Massive C*-algebras, Winter 2021. The third problem set. Solutions due March 19, 2021. Do any four of the following; note that some of the problems are closely related ($\mathcal{U}$ stands for a nonprincipal ultrafilter on $\mathbb{N}$ and A stands for a C*-algebra).

Exercise 1. *Prove that for every $\varepsilon > 0$ there exists $\delta > 0$ such that the formula $\varphi(x) := \max(\|x - x^*\|, \|x - x^2\|)$ satisfies the following. If A is a C*-algebra and $a \in A$, then $\varphi^A(a) < \delta$ implies there exists $b \in A$ such that $\varphi^A(b) = 0$ and $\|a - b\| < \varepsilon$.*

Use this to prove that $a \in A_{\mathcal{U}}$ is a projection if and only if it has a representing sequence (a_n) such that a_n is a projection for all n.

Exercise 2. *Prove that if A is unital then $a \in A_{\mathcal{U}}$ is a unitary if and only if it has a representing sequence (a_n) such that a_n is a unitary for all n.*

The zero set of a formula $\varphi(x)$ is $Z^A(\varphi) = \{a \in A | \varphi^A(a) = 0\}$.

Exercise 3. *Suppose that $\varphi(x)$ is a formula in the language of C*-algebras. Prove that the following are equivalent.*

(1) *For every $\varepsilon > 0$ there exists $\delta > 0$ such that in every C*-algebra A, for every $a \in A$, $|\varphi^A(a)| < \delta$ implies that there exists $b \in Z^A(\varphi)$ such that $\|b - a\| < \varepsilon$.*
(2) *For every A, every $\mathcal{U}$, and every $a \in A_{\mathcal{U}}$ we have $a \in Z^{A_{\mathcal{U}}}(\varphi)$, if and only if a has a representing sequence (a_n) such that $a_n \in Z^A(\varphi)$ for all n.*

Exercise 4. *Find a simple combinatorial condition on $\mathbf{E}$ and $\mathbf{F}$ in $\mathrm{Part}_{\mathbb{N}}$ equivalent to the conjunction of $\mathbf{E} \leq^* \mathbf{F}$ and $\mathbf{F} \leq^* \mathbf{E}$.*

Given an infinite cardinal κ, an ultrafilter $\mathcal{U}$ on a set X is *κ-regular* if there exists a family X_α, for $\alpha < \kappa$, of sets in $\mathcal{U}$ such that every $x \in \mathsf{X}$ belongs to at most finitely many of the X_α's.

Exercise 5. *Prove that an ultrafilter is $\aleph_0$-regular if and only if it is countably incomplete.*

Exercise 6. *Prove that if $\mathcal{U}$ is a κ-regular ultrafilter then the ultraproduct $\prod_{\mathcal{U}} A_\xi$ is κ^+-saturated for any family (A_ξ).*

Conclude that for every A and every consistent type $\mathsf{t}(\bar{x})$ over A there exists an elementary extension of A that realizes $\mathsf{t}(\bar{x})$.

Let $C_b([0,1), A)$ denote the algebra of all A-valued, bounded continuous functions on $[0,1)$. The algebra $C_0([0,1), A)$ is an ideal in $C_b([0,1), A)$, and A is identified with the subalgebra consisting of the equivalence classes of constant functions in

$$A_\infty := C_b([0,1), A)/C_0([0,1), A).$$

Exercise 7. *Prove that the density character of A_∞ is at least $\mathfrak{c}$ for every C*-algebra A.*

Exercise 8. *A subset $\mathcal{J}$ of $\mathcal{P}(\mathbb{N})$ is* hereditary *if $\mathsf{A} \subseteq \mathsf{B}$ and $\mathsf{B} \in \mathcal{J}$ implies $\mathsf{A} \in \mathcal{J}$. Suppose that $\mathcal{J} \subseteq \mathcal{P}(\mathbb{N})$ is hereditary and closed under making finite changes of its elements. Prove that $\mathcal{J}$ is meager if and only if there are pairwise disjoint $I(n) \Subset \mathbb{N}$ such that $\bigcup_{n \in \mathsf{A}} I(n) \in \mathcal{J}$ if and only if A is finite.*

Exercise 9. *For* $f \in \mathbb{N}^{\mathbb{N}}$ *let* $\Gamma_f := \{(m,n) | n \leq f(m)\}$. *A* coherent family of functions *indexed by* $\mathbb{N}^{\mathbb{N}}$ *is a family* $f_g \colon \Gamma_g \to \mathbb{N}$, *for* $g \in \mathbb{N}^{\mathbb{N}}$, *such that* $\{(m,n) \in \Gamma_g \cap \Gamma_h : f_g(m,n) \neq f_h(m,n)\}$ *is finite for all* g *and* h. *It is* trivial *if there exists* $f \colon \mathbb{N}^2 \to \mathbb{N}$ *such that* $\{(m,n) \in \Gamma_g : f_g(m,n) \neq f(m,n)\}$ *is finite for all* g.

Prove that $\mathrm{OCA_T}$ *implies that every coherent family indexed by* $\mathbb{N}^{\mathbb{N}}$ *is trivial.*