Massive C*-algebras, Winter 2021. The fourth problem set. Solutions due April 5, 2021. Do any four of the following; note that some of the problems are closely related ($\mathcal{U}$ stands for a nonprincipal ultrafilter on $\mathbb{N}$ and A stands for a C*-algebra).

Exercise 1. *Suppose that X and Y are Polish spaces and that f is a function from an uncountable $X_0 \subseteq X$ into Y. Prove that $\mathrm{OCA_T}$ there exists an uncountable $X_1 \subseteq X_0$ such that the restriction of f to X_1 is continuous.*

Exercise 2. *Suppose that X and Y are Polish spaces and that f is a function from an uncountable $X_0 \subseteq X$ into Y. Prove that there exists $f\colon X \to Y$ such that the restriction of f to $X_0 \subseteq X$ is discontinuous, for every X_0 of cardinality $2^{\aleph_0}$.*

Exercise 3. *On $\mathcal{B}(H)_{\leq 1}$ consider the following three topologies: weak operator topology, strong operator topology, and strong* operator topology. (Each of these spaces is Polish.) Prove that the Borel structures generated by these topologies coincide.*

Exercise 4. *Suppose that A is a σ-unital C*-algebra. Prove that if a *-homomorphism $\Phi\colon \ell_\infty/c_0 \to \mathcal{M}(A)/A$ has a C-measurable lifting then for an infinite $X \subseteq \mathbb{N}$ it has a continuous lifting on $\ell_\infty(X)_1$.*

(More is true: The assumptions imply that Φ has a continuous lifting on ℓ_∞, but this requires a more intricate construction.)

For every $n \geq 2$ fix an isomorphism $\mathcal{Q}(H) \cong M_n(\mathcal{Q}(H))$ and consider the following assumption:

(V) Assume that every endomorphism of $\mathcal{Q}(H)$ is unitarily equivalent to the endomorphism $\Phi \otimes 1_n\colon \mathcal{Q}(H) \to M_n(\mathcal{Q}(H))$ for some n.

Exercise 5. *Prove that (V) implies there are C*-algebras A and B such that $A \hookrightarrow \mathcal{Q}(H)$, $B \hookrightarrow \mathcal{Q}(H)$, but $A \otimes_\alpha B \not\hookrightarrow \mathcal{Q}(H)$ for any tensor product $\otimes_\alpha$.*

Exercise 6. *Prove that (V) implies there is a countable inductive system (A_n) such that $A_n \hookrightarrow \mathcal{Q}(H)$ for all n, but $\lim_n A_n \not\hookrightarrow \mathcal{Q}(H)$.*

The following exercise may be quite hard:

Exercise 7. *Is there a *-homomorphism $\Phi\colon \ell_\infty/c_0 \to \ell_\infty/c_0$ that has a σ-narrow lifting, but no continuous lifting?*