty)}(\Phi_*(a)q_\mathsf{Y} - q_\mathsf{X}\Phi_*(b))\|, \|p_{[n,\infty)}(q_\mathsf{Y}\Phi_*(a) - \Phi_*(b)q_\mathsf{X})\|) > 1/d$.

These conditions are symmetric and they define a partition $[\mathscr{X}]^2 = M_0^{d,n} \cup M_1^{d,n}$. In order to topologize $\mathscr{X}$, identify $(\mathsf{X},a) \in \mathscr{X}$ with

$$(\mathsf{B}(\mathsf{X}),\mathsf{X},a,q_\mathsf{X},\Phi_*(a)) \in \{0,1\}^{\mathbb{N}} \times \mathscr{P}(\{0,1\}^{<\mathbb{N}}) \times \mathsf{D} \times \mathscr{B}(H)^2_{\leq 1}$$

where $\{0,1\}^{\mathbb{N}}$, $\mathscr{P}(\{0,1\}^{<\mathbb{N}})$, and D are equipped with their standard compact metric topologies and $\mathscr{B}(H)_{\leq 1}$ is equipped with the weak operator topology. Consider $\mathscr{X}$ with respect to the subspace topology.

Claim. *For all d and n, the partition $[\mathscr{X}]^2 = M_0^{d,n} \cup M_1^{d,n}$ is open.*

Proof. The condition ($M_0^d 1$) is open. Once it is satisfied, $p_\mathsf{X} b$ and $p_\mathsf{Y} a$ are taking values in the finite set $\prod_{s \in \mathsf{X} \cap \mathsf{Y}} \mathsf{D}(s)$ and therefore ($M_0^d 2$) is open relative to ($M_0^d 1$). Since the set $\{b : \|p_{[n,\infty)} b\| > 1/d\}$ is WOT-open, the condition ($M_0^{d,n} 3$) is open.

It will be convenient to use OCA_∞ in place of OCA_T (see Theorem 8.6.6 and the discussion preceding it). The conclusion of the following claim is the negation of one of the alternatives of OCA_∞.

Claim. *For every d, there is no uncountable $\mathsf{Z} \subseteq \{0,1\}^{\mathbb{N}}$ such that some continuous $f\colon \mathsf{Z} \to \mathscr{X}$ satisfies $\{f(a),f(b)\} \in M_0^{d,\Delta(a,b)}$ for all distinct a and b in Z.*

Proof. Assume otherwise and fix d, $\mathsf{Z} \subseteq \{0,1\}^{\mathbb{N}}$, and a continuous $f\colon \mathsf{Z} \to \mathscr{X}$ as in (the negation of) the statement of the claim.

Since $ap_{\mathsf{Y}} = bp_{\mathsf{X}}$ for all (X,a) and (Y,b) in $\mathscr{H}$, there exists $c \in \mathsf{D}$ such that $cp_{\mathsf{X}} = a$ for all $(\mathsf{X},a) \in \mathscr{H}$ and $\|c\| \leq 1$. (Let $c(s) := a(s)$ for any $(\mathsf{X},a) \in \mathscr{H}$ such that $\mathsf{X} \in [s]$.)

Fix $\delta < 1/(2d)$. Since Φ_* lifts Φ, $q_{\mathsf{X}}\Phi_*(c) - \Phi_*(a)$ and $\Phi_*(c)q_{\mathsf{X}} - \Phi_*(a)$ are compact for every $(\mathsf{X},a) \in \mathscr{H}$. Fix $n = n((\mathsf{X},a)$ large enough so that

$$\max(\|p_{[n,\infty)}(\Phi_*(c)q_{\mathsf{X}} - \Phi_*(a))\|, \|p_{[n,\infty)}(q_{\mathsf{X}}\Phi_*(c) - \Phi_*(a))\|) < \delta.$$

