Massive C*-algebras, Winter 2021. the second problem set. Solutions due February 19, 20201. Do any four of the following ($\mathcal{U}$ stands for a nonprincipal ultrafilter on $\mathbb{N}$).

Exercise 1. *Find a separable* C**-subalgebra* A *of* $(M_{2^\infty})_{\mathcal{U}}$ *and an automorphism of* A *that does not extend to an automorphism of* $(M_{2^\infty})_{\mathcal{U}}$.

Exercise 2. *Prove that there exists a separable* C**-algebra* B *such that every separable* C**-algebra is isomorphic to a subalgebra of* $B_{\mathcal{U}}$ *(you don't need the Continuum Hypothesis for this).*[1]

Exercise 3. *Suppose that* A *and* B *are elementarily equivalent and that* $\mathtt{t}$ *is a type over the empty set. Prove that* $\mathtt{t}$ *is approximately satisfiable in* A *if and only if it is approximately satisfiable in* B.

In Exercise 4 you can use the conclusion of Exercise 3.

Exercise 4. *Let* F[2] *be a finite-dimensional* C**-algebra (hence it is a direct sum of full matrix algebras, by the Artin–Wedderburn Theorem). Prove that for a separable unital* C**-algebra* A *the following are equivalent.*[3]

(1) F *unitally embeds into* $A_{\mathcal{U}} \cap A'$.
(2) $\bigotimes_{\mathbb{N}} F$*unitally embeds into* $A_{\mathcal{U}} \cap A'$.
(3) $\bigotimes_{\aleph_1} F$ *unitally embeds into* $A_{\mathcal{U}} \cap A'$.

Exercise 5. *Prove that every countably saturated* C**-algebra of density character* $\aleph_1$ *has cardinality* $\aleph_1$. *(Hint: This is a trick question; you need to prove that the existence of such algebra implies CH.)*

Question 6 is about ultrapowers of Banach spaces. The language of Banach spaces is a subset (i.e., reduct) of the language of C*-algebras, obtained by removing the symbols for multiplication and the adjoint operation. (I.e., the terms are linear combinations.) A C*-algebra (Banach space, any metric structure) of density character κ is *saturated* if every approximately satisfiable type of cardinality smaller than κ is realized.

Exercise 6. *The ultrapower* X *of a Banach space* X *is defined as* $\ell_\infty(X)/c_{\mathcal{U}}(X)$. *Prove that* every[4] *ultrapowerof* $\ell_2(\mathbb{N})$ *is fully saturated.*

(Hint: If this looks like a difficult question, you are doing something wrong.)

Exercise 7. *(Assumes some familiarity with the spatial tensor product of* C**-algebras; see §2.4) Suppose that* A *and* B *are* C**-algebras. Prove that the natural embedding of* $A_{\mathcal{U}} \otimes B_{\mathcal{U}}$ *into* $(A \otimes B)_{\mathcal{U}}$ *is surjective if and only if at least one of* A *and* B *is finite-dimensional.*

[1] It is an open problem, due to Kirchberg, whether the Cuntz algebra $\mathcal{O}_2$ has the property of B.

[2] You can do this exercise for $F = M_2(\mathbb{C})$ for full marks, but the general case is, with a good setup, easier.

[3] $\bigotimes_{\mathbb{N}} F$ and $\bigotimes_{\aleph_1} F$ stand for the tensor product of $\aleph_0$ ($\aleph_1$, respectively) copies of F; since F is finite-dimensional, they are uniquely defined.

[4] I.e., associated with an arbitrary ultrafilter on an arbitrary set.