The final exam will take place at the previously announced time. It will be conducted using Zoom. You will need a web camera and reliable internet connection. If you cannot arrange for both of these conditions, then you should contact me asap. You will be required to present your student photo ID in order to take the exam. If during the exam you exhibit a behaviour that may be interpreted as an attempt at cheating, or if there are problems with your video, or if you do not present your student photo ID card, you will have to take an oral exam. The exam will cover chapters 1–6 of Wilson’s book (5th edition; the 4th edition contains all the material, but in a different order).
Menger’s theorem

Suppose $v$ and $w$ are distinct vertices in a graph $G$.

1. Two paths between $v$ and $w$ are edge-disjoint if they have no common edges.
2. A set $S$ of edges is called $vw$-disconnecting if in $G - S$ the vertices $v$ and $w$ do not belong to the same component.
Menger’s theorem

Suppose $v$ and $w$ are distinct vertices in a graph $G$.

1. Two paths between $v$ and $w$ are \textit{edge-disjoint} if they have no common edges.

2. A set $S$ of edges is called \textit{vw-disconnecting} if in $G - S$ the vertices $v$ and $w$ do not belong to the same component.

Last time we proved the following.

\textbf{Theorem 6.5} The maximum number of edge-disjoint paths connecting two distinct vertices $v$ and $w$ of a connected graph is equal to the minimum number of edges in a $vw$-disconnecting set.

What can we say about the situation when we are removing the vertices instead of the edges?
Suppose \( v \) and \( w \) are distinct vertices in a graph \( G \).

1. Two paths between \( v \) and \( w \) are *vertex-disjoint* if they have no common vertices.

2. A set \( S \) of edges is called \( vw \)-separating if in \( G - S \) the vertices \( v \) and \( w \) do not belong to the same component.

The following Menger’s Theorem was proved by Menger (unlike the first Menger’s Theorem).

**THEOREM 6.6** (Menger, 1927) *The maximum number of vertex-disjoint paths connecting two distinct non-adjacent vertices \( v \) and \( w \) of a graph is equal to the minimum number of vertices in a \( vw \)-separating set.*

The proof of this theorem is similar to the proof of the first Menger’s theorem, and therefore omitted.
By fixing a graph $G$ and looking at all pairs $v, w$ of distinct vertices of $G$, we obtain the following consequences of Menger’s theorems.

**COROLLARY 6.7** A graph $G$ is $k$-edge-connected if and only if any two distinct vertices of $G$ are connected by at least $k$ edge-disjoint paths.

**COROLLARY 6.8** A graph $G$ with at least $k + 1$ vertices is $k$-connected if and only if any two distinct vertices of $G$ are connected by at least $k$ vertex-disjoint paths.
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The digraph version also has a similar proof (omitted).

**Theorem 6.9** The maximum number of arc-disjoint paths from a vertex $v$ to a vertex $w$ in a digraph is equal to the minimum number of arcs in a $vw$-disconnecting set.
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**Theorem 6.9** The maximum number of arc-disjoint paths from a vertex $v$ to a vertex $w$ in a digraph is equal to the minimum number of arcs in a $vw$-disconnecting set.
Here is a curiosity.

**Theorem**

*The vertex form of Menger’s Theorem implies Hall’s Theorem.*
Network flows

A weighted digraph is called a *network*. The ‘weight’ of an arc $a$, denoted $c(a)$, is called its *capacity*. 
Network flows

A weighted digraph is called a \textit{network}. The ‘weight’ of an arc $a$, denoted $c(a)$, is called its \textit{capacity}.

The in-degree of a vertex $v$ is the sum of the capacities of all arcs of the form $wv$, and the out-degree is defined similarly.
Network flows

A weighted digraph is called a network. The ‘weight’ of an arc \(a\), denoted \(c(a)\), is called its capacity.

The in-degree of a vertex \(v\) is the sum of the capacities of all arcs of the form \(wv\), and the out-degree is defined similarly. The following is the Handshaking Dilemma for networks.

*The sum of the out-degrees of the vertices of a network is equal to the sum of the in-degrees.*
Flows, Maximum Flows

If $\text{in-deg}(v) = 0$ then $v$ is a *source*. If $\text{out-deg}(v) = 0$ then $v$ is a *sink*. We will assume that our network has exactly one source, $v$, and exactly one sink, $w$.

Easily reduced to this special case (see Exercise 6.12):

A *flow* in a network is a function $\phi$ that assigns to each arc $a$ a non-negative real number $\phi(a)$, called the *flow in $a$*, in such a way that

(i) for each arc $a$, $\phi(a) \leq c(a)$;
(ii) the out-degree and in-degree of each vertex, other than $v$ or $w$, are equal.

An arc $a$ for which $\phi(a) = c(a)$ is called *saturated*. 
Flows, Maximum Flows

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A flow in a network is a function $\varphi$ that assigns to each arc $a$ a non-negative real number $\varphi(a)$, called the flow in $a$, in such a way that

(i) for each arc $a$, $\varphi(a) \leq c(a)$;
(ii) the out-degree and in-degree of each vertex, other than $v$ or $w$, are equal.

An arc $a$ for which $\varphi(a) = c(a)$ is called saturated.
The value of the flow $\varphi$ is equal to the sum of $\varphi(a)$ for all arcs $a$ originating in $v$.
It is equal to the sum of $\varphi(a)$ for all arcs $a$ ending in $w$.
A flow is maximal if its value is as large as possible.
A cut is a set $S$ of arcs such that each path from $v$ to $w$ contains an edge in $S$.
The capacity of a cut is $\sum_{a \in S} c(a)$.
A minimum cut is a cut whose capacity is as small as possible.
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**Theorem 6.11** (Max-flow min-cut theorem) In any network, the value of any maximum flow is equal to the capacity of any minimum cut.
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The capacity of a cut is $\sum_{a \in S} c(a)$.
A minimum cut is a cut whose capacity is as small as possible.

**Theorem 6.11** (Max-flow min-cut theorem) In any network, the value of any maximum flow is equal to the capacity of any minimum cut.

We will give two different proofs.
First, we show that Theorem 6.6 is a corollary of

**Theorem 6.9** The maximum number of arc-disjoint paths from a vertex $v$ to a vertex $w$ in a digraph is equal to the minimum number of arcs in a $vw$-disconnecting set.
The second proof gives an algorithm that takes any flow as an input and ‘massages’ it into a maximal flow.