The final exam will take place at the previously announced time. It will be conducted using Zoom. You will need a web camera and reliable internet connection for this exam. If you cannot arrange for either, please contact me asap.
5.4. The Four Colour Theorem

Every planar map is 4-colourable (f).
This section will not be on the exam.
5.5 Colouring edges

A graph $G$ is \textit{k-colourable-(e)} (or $k$-edge colourable) if its edges can be coloured with $k$ colours so that no two adjacent edges have the same colour. If $G$ is $k$-colourable-(e) but not $(k - 1)$-colourable-(e), we say that the \textit{chromatic index} of $G$ is $k$, and write $\chi'(G) = k$. For example, Fig. 5.40 shows a graph $G$ for which $\chi'(G) = 4$. 

\textbf{Figure 5.40}

$\chi'(K_3) = 3$

$\chi'(C_4) = 2$

$\chi'(C_5)$
As before, $\Delta$ is the maximal degree of a vertex in $G$.

**Example**

Note that for any $C_n$, $\Delta = 2$. We have

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**Example**

Note that for any \( C_n \), \( \Delta = 2 \). We have

\[
\chi'(C_3) = \\
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\text{In general, } \chi'(C_{2n}) = \Delta = 2 \text{ and } \chi'(C_{2n+1}) = \Delta + 1 = 3.
\]
THEOREM 5.15 (Vizing, 1964) If \( G \) is a simple graph with largest vertex-degree \( \Delta \), then

\[
\Delta \leq \chi'(G) \leq \Delta + 1.
\]

\( \Delta \leq \chi'(G) \) is easy:

We will not prove the other inequality and it will not be on the exam.
Recall: \( \chi(K_n) = n \).
\[ \Delta(K_n) = n - 1 \]

**THEOREM 5.16** \( \chi'(K_n) = n \) if \( n \) is odd \( (n \geq 3) \), and \( \chi'(K_n) = n - 1 \) if \( n \) is even.
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Proof: Recall that \( m = n(n - 1)/2 \) and \( \Delta = n - 1 \).

If \( n \) is odd, \( n = 2k + 1 \),
at most $k$ edges can have the same colour $(2k \leq n)$. Therefore, $X'(K_{2k+1}) \geq 2k+1$. Let's prove $X'(K_{2k+1}) \leq 2k+1$. $2k+1 = 5$
This proves the case when \( n = 2k + 1 \) is odd.

If \( n = 2k \), even, \( K_{2k} \), we need to prove \( \chi'(K_{2k}) = 2k - 1 \).

\[ \Delta(K_{2k}) = 2k - 1 \]

So, \( \chi'(K_{2k}) \geq 2k - 1 \).
A proof that \( X'(K_{2h}) \leq 2h - 1 \).

\[ K_{2h} : \]

we have \( X'(K_{2h-1}) = 2h - 1 \).

\[ X(K_{2h}) = 2h - 1 . \]
Recall that a map is a 3-connected plane graph, and that a map is cubic if every vertex has degree equal to 3.

**Theorem 5.17** The four-colour theorem is equivalent to the statement that $\chi'(G) = 3$ for each cubic map $G$. 
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We will use addition modulo 2:

- $1 +_2 0 = 0 +_2 1 = 1$,
- $0 +_2 0 = 0$, and
- $1 +_2 1 = 0$. 

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We will use addition modulo 2:

\[
1 +_2 0 = 0 +_2 1 = 1, \\
0 +_2 0 = 0, \text{ and} \\
1 +_2 1 = 0.
\]

Proof: Fix a cubic map. Suppose that we have a 4-colouring $(f)$ with colours equal to $\alpha = (1, 0)$, $\beta = (0, 1)$, $\gamma = (1, 1)$ and $\delta = (0, 0)$. 

\[
\begin{cases}
\end{cases}
\]
\[ \alpha + 2\gamma = (1, 0) + 2(1, 1) = (0, 1) \quad \text{use } 0 \]

\[ \beta + 2 \delta = (0, 1) + 2(1, 1) = (1, 0) \quad \text{use } 0 \]

\[ \alpha + 2 \beta = (1, 0) + 2(0, 1) = (1, 1) \quad \text{use } 0 \]

\[ \delta + 2 \gamma = (0, 0) + 2(1, 1) = (1, 1) \]

Fact: \[ X + 2\xi = (0, 0) \]

(\Rightarrow ) \quad X = \xi.
Hence, only (1, 0) and (1, 1) appear as colours of edge. Why adjacent edges have distinct colours?

For \( e \): \( x + 2 \gamma = x + i \in \mathbb{C} \\setminus \{0\} \)

\[
\Rightarrow \gamma = z
\]
Proof \( \Theta = \) 

\[
x + 2 (x + 2) = x + 2 (x + 2)
\]

\[
(x + x + 2) = (x + 2 x + 2)
\]

\[
\begin{bmatrix}
0 & 0 \\
0 & 1
\end{bmatrix}
\]

\[
\gamma = \pi
\]

since the map is cubic, e and f assume distinct colours.
$(\Leftarrow)$ Assume $\chi'(G) = 3$.

Colours: $\alpha, \beta, \gamma, \delta$.

Let $G_1 = \{ e \mid \text{colour of } e \text{ is not } \delta \}$. $G_1$ has only even cycles. Also, $G_1$ is regular of degree 2.
By Theorem 5.8, \( G_2 \) can be 2-coloured (i.e.,

\[ \text{Repeat, removing } \alpha \text{ instead of } \beta, \gamma, G_2. \]
Each face of $G$ is an intersection of faces of $G_1$ and $G_2$.

Colors: pairs $(0, b)$

$a = \text{[colors]}$

$b = \text{[colors]}$

$f_1 = f_1 \cap f_2$

$f_2 \cap f_2' = f_2'$