By replacing Z with an uncountable set, we may assume that there is n such that $n = n(\mathsf{X},a)$ for all $(\mathsf{X},a) \in \mathsf{Z}$. Since Z is uncountable, it contains y and z such that $\Delta(y,z) > n$. Let (X,a) and (Y,b) to be the f-images of y and z. By the latest displayed formula,

$$p_{[n,\infty)}\Phi_*(a)q_{\mathsf{Y}} \approx^{\delta} p_{[n,\infty)}q_{\mathsf{X}}\Phi_*(c)q_{\mathsf{Y}} \approx^{\delta} p_{[n,\infty)}q_{\mathsf{X}}\Phi_*(b)$$

and therefore $\|p_{[n,\infty)}(\Phi_*(a)q_{\mathsf{Y}} - q_{\mathsf{X}}\Phi_*(b))\| < 1/d$. An analogous argument shows that $\|p_{[n,\infty)}(q_{\mathsf{Y}}\Phi_*(a) - \Phi_*(b)q_{\mathsf{X}})\| < 1/d$.

This implies $\{(\mathsf{X},a),(\mathsf{Y},b)\} \in M_1^{d,n}$, contradicting $\Delta(y,z) > n$.

By OCA_∞ and the latest Claim, there are sets $\mathscr{X}_n^d$, for $n \in \mathbb{N}$, so that each $\mathscr{X}_n^d$ is $M_1^{d,n}$-homogeneous and $\mathscr{X} \subseteq \bigcup_n \mathscr{X}_n^d$. For distinct (X,a) and (Y,b) in $\mathscr{X}$ and $k \in \mathbb{N}$ write

$$\Delta((\mathsf{X},a),(\mathsf{Y},b)) := \min\{k : (\exists s \in \{0,1\}^k)(s \in \mathsf{X}\Delta\mathsf{Y}$$
$$\text{or } (s \in \mathsf{X}\cap\mathsf{Y} \text{ and } a(s) \neq b(s)))\}.$$

For $k \in \mathbb{N}$ let

$$e_k := p_{[0,k)}.$$

For every n, fix a countable $\mathscr{E}_n^d \subseteq \mathscr{X}_n^d$ dense in the topology described after the definition of $M_0^{d,n}$. Since $M_0^{d,n}$ is open, the closure of any $M_1^{d,n}$-homogeneous set is $M_1^{d,n}$-homogeneous. Fix any branch $\tilde{\mathsf{B}}$ of $\{0,1\}^{<\mathbb{N}}$ that does not belong to the countable set $\{\mathsf{B}(\mathsf{X}) : (\mathsf{X},a) \in \bigcup_n \mathscr{E}_n^d\}$.

For all k and n, both sets $\{0,1\}^{<k}$ and $\mathsf{D}_{\{0,1\}^{<k}}$ are finite and the projection e_k has finite rank. Therefore the metric space $\{0,1\}^{<k} \times \mathsf{D}_{\{0,1\}^{<k}} \times \mathscr{B}(e_k[H])_1$ is compact. Choose $\mathsf{F}_{k,n} \Subset \mathscr{E}_n^d$ so that for every $(\mathsf{X},a) \in \mathscr{E}_n^d$ there exists $(\mathsf{Y},b) \in \mathsf{F}_{k,n}$ that satisfies $\Delta((\mathsf{X},a),(\mathsf{Y},b)) > k$ and

$$\max(\|(\Phi_*(p_{\mathsf{X}}) - \Phi_*(p_{\mathsf{Y}}))e_k\|, \|(\Phi_*(a) - \Phi_*(b))e_k\|) < 1/k.$$

Since $\mathscr{E}_n^d$ is dense in $\mathscr{X}_n^d$, for every $(\mathsf{X},a) \in \mathscr{X}_n^d$ there exists $(\mathsf{Y},b) \in \mathsf{F}_{k,n}$ with these properties.

