**Massive C$^*$-algebras, Winter 2021. the first problem set. Solutions due February 1, 20201.** Do any four of the following.

**Exercise 1.** *(15.6.5) Prove that for* C$^*$*-subalgebras* $A$ *and* $B$ *of a* C$^*$*-algebra* $C$ *the following are equivalent.*

1. $A$ *and* $B$ *are* orthogonal, *i.e.,* $ab = 0$ *for all* $a \in A$ *and* $b \in B$.
2. $ab = ba = 0$ *for all* $a \in A$ *and* $b \in B$.
3. $ab = 0$ *for all* $a \in A_+$ *and all* $b \in B_+$.

**Exercise 2.** *(13.4.1) Suppose* $X$ *is a locally compact Hausdorff space. Prove that the multiplier algebra of* $C_0(X)$ *is isomorphic to the algebra* $C_b(X)$ *of all bounded, continuous, complex valued functions on* $X$.

*(The latter algebra is naturally isomorphic to* $C(\beta X)$*; you don't need to prove this.)*

**Exercise 3.** *(13.4.15, the noncommutative Tietze Exetension Theorem) Prove that every surjective* $^*$*-homomorphism between* C$^*$*-algebras extends to a surjective* $^*$*-homomorphism between their multiplier algebras.*

*Prove that not every* $^*$*-homomorphism between* C$^*$*-algebras extends to a* $^*$*-homomorphism between their multiplier algebras.*

**Exercise 4.** *(15.6.11) Suppose that* $B$ *is a separable* C$^*$*-subalgebra of a countably degree-1 saturated* C$^*$*-algebra* $C$. *If* $J$ *is an ideal of* $B$ *then there is a positive contraction* $f \in C \cap B'$ *such that* $af = a$ *for all* $a \in J$.

*If* $c \in C$ *satisfies* $Jc = \{0\}$ *then* $f$ *can be chosen to satisfy* $fc = 0$ *and* $fJc = \{0\}$.

**Exercise 5.** *(15.6.12) A* C$^*$*-algebra* $C$ *is* $\sigma$-sub-Stonean *if for every separable* C$^*$*-subalgebra* $A$ *of* $C$ *and all positive* $c$ *and* $d$ *in* $C$ *such that* $cAd = \{0\}$ *there are contractions* $f$ *and* $g$ *in* $A' \cap C$ *such that* $fg = 0$, $fc = c$ *and* $gd = d$. *Prove that every countably degree-1 saturated* C$^*$*-algebra is* $\sigma$*-sub-Stonean.*

**Exercise 6.** *(15.6.15) Recall that* $\mathcal{Z}_0$ *denotes the ideal of asymptotic density zero subsets of* $\mathbb{N}$:

$$\mathcal{Z}_0 := \left\{ \mathsf{X} \subseteq \mathbb{N} | \limsup_n \frac{|X \cap n|}{n} = 0 \right\}.$$

*Let* $J$ *be the ideal in* $\ell_\infty(\mathbb{N})$ *generated by* $\{\mathrm{proj}_{\mathsf{X}} : \mathsf{X} \in \mathcal{Z}_0\}$. *Prove that* $\ell_\infty(\mathbb{N})/J$ *is not countably degree-1 saturated. (For bonus points, prove that for any separable, unital,* C$^*$*-algebra* $A$ *the analogously defined quotient* $\ell_\infty(A)/J$ *is not countably degree-1 saturated.)*