Math 3260, W20. Assignment #3 solutions

(1) Let $G$ be a simple graph with at least 11 vertices and let $\bar{G}$ be its complement.
(a) Prove that at least one of $G$ and $\bar{G}$ is not planar.

Solution. (We write $\#V(G)$ for $n$, $\#E(G)$ for $m$.) We have $\#V(G) = \#V(\bar{G}) = 11$ and

$$\#E(G) + \#E(\bar{G}) = \#E(K_8) = 11 \cdot 10/2 = 55.$$ 

It was proved in class that in a simple planar graph $H$ we have $\#E(H) \leq 3\#V(H) - 6$. So if both $G$ and $\bar{G}$ were planar, we would have

$$55 = \#E(G) + \#E(\bar{G}) \leq 3(\#V(G) + \#V(G)) - 12 = 3 \cdot 22 - 12 = 54$$

which is a contradiction.
(b) Find a simple graph $G$ with 8 vertices such that both $G$ and $\bar{G}$ are planar.

Solution. This is $G$ (with the addition of an edge connecting vertices 1 and 3; it can clearly be drawn so that it does not intersect the other vertices, but not using the program that I used to draw this graph):

![Graph](image)

This is $\bar{G}$ (note that the middle square was rotated by $180^\circ$ to make the drawing more transparent; also, vertices 2 and 4...
should be connected)

(2) Which complete tripartite graphs $K_{r,s,t}$ are planar? Justify your answer.

Solution. If $\max(r, s, t) \leq 2$ then $K_{r,s,t}$ is planar. The only planar graphs $K_{r,s,t}$ such that $\max(r, s, t) = N \geq 3$ are $K_{N,1,1}$, $K_{1,1,N}$ and $K_{1,N,1}$. (These three graphs are clearly isomorphic.) Here are diagrams showing that $K_{N,1,1}$ and $K_{2,2,2}$ are planar.

It remains to prove that any graph $K_{r,s,t}$ such that $r \geq 3$ and $s \geq 2$ is not planar. Note that $K_{r,s,t}$ is a subgraph of $K_{r,s,t}$; this is because if $V = V_1 \sqcup V_2 \sqcup V_3$ is the partition of the set of vertices of $K_{r,s,t}$ such that $V_1, V_2$ and $V_3$ have cardinality $r, s$, and $t$, respectively, then $V_2' = V_2 \sqcup V_3$ has $s + t$ elements and every vertex of $V_1$ is connected to every vertex of $V_2'$.

Therefore $K_{3,2,1}$ contains a copy of $K_{3,3}$ and is therefore not planar. Also, if $r \geq 3$ and $s \geq 2$ then $s + t \geq 3$ hence $K_{r,s+t}$ has $K_{3,3}$ as subgraph, and we are done.

(3) Assume $G$ is a countable graph such that every finite subgraph of $G$ is $k$-colourable. Prove that $G$ is $k$-colourable.

Solution. For $A \subseteq V(G)$ let the restriction of $G$ to $A$ be the graph whose vertex set is $A$ and whose edges are exactly those
edges of $G$ that connect two vertices in $A$. Denote this graph by $G \upharpoonright A$. Fix an enumeration $V(G) = \{a_n | n \in \mathbb{N}\}$. We form another graph $G'$ whose vertices are pairs $(B, c)$, where $B = \{a_n | n \leq m\}$ for some $m = m(B)$ and $c$ is a $k$-colouring of $G \upharpoonright B$. Add a vertex $e_0$ (formally corresponding to the case $m = 0$) that is connected exactly to all $(B, c)$ such that $m(B) = 1$.

Vertices $(B, c)$ and $(B', c')$ are adjacent if and only if $m(B) + 1 = m(B')$ and $c$ is the restriction of $c'$ to $B$, or vice versa. Then $H$ is a countable graph, and each vertex has finite degree, since if $m(B) = m$ then there are at most $k^m$ $k$-colourings of $G \upharpoonright B$. It is also infinite, since by our assumption for every $m \in \mathbb{N}$ there is at least one colouring of $(B, c)$, where $m(B) = m$.

We claim that $H$ is connected. It will suffice to show that each vertex $(B, c)$ is connected to $e_0$. But if $B = \{a_1, \ldots, a_m\}$, let $B_i = \{a_1, \ldots, a_i\}$ $(0 \leq i \leq m)$ and let $c_i$ be the restriction of $c$ to $B_i$. Then $e_0 \rightarrow (B_1, c_1) \rightarrow (B_2, c_2) \rightarrow \ldots \rightarrow (B_m, c_m)$ is a path connecting $e_0$ and $(B, c)$.

By König’s lemma, $H$ has an infinite one-sided path with initial vertex $e_0$: $e_0 \rightarrow (A_1, c_1) \rightarrow (A_2, c_2) \rightarrow \ldots$. Then $A_i = \{a_1, a_2, \ldots, a_i\}$ for each $i$, and $\bigcup_{i=1}^{\infty} A_i = V(G)$. So $\bigcup_{i} c_i$ defines a $k$-coloring of $V(G)$.

(4) Prove that the chromatic polynomial of the cycle graph $C_n$ is $P_{C_n}(k) = (k - 1)^n + (-1)^n(k - 1)$.

**Solution.** Let $P_n$ denote the tree with $n$ vertices in which the maximum degree of a vertex is 2. (i.e., a line.) We know that $P_{C_n}(k) = P_{C_n-e}(k) - P_{C_{n/e}}(k)$ for any edge of $C_n$. Also, $C_n - e$ is $P_n$ and $C_{n/e}$ is $C_{n-1}$. So we have

$$P_{C_n}(k) = P_{P_n}(k) - P_{C_{n-1}}(k).$$

We know that $P_{P_n}(k) = k(k - 1)^n$.

We are going to show by induction on $n$ that the chromatic polynomial is given by the equation above. For $C_2$, the chromatic polynomial is $k(k - 1) = (k - 1)^2 + (-1)^2(k - 1)$. Assume that the chromatic polynomial for $C_{n-1}$ is given by $(k - 1)^{n-1} + (-1)^{n-1}(k - 1)$. It follows that

$$P_{C_n}(k) = P_{P_n}(k) - P_{C_{n-1}}(k)$$

$$= k(k - 1)^n - ((k - 1)^{n-1} + (-1)^{n-1}(k - 1))$$

$$= (k - 1)^{n-1} + (-1)^n(k - 1)$$

Therefore by induction we know that the formula holds for all $n$. 