For any infinite $\mathsf{Y} \subseteq \mathbb{N}$, the set $\bigcup_{k \in \mathsf{Y}} \mathsf{F}_{k,n}$ is dense in $\mathscr{X}_n^d$ with respect to the topology defined after $M_0^{d,n}$. Let $k(0) := 0$ and for $j \in \mathbb{N}$ choose $k(j+1) > k(j)$ to be the minimal such that (writing $\tilde{\mathsf{B}} \upharpoonright k$ for the unique s in $\tilde{\mathsf{B}}$ which satisfies $|s| = k$)

$$\tilde{\mathsf{B}} \upharpoonright k(j+1) \neq \mathsf{B}(\mathsf{Y}) \upharpoonright k(j+1) \text{ for all } (\mathsf{Y},b) \in \bigcup_{n \leq k} \mathsf{F}_{k(j),n}.$$

Let
$$\tilde{\mathsf{X}} := \{\tilde{\mathsf{B}} \upharpoonright k(j) : j \in \mathbb{N}\}$$
and consider the compact metrizable space $\mathsf{D}_{\tilde{\mathsf{X}}} \times \mathscr{B}(H)_{\leq 1}$; it's descriptive set theory time again! For $(\mathsf{Y}, b) \in \mathscr{X}_n^d$ and $k \in \mathbb{N}$ the set

$$\mathscr{W}(\mathsf{Y}, b, k) := \{(a, c) \in \mathsf{D}_{\tilde{\mathsf{X}}} \times \mathscr{B}(H)_{\leq 1} : (\tilde{\mathsf{X}}, a) \in \mathscr{X}_n^d, \ \Delta((\tilde{\mathsf{X}}, a), (\mathsf{Y}, b)) > k,$$
$$\max(\|(\Phi_*(b) - c)e_k\|, \|(q_{\tilde{\mathsf{X}}} - q_{\mathsf{Y}})e_k\|) \leq 1/k\}$$

is closed. Therefore the set

$$\mathscr{Z}_n := \bigcap_j \bigcup \{\mathscr{W}(\mathsf{Y}, b, k(j)) : (\mathsf{Y}, b) \in \mathsf{F}_{k(j),n}\}$$

is Borel.

Claim. *For every $n \in \mathbb{N}$ the following holds.*
  (1) If $(\tilde{\mathsf{X}}, a) \in \mathscr{X}_n^d$ then $(a, \Phi_*(a)) \in \mathscr{Z}_n$.
  (2) If $(a, c) \in \mathscr{Z}_n$, then $q_{\tilde{\mathsf{X}}} c \approx_{1/d}^{\mathscr{K}} \Phi_*(a)$.
  (3) The set $\mathscr{Z}_n' := \{(a, q_{\tilde{\mathsf{X}}} c) : (a, c) \in \mathscr{Z}_n\}$ is $2/d$-narrow.

Proof. (1) The definition of $\mathsf{F}_{k(j),n}$ implies that for every j there is $(\mathsf{Y}, b) \in \mathsf{F}_{k(j),n}$ such that $(a, \Phi_*(a)) \in \mathscr{W}(\mathsf{Y}, b, k(j))$.

(2) Suppose that $(a, c) \in \mathscr{Z}_n$. We will prove $\|p_{[n,\infty)}(\Phi_*(a)q_{\tilde{\mathsf{X}}} - q_{\tilde{\mathsf{X}}} c)\| \leq 1/d$. Assume otherwise, and fix $\delta > 0$ such that

$$\|p_{[n,\infty)}(\Phi_*(a)q_{\tilde{\mathsf{X}}} - q_{\tilde{\mathsf{X}}})c\| > 2\delta + 1/d.$$

for some $\delta > 0$. Choose j large enough to have

$$\|p_{[n,\infty)}(\Phi_*(a)q_{\tilde{\mathsf{X}}} - q_{\tilde{\mathsf{X}}} c)e_{k(j)}\| > 2\delta + 1/d$$

and $j \geq \max(n, 2/\delta)$. Fix $(\mathsf{Y}, b) \in \mathsf{F}_{k(j),n}$ such that $(a, c) \in \mathscr{W}(\mathsf{Y}, b, k(j))$. Then

$$\max(\|p_{[n,\infty)}(\Phi_*(b) - c)e_{k(j)}\|, \|p_{[n,\infty)}(q_{\tilde{\mathsf{X}}} - q_{\mathsf{Y}})e_{k(j)}\|) \leq 1/k(j) < \delta.$$

