Massive C*-algebras, Winter 2021. The third problem set. Solutions due March 19, 2021. Do any four of the following; note that some of the problems are closely related (\(\mathcal{U}\) stands for a nonprincipal ultrafilter on \(\mathbb{N}\) and \(A\) stands for a C*-algebra).

**Exercise 1.** Prove that for every \(\varepsilon > 0\) there exists \(\delta > 0\) such that the formula \(\varphi(x) := \max(\|x - x^*\|, \|x - x^2\|)\) satisfies the following. If \(A\) is a C*-algebra and \(a \in A\), then \(\varphi^A(a) < \delta\) implies there exists \(b \in A\) such that \(\varphi^A(b) = 0\) and \(\|a - b\| < \varepsilon\).

Use this to prove that \(a \in A_\mathcal{U}\) is a projection if and only if it has a representing sequence \((a_n)\) such that \(a_n\) is a projection for all \(n\).

**Exercise 2.** Prove that if \(A\) is unital then \(a \in A_\mathcal{U}\) is a unitary if and only if it has a representing sequence \((a_n)\) such that \(a_n\) is a unitary for all \(n\).

The zero set of a formula \(\varphi(x)\) is \(Z^A(\varphi) = \{a \in A| \varphi^A(a) = 0\}\).

**Exercise 3.** Suppose that \(\varphi(x)\) is a formula in the language of C*-algebras. Prove that the following are equivalent.

1. For every \(\varepsilon > 0\) there exists \(\delta > 0\) such that in every C*-algebra \(A\), for every \(a \in A\), \(|\varphi^A(a)| < \delta\) implies that there exists \(b \in Z^A(\varphi)\) such that \(\|b - a\| < \varepsilon\).
2. For every \(A\), every \(\mathcal{U}\), and every \(a \in A_\mathcal{U}\) we have \(a \in Z^A(\varphi)\), if and only if \(a\) has a representing sequence \((a_n)\) such that \(a_n \in Z^A(\varphi)\) for all \(n\).

**Exercise 4.** Find a simple combinatorial condition on \(E\) and \(F\) in Part N equivalent to the conjunction of \(E \leq^* F\) and \(F \leq^* E\).

Given an infinite cardinal \(\kappa\), an ultrafilter \(\mathcal{U}\) on a set \(X\) is \(\kappa\)-regular if there exists a family \(X_\alpha\), for \(\alpha < \kappa\), of sets in \(\mathcal{U}\) such that every \(x \in X\) belongs to at most finitely many of the \(X_\alpha\)'s.

**Exercise 5.** Prove that an ultrafilter is \(\aleph_0\)-regular if and only if it is countably incomplete.

**Exercise 6.** Prove that if \(\mathcal{U}\) is a \(\kappa\)-regular ultrafilter then the ultraproduct \(\Pi_{\mathcal{U}} A_\xi\) is \(\kappa^+\)-saturated for any family \((A_\xi)\).

Conclude that for every \(A\) and every consistent type \(t(\bar{x})\) over \(A\) there exists an elementary extension of \(A\) that realizes \(t(\bar{x})\).

Let \(C_b([0, 1], A)\) denote the algebra of all \(A\)-valued, bounded continuous functions on \([0, 1]\). The algebra \(C_0([0, 1], A)\) is an ideal in \(C_b([0, 1], A)\), and \(A\) is identified with the subalgebra consisting of the equivalence classes of constant functions in \(A_\infty := C_b([0, 1], A)/C_0([0, 1], A)\).

**Exercise 7.** Prove that the density character of \(A_\infty\) is at least \(\varepsilon\) for every C*-algebra \(A\).

**Exercise 8.** A subset \(\mathcal{J}\) of \(\mathcal{P}(\mathbb{N})\) is hereditary if \(A \subseteq B\) and \(B \in \mathcal{J}\) implies \(A \in \mathcal{J}\). Suppose that \(\mathcal{J} \subseteq \mathcal{P}(\mathbb{N})\) is hereditary and closed under making finite changes of its elements. Prove that \(\mathcal{J}\) is meager if and only if there are pairwise disjoint \(I(n) \in \mathbb{N}\) such that \(\bigcup_{n \in A} I(n) \in \mathcal{J}\) if and only if \(A\) is finite.
Exercise 9. For $f \in \mathbb{N}^\mathbb{N}$ let $\Gamma_f := \{(m,n)|n \leq f(m)\}$. A coherent family of functions indexed by $\mathbb{N}^\mathbb{N}$ is a family $f_g : \Gamma_g \to \mathbb{N}$, for $g \in \mathbb{N}^\mathbb{N}$, such that $\{(m,n) \in \Gamma_g \cap \Gamma_h : f_g(m,n) \neq f_h(m,n)\}$ is finite for all $g$ and $h$. It is trivial if there exists $f : \mathbb{N}^2 \to \mathbb{N}$ such that $\{(m,n) \in \Gamma_g : f_g(m,n) \neq f(m,n)\}$ is finite for all $g$.

Prove that OCA$^\text{T}$ implies that every coherent family indexed by $\mathbb{N}^\mathbb{N}$ is trivial.