Massive C*-algebras, Winter 2021. The fourth problem set. Solutions due April 5, 2021. Do any four of the following; note that some of the problems are closely related (U stands for a nonprincipal ultrafilter on N and A stands for a C*-algebra).

Exercise 1. Suppose that X and Y are Polish spaces and that f is a function from an uncountable $X_0 \subseteq X$ into Y. Prove that OCA$_T$ there exists an uncountable $X_1 \subseteq X_0$ such that the restriction of f to $X_1$ is continuous.

Exercise 2. Suppose that X and Y are Polish spaces and that f is a function from an uncountable $X_0 \subseteq X$ into Y. Prove that there exists $f : X \to Y$ such that the restriction of f to $X_0 \subseteq X$ is discontinuous, for every $X_0$ of cardinality $2^{\aleph_0}$.

Exercise 3. On $\mathcal{B}(H)_{\leq 1}$ consider the following three topologies: weak operator topology, strong operator topology, and strong$^*$ operator topology. (Each of these spaces is Polish.) Prove that the Borel structures generated by these topologies coincide.

Exercise 4. Suppose that A is a $\sigma$-unital C*-algebra. Prove that if a *-homomorphism $\Phi : \ell_\infty/c_0 \to \mathcal{M}(A)/A$ has a C-measurable lifting then for an infinite $X \subseteq \mathbb{N}$ it has a continuous lifting on $\ell_\infty(X)_1$.

(More is true: The assumptions imply that $\Phi$ has a continuous lifting on $\ell_\infty$, but this requires a more intricate construction.)

For every $n \geq 2$ fix an isomorphism $Q(H) \cong M_n(Q(H))$ and consider the following assumption:

(V) Assume that every endomorphism of $Q(H)$ is unitarily equivalent to the endomorphism $\Phi \otimes 1_n : Q(H) \to M_n(Q(H))$ for some $n$.

Exercise 5. Prove that (V) implies there are C*-algebras A and B such that $A \hookrightarrow Q(H)$, $B \hookrightarrow Q(H)$, but $A \otimes_\alpha B \not\hookrightarrow Q(H)$ for any tensor product $\otimes_\alpha$.

Exercise 6. Prove that (V) implies there is a countable inductive system $(A_n)$ such that $A_n \hookrightarrow Q(H)$ for all n, but $\lim_n A_n \not\hookrightarrow Q(H)$.

The following exercise may be quite hard:

Exercise 7. Is there a *-homomorphism $\Phi : \ell_\infty/c_0 \to \ell_\infty/c_0$ that has a $\sigma$-narrow lifting, but no continuous lifting?