Also, $\{(\tilde{\mathsf{X}}, a), (\mathsf{Y}, b)\} \in M_1^d$ and $(M_0^d 1)$ holds. By the choices of $k(j+1)$ and $\tilde{\mathsf{X}}$, for every $s \in \tilde{\mathsf{X}}$ we have $|s| \leq k(j)$ or $|s| \geq k(j+1)$, hence $\Delta((\tilde{\mathsf{X}}, a), (\mathsf{Y}, b)) > k(j)$ and $(M_0^d 2)$ holds as well.

Therefore $(M_0^{d,n} 3)$ fails, and $\|p_{[n,\infty)}(\Phi_*(a)q_{\mathsf{Y}} - q_{\tilde{\mathsf{X}}}\Phi_*(b))\| \leq 1/d$. Temporarily writing $x \approx_{\delta,k} y$ if $\|(x - y)e_k\| < \delta$, we have

$$p_{[n,\infty)}\Phi_*(a)q_{\tilde{\mathsf{X}}} \approx_{\delta,k(j)} p_{[n,\infty)}\Phi_*(a)q_{\mathsf{Y}} \approx_{1/d} p_{[n,\infty)}q_{\tilde{\mathsf{X}}}\Phi_*(b) \approx_{\delta,k(j)} p_{[n,\infty)}q_{\tilde{\mathsf{X}}} c.$$

Therefore $\|p_{[n,\infty)}(\Phi_*(a)q_{\tilde{\mathsf{X}}} - q_{\tilde{\mathsf{X}}} c)e_{k(j)}\| \leq 2\delta + 1/d$; contradiction. Since $\delta > 0$ was arbitrary, $\|p_{[n,\infty)}(\Phi_*(a)q_{\tilde{\mathsf{X}}} - q_{\tilde{\mathsf{X}}} c)\| \leq 1/d$. Since $\Phi_*(a) - \Phi_*(a)q_{\tilde{\mathsf{X}}}$ is compact, $\Phi_*(a) \approx_{1/d}^{\mathscr{K}} q_{\tilde{\mathsf{X}}} c$ follows.

(3) To prove that $\mathscr{Z}_n'$ is $2/d$-narrow, note that by (2) for all (a, c) and (a, c') in $\mathscr{Z}_n$ we have $q_{\tilde{\mathsf{X}}} c \approx_{1/d}^{\mathscr{K}} \Phi_*(a) \approx_{1/d}^{\mathscr{K}} q_{\tilde{\mathsf{X}}} c'$.

The sets $\mathscr{Z}_n'$, for $n \in \mathbb{N}$, defined in Claim are Borel, each one of them is $2/d$-narrow, and they cover the graph of the restriction of Φ_* to $\mathsf{D}_{\tilde{\mathsf{X}}}[\mathbf{E}]$. Since $2/d \leq \varepsilon$, this restriction is a σ-narrow ε-approximation of Φ on $\mathsf{D}_{\tilde{\mathsf{X}}}[\mathbf{E}]$.

Lemma 17.7.1. *There is nothing wrong here, although the statement of this lemma may appear lopsided. Note that* $1/d$*-narrow analytic sets exist and that in* (2) *there is no assumption that* $\mathscr{Z}$ *has any relation to* Φ *whatsoever. Thus the proof proceeds by fixing any* $1/d$*-narrow analytic set and showing that if (1) fails then (2) holds.*

p. 445–446. *The sentence*

Since $\mathscr{W}(a)$ *is, being a continuous image of an analytic set, analytic, it has the Property of Baire (§B.2.1).*

is, although correct, out of place and should be deleted.

page 446. The last displayed formula on this page,

$$\Xi_0(a(0)+b(0)+c)q_{A(0)} \not\preccurlyeq_{1/d}^{\mathscr{K}} \Phi_*(a(0)),$$

(not the previous one) should be tagged as (17.5).

p. 448. Both instances of $\mathbb{F}(d) \subseteq \mathbb{F}(d+1)$ on this page should be replaced with $\mathbb{F}(d) \supseteq \mathbb{F}(d+1)$.

On the same page, lines -7 and -9, in the subscript $k(d+1)$ should be $k(d)$.

Corollary 17.8.4. *Before the last sentence ('Since* $\Phi_1 = \Phi_{\mathbb{F}}\ldots$*') insert the following sentence (it will be Lemma 17.5.8, not Lemma* 17.5.8*):*

By Lemma 17.5.8, the restriction of Φ to $\mathscr{F}[\mathbf{E}]$ is implemented by a unitary for every $\mathbf{E} \in \mathrm{Part}_{\mathbb{N}}$.

Exercise 17.9.11. Each of the instances of ℓ_∞/c_0 should be changed to $\mathscr{P}(\mathbb{N})/\mathrm{Fin}$.

Notes to Chapter 17. (i) There is a gap in the proof of the main result of [1] (see [3, Remark 11.A.6]). Fortunately, the result is correct since it has been proved by Truss in [13]. To make the long story short:

In the last sentence in the comments on §17.1, the reference to [1] (i.e., **[11]** in the text) should be replaced by a reference to (the main result of) [13].

(ii) *The last sentence of the paragraph commenting on §17.2 should read as follows.*

Theorem 17.2.3 and Theorem 17.2.6 are taken from [8]. Weaker versions of these results first appeared in [4, §5].

p. 456, item 5. $\wedge z) \in x$ should be $\wedge z \in x$).

p. 456, item 13. 'whose elements are' should be 'that includes' (by applying the Comprehension Scheme, one proves that the statement in item 13 as printed is correct).

§A.4. At the end of the second sentence add 'if $X \neq \emptyset$'.

§A.6. In the first line, (M, ε) should be (M, E).

Example B.1.3. $\sum_{n \in x} 3^{-n}$ should be $\sum_{x \in x} 2 \cdot 3^{-n}$.

Example B.1.4. replace the end with something like '$n_j \in \mathbb{N} \setminus \{0\}$, for $j \in \mathbb{N}$, corresponding to the continued fraction $x = \frac{1}{n_0 + \frac{1}{n_1 + \cdots}}$. Alternatively, for a natural bijection between $\mathbb{Q}$ and $\mathbb{Z}^{\mathbb{N}}$ see [9, Theorem 1.1].'

§B.2.2. In the second line, $f\colon \pi_{\mathsf{X}}[\mathsf{A}] \to \mathsf{X}$ should be $f\colon \pi_{\mathsf{X}}[\mathsf{A}] \to \mathsf{Y}$.

Theorem B.2.14: *in addition to the set $\{x \in \mathsf{X} | \mathsf{A}_x$ is nonmeagre$\}$, the set $\{x \in \mathsf{X} | \mathsf{A}_x$ is comeagre$\}$ is also analytic (*[7, Theorem 29.22]*) and this was used in the proof of Lemma 17.7.1.*

Example C.1.5 (1). $f\colon X \to \mathbb{C}$ should be $f\colon X \to \mathbb{K}$.

The sentence following Theorem C.5.3. $\mu(a) = 0$ should be $\mu(A) = 0$.

Proposition C.6.7. $z \in \mathbb{Z}$ should be $z \in \mathbb{C}$.

Definition D.1.1 (9). Add 'is a term' at the end.

Definition D.2.10. In the second line, the second $\mathscr{L}_1$ should be $\mathscr{L}_0$.

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